Lecture 5: Combinatorics

MA208 Quantitative Techniques for Business

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Unit II: Counting and Probability

In this unit we will cover

- Combinatorics/ Counting Techniques
- Probability
- Random Variables

Lecture 5 - Outline

To pursue our study more we need to learn some basic principles from Probability, Counting and Combinatorics.

An efficient way of counting is necessary to handle large masses of data.
Combinatorics is the study of the number of ways a set of objects can be arranged, combined, or chosen.

Today we will introduce

- The Additive Principle
- The Multiplicative Principle
- Permutations
- Combinations.

Combinatorics / Counting Techniques

Example

1. The Burger Bar offers the following items on its menu:

Burger	**Sides**	**Beverages**	**Desserts**
Single Meat	Fries	Tea	Cheesecake
Double Meat	Onion Rings	Coffee	Brownie
	Fruit Bowl	Soda	Cookie
	Mixed Salad		Ice Cream

   (a) If a customer wants only one item, how many choices does s/he have?

   (b) If a customer chooses one item from each category, how many meals can be made?

2. Picture a room in a building. The room has two exits and the building has three exits. In how many ways can we exit the building?

Solution

1. a) $2+4+3+4=13$.
   b) $2\times 4\times 3\times 4=96$.
2. Presumably $6=2\times 3$.

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   To be a bit cheeky, we can also argue the answer to be 3, in 2 different ways:
   - Room exit 1 = Building exit 1. (Note, in that case, it matters not if Room exit 2 = Building exit 2 as well.)
   - We are not in the room :).

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   Please don't do that during an exam! :) If you have questions and are unsure, please ask!

The Additive Principle

Example

The student union shop offers 4 different University of Galway Hoodies and 7 different University of Galway T-Shirts. You would like to buy a top with University of Galway logo for your friend, an Erasmus student from Hungary who is going home next week.

You can afford only one item, how many choices do you have?

Solution

$4 + 7 = 11$ choices.

The Additive Principle

The Additive Principle

If event A can occur in m ways and event B can occur in n (disjoint) ways, then event "A **or** B" can occur in $m + n$ ways.

Example

By "word" we mean any 2 letter combinations. E.g., "FR" would be a "word".

1. Can we use the additive principle to determine how many two letter "words" (2LW) begin with either F **or** K?
2. Can we use the additive principle to determine how many two letter "words" (2LW) contain either F **or** K?

Solution

1. Yes. There are 26 2LWs beginning with "F", 26 2LWs beginning with K and $52 = 26 + 26$ 2LWs beginning with F or K.
2. No (not directly). "FK" and "KF" belong in both, hence these events are **not** disjoint.

The Multiplicative Principle

The Multiplicative Principle

If event A can occur in m ways and each possibility allows for event B to occur in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

Examples

1. When buying a car you can pick four types of engine and six colours. How many choices do you have?
2. How many non-repeating two letter combinations of the vowels a, e, i, o, u can you find?
   (non-repeating = letters are supposed to be different.)

Solution

1. There are $4 \times 6 = 24$ possible choices.
2. There are $5 \times (5 - 1) = 20$ possible choices.

Factorial notation

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$$n! = (n)(n-1)(n-2) \cdot \ldots \cdot (3)(2)(1)$$

Examples

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$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ 10! &= 10 \times 9 \times 7 \times \ldots \times 3 \times 2 \times 1 \end{aligned}$$

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Examples

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ 10! &= 10 \times 9 \times 7 \times \ldots \times 3 \times 2 \times 1 \\ 0! &= 1 \quad \text{This is a convention.} \end{aligned}$$

(There are axtually mathematical reasons...)

Permutations

An arrangement (or ordering) of a set of objects is called a permutation. In a permutation, the **order** that we arrange the object in, **is important**.

Example

Consider arranging three letters. In how many ways can this be done?

Arranging *n* objects

In general, n distinct objects can be arranged in $n!$ ways.

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Arranging *n* objects

In general, n distinct objects can be arranged in $n!$ ways.
Can you argue why?

In our example above we have

$$3! = 3 \times 2 \times 1 = 6$$

possible ways to arrange the letters.

Permutations

Examples

1. Find all two letter permutations of $\{a, b, c, d\}$.
2. Find all three letter permutations of $\{a, b, c, d\}$.

Solution

1. $\left\{ \begin{array}{lll} AB, & AC, & AD, \\ BA, & BC, & BD, \\ CA, & CB, & CD, \\ DA, & DB, & DC \end{array} \right\}$

2. $\left\{ \begin{array}{llll} abc, & abd, & acb, & acd, \\ adb, & bac, & bad, & bca, \\ bcd, & bda, & cab, & cad, \\ cba, & cbd, & cda, & dab, \\ dac, & dba, & dbc, & dca \end{array} \right\}$

Permutations

Number of Permutations

The number of permutations of of length r, chosen from (fixed) n distinct objects is denoted by P_r^n. Repetitions are **not** allowed here. We have

$$P_r^n = (n)(n-1)(n-2)\cdot\cdots\cdot(n-r+1) = \frac{n!}{(n-r)!}.$$

Note: Other common notations are $P(n,r)$ and ${}^nP_r$.

Use this formula for the examples above.

Solution

1. $P_2^4 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4\times3\times2}{2} = 12.$
2. $P_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{4\times3\times2}{1} = 24.$

Combinations

Let us fix n objects. A selection of r objects from the fixed n objects, which does not take into account the arrangements of these objects is called a **combination**. That is to say, the **order** is **not important**.

Example

Students are asked to choose four different letters from the English alphabet. David selects $\{A, E, R, T\}$, Sarah selects $\{D, E, N, Q\}$ and Anna selects $\{R, E, A, T\}$.

1. How many sets of four letters have the three students chosen together?
   - 2 Sets, as $\{A, E, R, T\} = \{R, E, A, T\}$.
2. How many different sets of four letters can be selected from the alphabet?
   - We can write it all out ... Or we can remember the formulas for P_r^n and divide by how many different permutations of size r we have (since after forgetting the order, they will all give the same combination).

Combinations

If we did the second example right, we should have derived the following formula:

Number of Combinations

The number of ways (or combinations) in which r objects can be selected from a set of n objects, where repetition is **not** allowed, is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Note: C_r^n is often denoted by $\binom{n}{r}$ ("n choose or r") or ${}^nC_r$.

Use this formula to calculate the number of different sets of four letters which can be chosen from the alphabet.

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Use this formula to calculate the number of different sets of four letters which can be chosen from the alphabet.

$$C_4^{26} = \binom{26}{4} = \frac{26!}{4!(26-4)!} = \frac{26!}{4! \times 22!} = 14950$$

Exercises

1. You are a salesperson for a certain company and next month you must travel to eight destinations in U.S.A.: Dallas, Los Angeles, Miami, Boston, Chicago, Seattle, Omaha, and Kansas City. How many different possible routes can you take?
2. Suppose you plan to invest equal amounts of money in each of five business ventures. If you have 20 ventures from which you make the selection, how many different samples of five ventures can be selected from the 20?
3. In how many ways can the tax office pick five out of 36 companies to audit?
4. Out of 5 mathematicians and 7 engineers a committee is to be formed that must consist of 2 mathematicians and 3 engineers. In how many ways can this be done?

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