Lecture 11: Present Value of an Annuity / Amortization

MA208 Quantitative Techniques for Business

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Lecture 12 - Outline

Today we will talk about

- Present value of an Annuity
- Amortization

Recall from last lecture:

In our last lecture we used the Compound Interest Formula to develop the formula below to calculate the future value of an annuity or sinking funds.

Future Value of an Ordinary Annuity

$$\text{FV} = \text{PMT}\frac{(1+i)^n - 1}{i}$$

where

- FV $=$ future value (amount),
- PMT $=$ periodic payment,
- $i =$ rate per period,
- $n =$ number of payments,

with the understanding that payments are made at the end of each period.

Example

A company estimates that it will need to replace a piece of equipment at a cost of €$50,000$ in 5 years. To have this money available in 5 years it establishes a sinking fund by making equal monthly payments into an account paying $6.6\%\,\mathrm{p/a}$ compounded monthly.

(i) How much should each payment be?

(ii) How much interest is earned during the last year?

Solution

(i) $\mathrm{FV} = 50000, \quad \mathrm{i} = \frac{0.06}{12} = 0.0055, \quad , \mathrm{n} = 60, \quad \mathrm{PMT} = ?$

$$50000 = \mathrm{PMT}\frac{(1+0.055)^{60}-1}{0.0055}$$

which implies that

$$\mathrm{PMT} = \text{€}\,705.65 \text{ per month.}$$

Interest during last years

Amount after 4 years:

$$\text{PMT} = 705.65, \quad \mathrm{i} = 0.0055, \quad \mathrm{n} = 4 \cdot 12 = 48$$

$$\text{FV} = 705.65 \frac{1.0055^{48} - 1}{0.0055} = \text{€}\, 38642.25.$$

Growth in the last years $=$ Total - first 4 years:

$$50000 - 38642.25 = 11357.75$$

Payments during 5th year:

$$12 \times 705.65 = 8467.80.$$

Interest earned during 5th year:

$$11357.75 - 8467.80 = \text{€}\, 2889.95.$$

Present value of an Annuity

Example

How much should you deposit in an account paying 6% compounded semiannually in order to be able to withdraw €1000 every six months for the next three years?

Note: After the last payment is made, no money is to be left in the account.

We might think its €5024.9, which we get by solving for P in

$$6000 = P\left(1 + \frac{0.06}{2}\right)^{3\cdot 2} = P(1.03)^6 = P \cdot 1.194$$

but that is not correct. We need to deposit €5024.9 to have €6000 after 3 years. But in our question, we are withdrawing €1000 every 6 months. These are subsequently no longer earning interest.
Note that the first €1000 withdrawal has a higher value than the last.

In our example, the 'zeroth' €1000 (that is to say, if we just put €1000 less in the bank) has a present value of €1000 (of course) and a future value of

$$FV_0 = 1000 \cdot (1.03)^6 \approx 1194.$$

The first €1000 has a present value of

$$P_1 = 1000 \cdot (1.03)^{-1} \approx 970, \quad FV_1 = 1000 \cdot (1.03)^5 \approx 1159.$$

For the second €1000, we have

$$P_2 = 1000 \cdot (1.03)^{-2} \approx 942, \quad FV_2 = 1000 \cdot (1.03)^4 \approx 1126.$$

To solve the example, it is easier to look at the present values, as we want:

$$P = P_1 + P_2 + P_3 + P_4 + P_5 + P_6.$$

Present value of an Annuity

Let us go through this a bit slower:
Here we are interested in finding the present value of each €1000 that is paid out during the three years.

We can find this by using the compound interest formula and solving for P:

$$\begin{aligned} A &= P(1+i)^n \\ P &= \frac{A}{(1+i)^n} = A(1+i)^{-n}, \end{aligned}$$

where the rate per period is $i = \frac{0.06}{2} = 0.03$.

The present value for the **first** payment is

$$P_1 = 1000\,(1.03)^{-1}$$

The present value for the **second** payment is

$$P_2 = 1000\,(1.03)^{-2}$$

...

The present value for the **sixth** payment is

$$P_6 = 1000\,(1.03)^{-6}$$

So the present value **after** 3 **years** (i.e. 6 payments) is

$$P = 1000(1.03)^{-1} + 1000(1.03)^{-2} + ... + 1000(1.03)^{-6} \qquad (1)$$

Multiplying equation **(1)** by 1.03, we get

$$P = 1000(1.03)^{-1} + 1000(1.03)^{-2} + \ldots + 1000(1.03)^{-6} \quad \textbf{(1)}$$

$$1.03P = 1000 + 1000(1.03)^{-1} + \ldots + 1000(1.03)^{-5} \quad \textbf{(2)}$$

Now we subtract equation **(1)** from equation **(2)**:

$$\begin{aligned} 1.03P - P &= 1000 - 1000(1.03)^{-6} \\ 0.03P &= 1000 - 1000(1.03)^{-6} \\ P &= \frac{1000 - 1000(1 + 0.03)^{-6}}{0.03} \\ P &= 1000\,\frac{1 - (1 + 0.03)^{-6}}{0.03} \end{aligned}$$

Present value of an Annuity

In general the Present Value of an Ordinary Annuity is

$$PV = PMT \frac{(1-(1+i)^{-n})}{i}$$

where $PV =$ present value of all payments, $PMT =$ periodic payment, $i =$ rate per period, and $n =$ number of periods.
Note: Payments are made at the end of each period.

So in our example, we get

$$PV = 1000 \frac{(1-(1.03)^{-6})}{0.03} = €5417.19.$$

Example

What is the present value of an annuity that pays €200 per month for five years if money is worth 6% compounded monthly?

Solution

$$PV = PMT\frac{1-(1+i)^{-n}}{i}$$

$$PMT = 200, \quad , i = \frac{0.06}{12} = 0.005, \quad n = 5 \cdot 12 = 60.$$

We get

$$PV = 200 \cdot \frac{1-(1.005)^{-60}}{0.005} = € 10345$$

Amortization

The present value formula for an ordinary annuity is also used to determine the distribution of loan repayments, as determined by an **amortization** schedule.

Example

Suppose you borrow €5000 from a bank to buy a car and agree to repay the loan in 36 equal monthly payments, including all interest due. If the bank charges 1% per month on the unpaid balance (i.e. 12% per year compounded monthly), how much should each payment be?

Here, the bank has bought an annuity from you. So the question is:

If the bank pays you $PV = €\,5000$ for an annuity, what are the periodic payments PMT?

Solution

We have $PV = 5000, \quad i = 0.01, \quad n = 36.$

$$\begin{aligned} PV &= PMT\frac{1-(1+i)^{-n}}{i} \\ 5000 &= PMT\frac{1-(1.01)^{-36}}{0.01} \\ PMT &= 5000\frac{0.01}{1-(1.01)^{-36}} \\ PMT &= €\,166.07 \end{aligned}$$