Lecture 14: The Inverse Matrix

MA208 Quantitative Techniques for Business

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Recall from last lecture:

In our last lecture we discussed **linear systems** and learned that we can use **matrices** to solve such systems. So we introduced matrices and **matrix operations**.

In particular, we learned that the identity matrix I behaves a lot like the number 1 in "ordinary" multiplication of numbers, and introduced:

The Inverse Matrix

Let A be a square matrix. We call A^{-1} the inverse matrix of A if

$$AA^{-1} = A^{-1}A = \mathrm{I}$$

In some cases the inverse matrix can be used to solve linear systems of equations.

Solving a system of linear equations (*Inverse-method*)

Consider the system of linear equations from last lecture:

$$\begin{array}{rcrcl} 2x & + & y & = & 8 \\ x & + & 3y & = & 9 \end{array}$$

We can write system this in matrix form as follows:

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}.$$

We may write this as

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \qquad \text{where } A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

The square matrix A has an inverse, A^{-1}. Let us multiply each side of the above equation by A^{-1} from the left.

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 8 \\ 9 \end{pmatrix},$$

giving

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 8 \\ 9 \end{pmatrix}.$$

This states that the values of x and y may be found by multiplying the column vector $\begin{pmatrix} 8 \\ 9 \end{pmatrix}$ by the inverse of A.

How do we find A^{-1}?

Finding the inverse of a 2 x 2 matrix

Inverse of a 2 x 2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

provided that $ad \neq bc$.

Example

(i) Find the inverse matrix of $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.

(ii) Check your result from (i), i.e. is $A \cdot A^{-1} = I$?

Solution

Inverse method, ctd.

Now we can solve the system $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$ from above.

$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, so we get

$$A^{-1} = \frac{1}{6-1} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

Let's continue our calculations from above:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{15}{5} \\ \frac{10}{5} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

So the solution is $x = 3$ and $y = 2$ as before.

Finding the inverse of an n x n matrix

Finding A^{-1} for any (i.e. $n \times n$) matrix is more complicated.

There are several methods to do this. In this course we will introduce the Gauss-Jordan method.

Gauss-Jordan method

The Gauss-Jordan method finds the inverse A^{-1} of a square matrix A:

$$(A|I) \overset{\text{row ops}}{\longrightarrow} (I|A^{-1}).$$

Row operations

We can

- Swap two rows,
- Multiply a row with a number,
- Add a multiple of one row to another.

Example

Find the inverse of $A = \begin{pmatrix} 4 & -1 \\ -6 & 2 \end{pmatrix}$.

Solution

Example

Find the inverse of $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ -1 & 3 & 0 \end{pmatrix}$.

Solution