Semester II 2026

MA135/MA160 - Algebra II



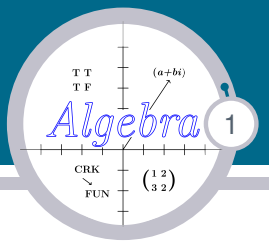
OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Ilia Pirashvili, PhD

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

Lecture 8: Complex numbers

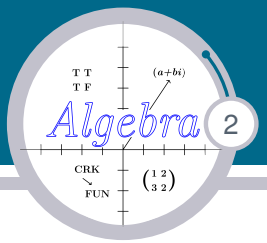
Recap



Number sets:

- $\mathbb{N} := \{1, 2, 3, ...\}$ (positive integers),
- $\mathbb{Z} := \{0, 1, -1, 2, -2, 3, -3, ...\}$ (integers),
- $\mathbb{Q} := \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ (rational numbers),
- irrational numbers i.e. those real numbers that cannot be expressed as fractions (e.g., $\sqrt{2}, \pi$),
- $\mathbb{R}$, the set of all real numbers (all rational and irrational numbers), and
- the set of complex numbers $\mathbb{C} := \{x + yi : x, y \in \mathbb{R}\}$, where $i^2 = -1$.

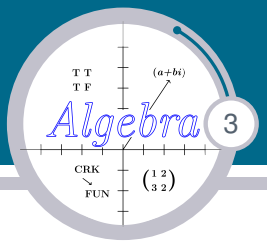
Recap (arithmetic)



Let $z = x + yi$ and $w = a + bi$, then

$$z + w = (x + yi) + (a + bi) = (x + a) + (y + b)i$$
$$z - w = (x + yi) - (a + bi) = (x - a) + (y - b)i$$
$$zw = (x + yi)(a + bi) = (xa - yb) + (xb + ya)i$$

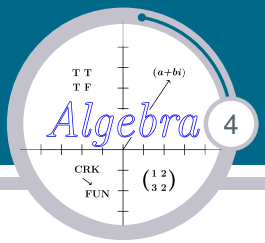
Exercises



Let's keep up our practice:

$$\text{(i)}(3+4i)+(7+2i),$$
$$\text{(ii)}(4-3i)-(5+2i),$$
$$\text{(iii)}(5-4i)(-3+2i).$$

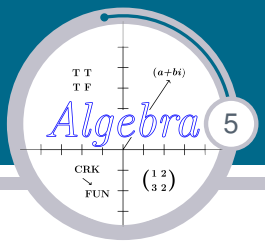
Exercises - solution (i)



(i) $(3 + 4i) + (7 + 2i)$

$$\begin{aligned}(3 + 4i) + (7 + 2i) &= (3 + 7) + (4 + 2)i \\ &= 10 + 6i.\end{aligned}$$

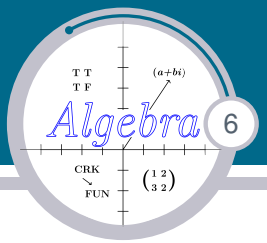
Exercises - solution (ii)



(ii) $(4 - 3i) - (5 + 2i)$

$$\begin{aligned}(4 - 3i) - (5 + 2i) &= (4 - 5) + (-3 - 2)i \\ &= -1 - 5i.\end{aligned}$$

Exercises - solution (iii)



(ii) $(5-4i)(-3+2i)$

$$\begin{aligned}(5-4i)(-3+2i) &= 5(-3+2i)-4i(-3+2i)\\ &= -15+10i+12i-8i^2\\ &= -15+22i+8\\ &= -7+22i.\end{aligned}$$

Division (page 1)

How do we divide complex numbers z/w?

Let's look at some examples

(i)

$$\frac{8+12i}{4} = \frac{8}{4} + \frac{12i}{4} = 2+3i.$$

Note: $4 = 4 + 0i$.

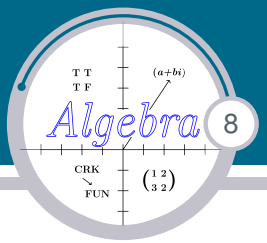
(ii)

$$\frac{8+12i}{3.7} = \frac{8}{3.7} + \frac{12}{3.7}i.$$

Note: we want the solution in the form $x + yi$.

Dividing by a real number is not a problem.

Division (page 2)



How do we make sense of

$$\frac{2+3i}{3-5i},$$

and how can we express it in the form of $x + yi$?
We can make use of the fact that

$$\begin{aligned}(a+bi)(a-bi) &= a(a-bi) + bi(a-bi)\\ &= a^2 - abi + abi - b^2 i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2,\end{aligned}$$

where $(a^2 + b^2)$ is a real number. Here $(a - bi)$ is called the conjugate of $a + bi$ (only the sign on the imaginary term is changed).

Division (page 3)

Therefore we can rewrite

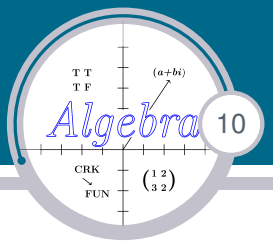
$$\frac{(2+3i)}{(3-5i)} \text{ as } \frac{(2+3i)}{(3-5i)}\frac{(3+5i)}{(3+5i)},$$

since we know $\frac{(3+5i)}{(3+5i)} = 1$.
We also know that when we multiply out the denominator we will get a real number and we can divide a complex number by a real number without difficulty.
So

$$\frac{(2+3i)}{(3-5i)}\frac{(3+5i)}{(3+5i)} = \frac{(2+3i)(3+5i)}{(3^2+5^2)} = \frac{(2+3i)(3+5i)}{34},$$

which we can solve.

Division (page 4)



$$\begin{aligned}
\frac{(2+3i)(3+5i)}{(34)} &= \frac{2(3+5i)+3i(3+5i)}{34} \\
&= \frac{6+10i+9i+15i^2}{34} \\
&= \frac{6+10i+9i-15}{34} \\
&= \frac{-9+19i}{34} \\
&= -\frac{9}{34}+\frac{19}{34}i.
\end{aligned}$$

Examples of division

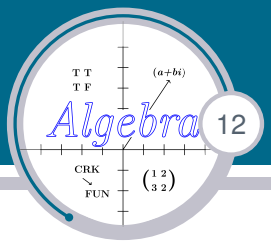
Solve:

$$\frac{1+i}{1-i}.$$

Solution:

$$\begin{aligned}
\frac{1+i}{1-i} &= \frac{(1+i)}{(1-i)}\frac{(1+i)}{(1+i)} \\
&= \frac{(1+i)(1+i)}{(1-i)(1+i)} \\
&= \frac{(1(1+i)+i(1+i)}{(1+1)} \\
&= \frac{1+i+i+i^2}{2} \\
&= \frac{1+2i-1}{2} = \frac{2i}{2} = i.
\end{aligned}$$

Examples of division



Solve:

$$\frac{-3-3i}{2+2i}.$$

Solution:

$$\begin{aligned}
\frac{-3-3i}{2+2i} &= \frac{-3-3i}{2+2i}\frac{2-2i}{2-2i} \\
&= \frac{(-3-3i)(2-2i)}{(2+2i)(2-2i)} \\
&= \frac{(-3(2-2i)-3i(2-2i)}{(4+4)} \\
&= \frac{-6+6i-6i+6i^2}{8} \\
&= \frac{-6-6}{8} = -\frac{12}{8} = -\frac{3}{2}.
\end{aligned}$$

Examples of division

Solve:

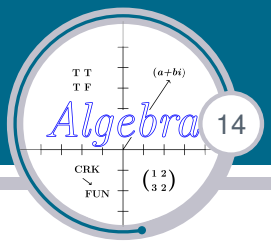
$$\frac{-3-3i}{2+2i}.$$

Alternative method

$$\frac{-3(1+i)}{2(1+i)} = -\frac{3}{2}.$$

It is a good idea to check if the equation can be easily simplified.

Exercise



Solve

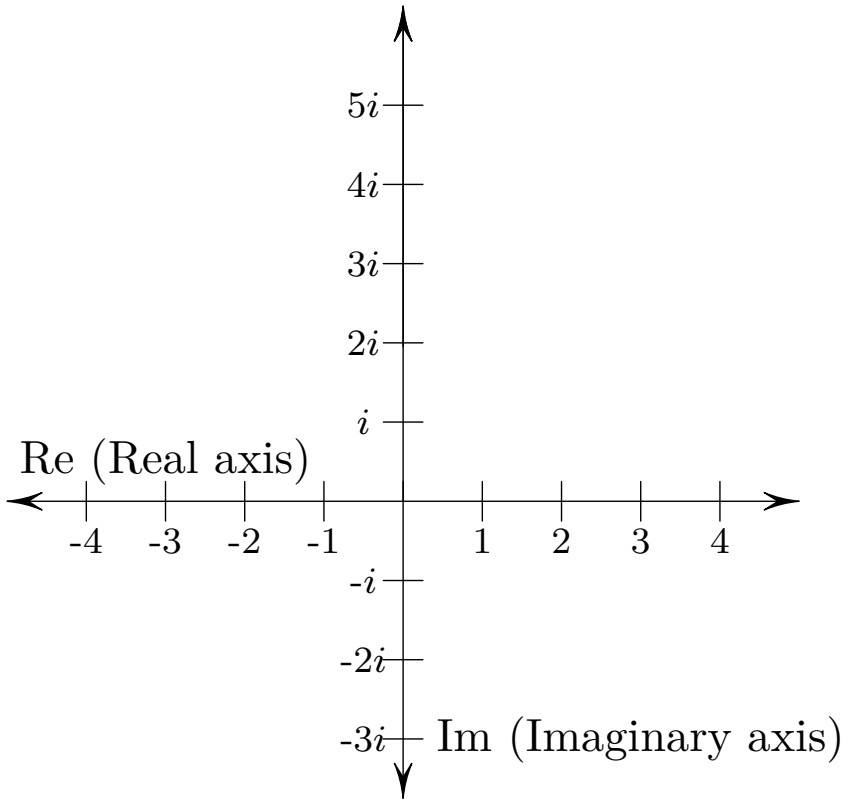
$$z = \frac{4+3i}{5-2i}.$$

Solution:

$$\begin{aligned}
\frac{4+3i}{5-2i} &= \frac{(4+3i)}{(5-2i)}\frac{(5+2i)}{(5+2i)} \\
&= \frac{(4+3i)(5+2i)}{5^2+2^2} = \frac{4(5+2i)+3i(5+2i)}{25+4} \\
&= \frac{20+8i+15i+6i^2}{29} = \frac{20+23i-6}{29} \\
&= \frac{14+23i}{29} = \frac{14}{29}+\frac{23}{29}i.
\end{aligned}$$

Plotting complex numbers

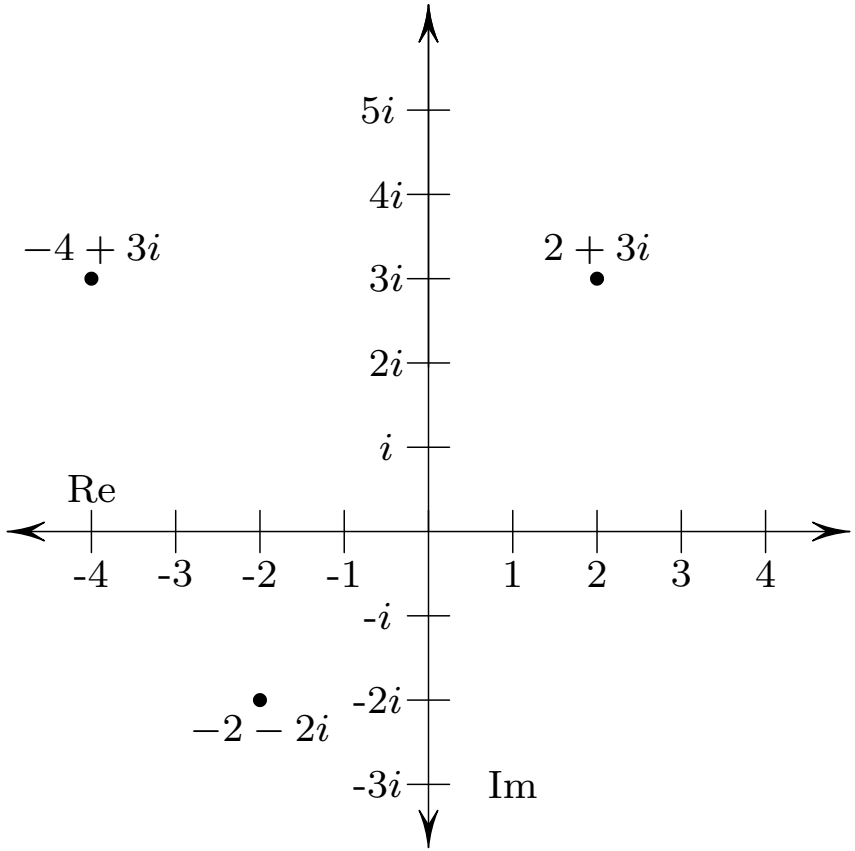
We plot complex numbers on an Argand diagram (or Gaussian plane).



Lets plot $2 + 3i$, $-4 + 3i$ and $-2 - 2i$.

Plotting complex numbers

Lets plot $2 + 3i$, $-4 + 3i$ and $-2 - 2i$ on an Argand diagram.



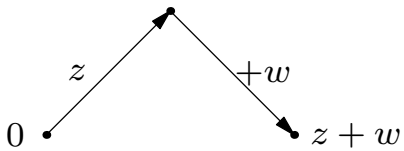
Example

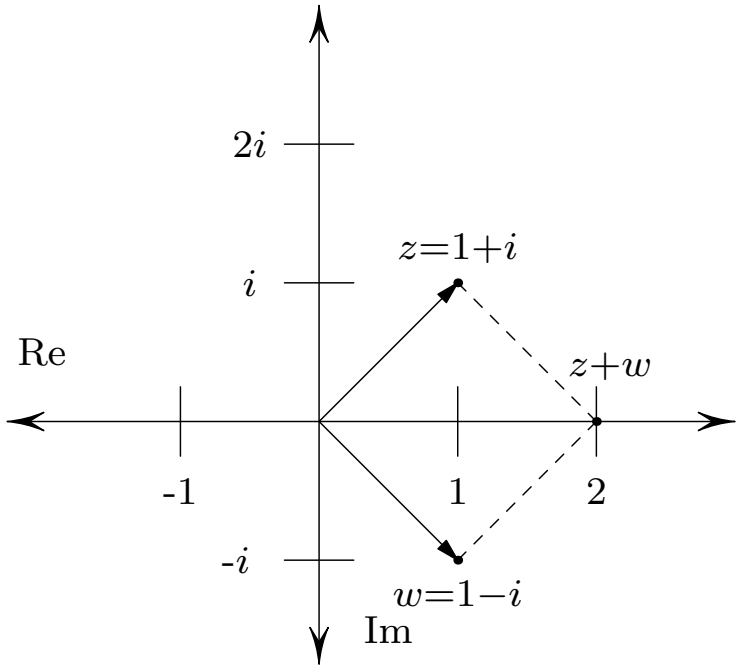
Let $z = 1 + i$ and $w = 1 - i$. Find $z + w$.

$$\begin{aligned} z + w &= (1 + i) + (1 - i) \\ &= (1 + 1) + (i - i) = 2. \end{aligned}$$

Note that $o := (0, 0)$, $w, z, z + w$ form the vertices of a parallelogram.

For those of you that know about vectors. Addition and subtraction of complex numbers *is* vector addition and subtraction. ($\mathbb{C}$ is $\mathbb{R}^2$ as a vector space, but we will not get into that.)





Example

Let $u = 3 - 4i$ and $w = 1 + i$. Find $u + w$.

$$\begin{aligned} u + w &= (3 - 4i) + (1 + i) \\ &= (3 + 1) + (-4 + 1)i \\ &= 4 - 3i. \end{aligned}$$

