Lecture 10: Future Value of an Annuity / Sinking Funds

MA208 Quantitative Techniques for Business

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Lecture 11 - Outline

Today we will talk about

- Continuous Compound Interest,
- Future Value of an Annuity,
- Sinking Funds.

Recall from last lecture:

In our last lecture we used the Simple Interest Formula to develop:

Compound Interest Formula

The amount A to which a principal P will grow after n compounding periods at a rate of i per compounding period is given by

$$A = P(1+i)^n.$$

Continuous Compound Interest Formula

If a principal P is invested at an annual rate r, then the amount A at the end of t years is given by

$$A = Pe^{rt}$$

Example

What amount will an account have after two years if €5000 is invested at an annual rate of 8%

(i) compounded daily?

(ii) compounded continuously?

Solution

(i) Use the compound interest formula:

$$A = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

In our case: $P = 5000, \quad r = 0.08, \quad m = 365, \quad t = 2$. This gives us

$$A = 5000\left(1+\frac{0.08}{365}\right)^{(365\cdot 2)} = 5867.45$$

The compound interest formula:

$$\begin{aligned} A &= Pe^{rt} \\ &= 5000e^{0.08 \cdot 2} \\ &= 5867.55 \end{aligned}$$

Example

Bank A is offering a rate of 4.6% to savers, compounded annually. Bank B is offering 4.5%, compounded continuously. Which offer is better?

Solution

Bank A : 4.6% Compounded annually
Bank B : 4.5% Compounded continuously

$$\begin{aligned} A = A_A &= P(1+i)^n \\ A = A_A &= P(1+r)^t \\ &= P \cdot 1.046^t \end{aligned}$$

Note, we changed $i = r$, $n = t$ to highlight that its compounded annually. For quarterly, we would have written $i = r/4$, $t = 4n$.

$$\begin{aligned} B = A_B &= P(e)^{rt} \\ &= Pe^{0.045t} \end{aligned}$$

How do we compare $P \cdot 1.046^t$ and $Pe^{0.045t}$?

A_A	A_B
$P \cdot 1.046^t$	$Pe^{0.045t}$
1.046^t	$e^{0.045t}$
$\ln(1.046^t)$	$\ln(e^{0.045t})$
$t \cdot \ln(1.046)$	$0.045t$
$\ln(1.046)$	0.045
0.04497	0.045

and as $0.04497 < 0.045$, Bank B's offer is better!

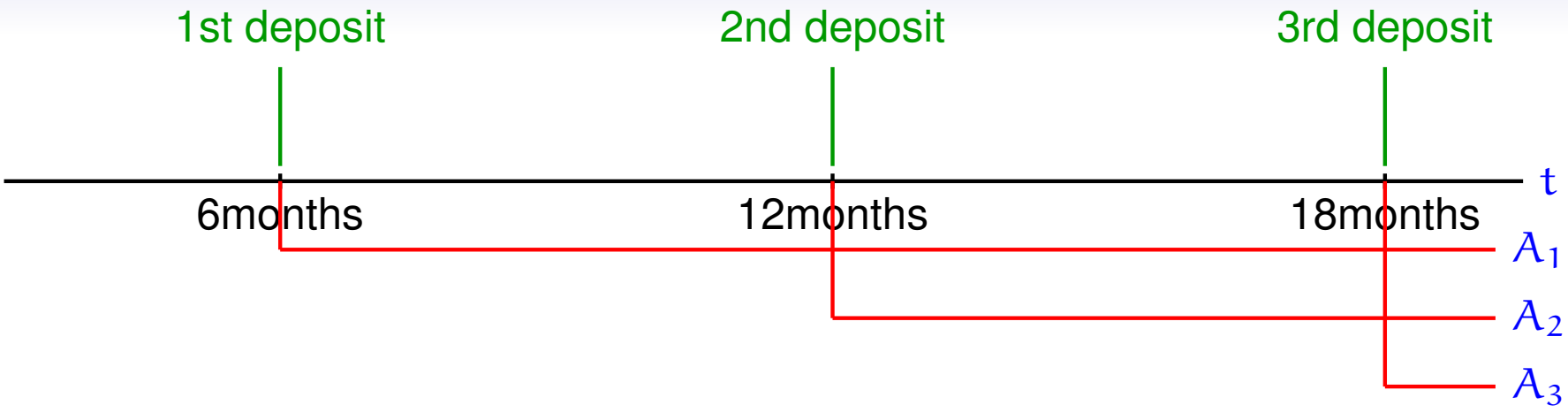
Future Value of an Annuity

An annuity is any sequence of equal periodic payments. The amount, or future value, of an annuity is the sum of all payments made plus all interest earned. (N.B. payments are made at the end of each time period, known as an ordinary annuity.)

Example

Suppose you decide to deposit €100 every 6 months into an account that pays 6% p/a compounded semiannually. If you make three deposits, one at the end of each interest payment period, over eighteen months, how much money will be in the account after the last deposit is made?

By analyzing this example we will derive a formula for the future value of an annuity.



Total $S = A_1 + A_2 + A_3$

We can use the compound interest formula $A = P(1+i)^n$ to calculate the interest earned by each deposit over the eighteen month period.

- At the end of the first 6 month period $€100$ is deposited. This sum earns compound interest at the end of the second and third 6 month period (i.e. twice). It follows then that this $€100$ deposit grows to

  $$A = 100(1.03)^2,$$

  by the end of the 18 months.

- At the end of the second 6 month period another $€100$ is deposited. This sum earns interest at the end of the third 6 month period (i.e. once). It follows then that this $€100$ deposit grows to

$$A = 100(1.03)^1$$

by the end of the eighteen months.

- At the end of the third six month period a third deposit of $€100$ is made. This sum does not earn any interest.

The asmount in the account at the end of the 18 months is

$$S = 100 + 100(1.03)^1 + 100(1.03)^2 = €309.09$$

We would like to derive a formula to compute this amount.

$$
\begin{aligned}
S &= 100 + 100(1.03)^1 + 100(1.03)^2 \\
1.03S &= 100(1.03)^1 + 100(1.03)^2 + 100(1.03)^3
\end{aligned}
$$

Now we subtract equation (1) from equation (2)

$$
\begin{aligned}
-S &= -100 - 100(1.03)^1 - 100(1.03)^2 && (1)\\
1.03S &= 100(1.03)^1 + 100(1.03)^2 + 100(1.03)^3 && (2)
\end{aligned}
$$

Now we subtract equation (1) from equation (2) to obtain

$$
\begin{aligned}
1.03S - S &= 100(1.03)^3 - 100\\
0.03S &= 100((1.03)^3 - 1)\\
S &= \frac{100((1+0.03)^3 - 1)}{0.03} = €309.09
\end{aligned}
$$

This gives us the formula we are looking for.

What is going to happen if we went a year further?

$$S = 100 + 100(1.03)^1 + 100(1.03)^2 + 100(1.03)^3$$

Hence

$$1.03S = 100(1.03)^1 + 100(1.03)^2 + 100(1.03)^3 + 100(1.03)^4$$

and their difference is again

$$\begin{aligned} 1.03S &= 100(1.03)^1 + 100(1.03)^2 + 100(1.03)^3 + 100(1.03)^4 \\ -S &= -100 - 100(1.03)^1 - 100(1.03)^2 - +100(1.03)^3 \end{aligned}$$

Future Value of an Ordinary Annuity

$$\mathrm{FV} = \mathrm{PMT}\frac{(1+i)^n - 1}{i}$$

where

- $\mathrm{FV} =$ future value (amount),
- $\mathrm{PMT} =$ periodic payment,
- $i =$ rate per period,
- $n =$ number of payments,

with the understanding that payments are made at the end of each period.

Example

What is the value of an annuity at the end of 20 years if $€2,000$ is deposited each year into an account earning $8.5\%\,\mathrm{p/a}$ compounded annually?

Solution

$$\mathrm{FV} = \mathrm{PMT}\frac{(1+\mathrm{i})^{\mathrm{n}}-1}{\mathrm{i}}$$

Here, $\mathrm{FV} = ?, \quad \mathrm{PMT} = 200, \quad \mathrm{i} = 0.085, \quad \mathrm{n} = 20.$

$$\mathrm{FV} = 2000\frac{(1+0.0865)^{20}-1}{0.085} = €96754.03$$

Note, of these

$$96754.03 - 20 \times 2000 = €56754.03$$

are interest earnings.

Sinking Funds

Any account that is established to accumulate funds to meet future obligations or debts is called a sinking fund. If the payments are to be made in the form of an ordinary annuity we only need to solve the annuity formula for the payment PMT.

Example

The parents of a newborn child wish to deposit $€x$ every year into an account that pays 6% compounded annually. Compute x, if they want $€80,000$ for when the child is 17.

Solution

$$\text{FV} = 80000, \quad \text{PMT} = x, i = 0.06, n = 17 \quad \text{FV} = \text{PMT}\frac{(1+i)^n - 1}{i}.$$

$$80000 = x \times \frac{(1+0.06)^{17} - 1}{0.06}$$

implying

$$x = \frac{80000 \cdot 0.06}{1.06^{17} - 1} = 3835.58$$

Example

A company estimates that it will need to replace a piece of equipment at a cost of €$50,000$ in 5 years. To have this money available in 5 years it establishes a sinking fund by making equal monthly payments into an account paying $6.6\%\,\mathrm{p/a}$ compounded monthly.

(i) How much should each payment be?

(ii) How much interest is earned during the last year?

Solution

(i) $\mathrm{FV} = 50000, \quad \mathrm{i} = \frac{0.06}{12} = 0.0055, \quad , \mathrm{n} = 60, \quad \mathrm{PMT} = ?$

$$50000 = \mathrm{PMT}\frac{(1+0.055)^{60}-1}{0.0055}$$

which implies that

$$\mathrm{PMT} = €705.65 \text{ per month.}$$

Interest during last years

Amount after 4 years:

$$\text{PMT} = 705.65, \quad \mathrm{i} = 0.0055, \quad \mathrm{n} = 4 \cdot 12 = 48$$

$$\text{FV} = 705.65 \frac{1.0055^{48} - 1}{0.0055} = €38642.25.$$

Growth in the last years $=$ Total - first 4 years:

$$50000 - 38642.25 = 11357.75$$

Payments during 5th year:

$$12 \times 705.65 = 8467.80.$$

Interest earned during 5th year:

$$11357.75 - 8467.80 = €2889.95.$$