Lecture 16: Using Gaussian Elimination to solve a linear system

MA208 Quantitative Techniques for Business

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Recall from last lecture:

We learned that in some cases the inverse matrix can be used to solve linear systems of equations. However, finding the inverse matrix can be a challenge.

We learned a formula to calculate the inverse of a 2×2 matrix and the *Gauss Jordan method* to calculate the inverse of an $n \times n$ matrix. This method works for any $n \times n$ matrix (provided it has an inverse), but it takes time to calculate the inverse of a large matrix.

However, sometimes we are lucky. Consider the following example...

Example

Let $A = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 8 & -6 \\ -7 & -1 & 12 \\ 8 & -4 & -6 \end{pmatrix}$.

Compute $A \cdot B$.

Solution

$$AB = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 8 & -6 \\ -7 & -1 & 12 \\ 8 & -4 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \cdot 2 + 4(-7) + 5 \cdot 8 & 3 \cdot 8 + 4(-1) + 5(-4) & 3(-6) + 4 \cdot 12 + 5(-6) \\ 3 \cdot 2 + 2(-7) + 1 \cdot 8 & 3 \cdot 8 + 2(-1) + 1(-4) & 3(-6) + 2 \cdot 12 + 1(-6) \\ 2 \cdot 2 + 4(-7) + 3 \cdot 8 & 2 \cdot 8 + 4(-1) + 3(-4) & 2(-6) + 4 \cdot 12 + 3(-6) \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix} = 18$$

So we have

$$A \cdot B = 18\, I$$

or

$$A \cdot \frac{1}{18}\, B = I$$

Recall: If $A \cdot M = I$, then $M = A^{-1}$.

Thus we get

$$\frac{1}{18}\, B \;=\; A^{-1} \qquad (*)$$

We can use this to solve the following problem. How?

Example

A bakery wants to produce three types of bread. Each morning, one bag of wheat flour, one bag of rye flour and one bag of barley flour are opened. The required quantities of each flour for one loaf of bread in each of the three recipes are specified in the table below, in terms of multiples of the measuring cup that is used for taking flour out of the bags.

Grain type	Bread A	Bread B	Bread C	Size of each flour bag
Wheat	3 cups	4 cups	5 cups	86 cups
Rye	3 cups	2 cups	1 cup	40 cups
Barley	2 cups	4 cups	3 cups	64 cups

Example (ctd.)

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(i) Let x, y and z be the numbers of bread loaves A, B and C produced in one morning. Write down a system of three linear equations which hold precisely when all the three resources are fully used.

(ii) Find the values of x, y and z which ensure that the bags of flour are fully used.

Solution

(i) System of equations

Let x, y, z *be the numbers of loaves of bread A, B and C.*

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The sys. of linear equations is

$$\begin{aligned} 3x + 4y + 5z &= 86 \quad \textit{(wheat)} \\ 3x + 2y + z &= 40 \quad \textit{(rye)} \\ 2x + 4y + 3z &= 64 \quad \textit{(barley)} \end{aligned}$$

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In matrix form, this is

$$\begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}.$$

Solution (ctd.)

(ii) Solving the matrix equation

$$\begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix} :$$

Solution (ctd.)

(ii) Solving the matrix equation

$$\begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix} :$$

We essentially have

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix} \qquad \textit{with} \quad A = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix}$$

Multiplying by A^{-1} *gives us*

$$A^{-1}A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}.$$

Solution

$$A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}.$$

Using the fact that we know A^{-1} *from the last example, we have*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 2 & 8 & -6 \\ -7 & -1 & 12 \\ 8 & -4 & -6 \end{pmatrix}\begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}$$

$$= \frac{1}{18}\begin{pmatrix} 2\cdot 86 + 8\cdot 40 - 6\cdot 64 \\ -7\cdot 86 - 1\cdot 40 + 12\cdot 64 \\ 8\cdot 86 - 4\cdot 40 - 6\cdot 64 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 108 \\ 126 \\ 144 \end{pmatrix}$$

$$(x, y, z) = (6, 7, 8).$$

Recap

We introduced systems of linear equations and revised methods to solve them efficiently by eliminating variables.

We then showed how systems of linear equations can be represented by matrices. Many problems can be **expressed** in terms of matrices, and **solved** by operations on matrices. Matrices can be added and multiplied (provided they have matching sizes). Elementary row operations can be used to find the inverse of a **square** matrix, and the inverse matrix can be used to solve systems of linear equations, provided they have the same amount of variables and equations.

We will now learn how elementary row operations can be used to solve **any** systems of linear equations: we will introduce an algorithm called **Gaussian Elimination**.

Let's look at an example to introduce this algorithm.

Gaussian Elimination

Consider the following linear system

(i) $a + b + c = 4$

(ii) $8a + 2b + c = -1$

(iii) $-a + b + c = 2$

This set of equations is represented by the **augmented matrix**

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & 1 & 1 & 2 \end{pmatrix}.$$

By a sequence of elementary row operations this matrix can be changed to

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

We will show later how this transformation has been achieved.

Comparing this with the original set of equations it is clear that the third row of the matrix represents the equation

$$c = 3.$$

Substitution of this value of c into the equation represented by Row 2 gives

$$b + \frac{7}{6}c = \frac{33}{6}$$
$$b = \frac{33}{6} - \frac{21}{6} = 2.$$

Finally Row 1 of the simplified matrix represents the equation

$$a + b + c = 4$$
$$a + 2 + 3 = 4$$
$$a + 5 = 4$$
$$a = -1$$

Thus the solution to the linear system is $a = -1, b = 2, c = 3$.

The simplified matrix

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

is said to be in the **row echelon form**.

It represents the same linear system as the original matrix, but it enables us to arrive at the solution to the linear system in just a few calculations as seen before.

We will now learn how to reduce the matrix to row-echelon form by using **Gaussian Elimination**.

Solve the linear system

$$\begin{array}{rcrcrcr} a & + & b & + & c & = & 4, \\ 8a & + & 2b & + & c & = & -1, \\ -a & - & b & + & c & = & 2. \end{array}$$

First, represent the linear system by the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix}$$

Now reduce the matrix to *row echelon form* by using *Gaussian Elimination*.

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 8R_1,\, R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -6 & -7 & -33 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{6}R_2,\, R_3 \leftarrow \frac{1}{2}R_3} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

From here we get the solution of the linear system:

- by **back-substitution** (as we did in the beginning of the lecture), or
- by finding the ***reduced row echelon form*** (eliminating values above the main diagonal).

Note: In this course we will use back-substitution.

Example

A corporation wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15 and 20 passengers, respectively. How many of each type of plane could be leased?

Note: This problem has more than one solution. You are required to find **all** solutions!

Solution

$$x + y + z = 12$$

and

$$10x + 15y + 20z = 220.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 10 & 15 & 20 & 220 \end{array}\right)$$

...

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -8 \\ 0 & 1 & 2 & 20 \end{array}\right)$$

$$R1 \Rightarrow x - z = -8$$

$$R2 \Rightarrow y + 2z = 20$$

Solution

x, y, z *need to be non-negative integers satisfying*

$$x = z - 8 \geqslant 0$$

$$y = -2z + 20 \geqslant 0$$

hence

$$z \geqslant 8$$

$$z \leqslant 10$$

To sum up

$$8 \leqslant z \leqslant 10$$

and so the possible solutions are

x	0	1	2
y	4	2	0
z	9	9	10

Example

Example

The University of Tennessee Lady Volunteers defeated the Rutgers University Scarlet Knights 59 to 46. The Lady Volunteers' scoring resulted from a combination of three-point baskets, two-point baskets, and one-point free throws. There were three times as many two-point baskets as three-point baskets. The number of free throws was one less than the number of two-point baskets. (Source: American National Collegiate Athletic Association)

(a) Set up a system of linear equations to find the numbers of three-point baskets, two-point baskets, and one-point free throws scored by the Lady Volunteers.

(b) Solve your system.

Solution