Argument of a Complex Number

Let

$$z = x + iy \in \mathbb{C}, \quad z \neq 0.$$

1. Polar form and definitions

The argument $\theta = \arg(z)$ is defined by writing z in polar form:

$$z = r(\cos\theta + i\sin\theta), \quad r = |z| = \sqrt{x^2 + y^2}.$$

Equating real and imaginary parts gives:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

2. Using tangent

By definition,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Dividing the real and imaginary parts:

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta.$$

Thus,

$$\tan\theta = \frac{y}{x}.$$

3. Using the inverse tangent

The inverse tangent gives

$$\theta = \arctan\left(\frac{y}{x}\right), \quad x > 0.$$

4. Quadrant adjustment

To cover all quadrants:

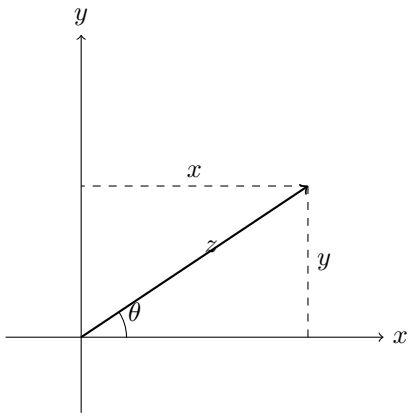
$$\arg(z) = \begin{cases} \arctan\left(\frac{y}{x}\right), & x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi, & x < 0,\ y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi, & x < 0,\ y < 0, \\ \frac{\pi}{2}, & x = 0,\ y > 0, \\ -\frac{\pi}{2}, & x = 0,\ y < 0. \end{cases}$$

This is equivalent to using the two-argument arctangent:

$$\arg(z) = \operatorname{atan2}(y, x).$$

5. Geometric interpretation

The complex number $z = x + iy$ corresponds to the point (x, y) in the plane. The argument θ is the angle between the positive x-axis and the vector (x, y).



From the right triangle formed:

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x}.$$

Hence, the argument is determined by the slope of the vector:

$$\boxed{\arg(x+iy) = \arctan\left(\frac{y}{x}\right) \text{ (with quadrant adjustment)}}.$$