# Semester II 2026

# MA135/MA160 - Algebra II




OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

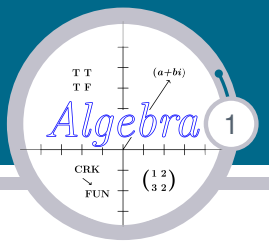
**Ilia Pirashvili, PhD**

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

# Lecture 7: Complex numbers

# Why do we need complex numbers?

## Numbers through completion.




Suppose the only numbers we had were

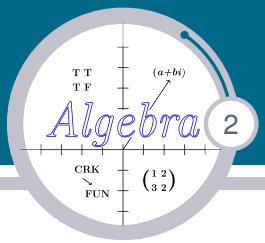
$$\mathbb{N}_+ = \{1, 2, 3, \ldots\}, \text{ the positive integers.}$$

We can easily solve $x - 5 = 0$, $x - 17 = 0$ or $x - n = 0$ for any $n \in \mathbb{N}$. The solutions all exist in $\mathbb{N}$, i.e., 5, 17 and $n$.

But if we wanted to solve $x + 5 = 0$, for example, then we don't have a solution in $\mathbb{N}$. But we can solve the problem by widening the set of available numbers to $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, ...\}$ (all integers). We can now solve both $x + 5 = 0$ and $x - 5 = 0$, etc.

# Why do we need complex numbers?

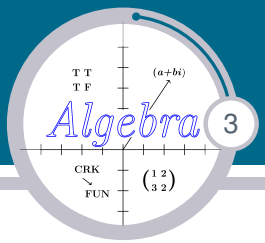
## Numbers through completion.




But if we need to solve an equation like $2x - 5 = 0$, we cannot find a solution in $\mathbb{Z}$, unless we widen $\mathbb{Z}$ to include numbers like $5/2$, i.e., fractions. So we need $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ (the rational numbers). So the number systems evolve from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$ as the equations we want to solve demand.

# Why do we need complex numbers?

## Numbers through completion.




However, the equation $x^2 = 2$ has no solution in $\mathbb{Q}$ as its solutions $\pm\sqrt{2}$ is not in $\mathbb{Q}$. They are irrational, so we next have to add all the irrational numbers into the mix. (Irrational numbers cannot be written as a fraction.) This gives us $\mathbb{R}$, the set of all real numbers..

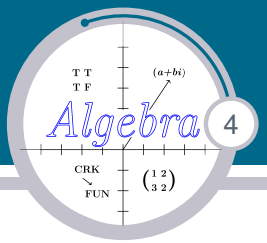
**Note:**

Technically, these kinds of completions only give us *algebraic numbers*. We need an analytic type of completion (e.g., *Cauchy sequences* or *Dedekind cuts*) to get to real numbers.

Yet problems still persist in solving simple equations. If $x \in \mathbb{R}$, then $x^2 \geq 0$ (it is never negative). So once again our number system $\mathbb{R}$ won't let us solve $x^2 + 1 = 0$.

So how do we get out of this one?.....we introduce $i$ where $i$ satisfies $i^2 + 1 = 0$, i.e., $i^2 = -1$, so clearly $i \notin \mathbb{R}$.

# Complex numbers



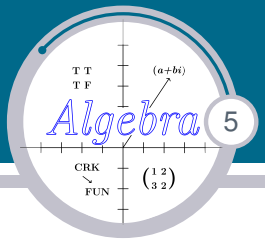

We define a complex number, $z$, to be an expression of the form $z = x + yi$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

We denote the set of all complex numbers as $\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$. For example,

$$z = 2 + 3i,$$
$$z = \frac{2}{3} + \frac{4}{7}i,$$
$$z = \sqrt{2} + 3i$$

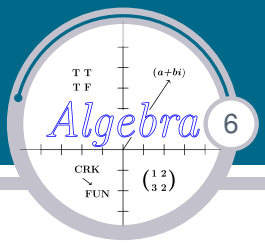
are all in $\mathbb{C}$ i.e. they are all examples of complex numbers.

# Practical applications of complex numbers?




- Complex numbers are used in electronics and electromagnetism.
- The laws of electricity can be expressed using the addition and multiplication of complex numbers.
- They are used in computer science, for example, for rotation in graphics applications.

# Arithmetic of complex numbers




We also define arithmetic for complex numbers: Addition:

$$(a+bi)+(c+di)=(a+b)+(c+d)i.$$

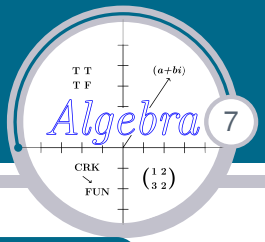
Subtraction:

$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

Multiplication:

$$\begin{aligned}(a+bi)(c+di) &= a(c+di)+bi(c+di)\\ &= ac+adi+bci+bd\underbrace{i^2}_{=-1}\\ &= ac+adi+bci-yd\\ &= (ac-bd)+(ad+bc)i.\end{aligned}$$

# Arithmetic (cont)




## Suggestion:

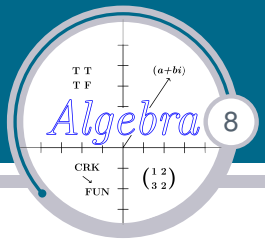
If the multiplication is not immediately clear, try using the more familiar symbol $x$ instead of $i$ and multiply as you would polynomials:

$$\begin{aligned}(a+bx)(c+dx) &= a(c+dx)+bx(c+dx)\\ &= ac+adx+bcx+bdx^2\\ &= ac+(ad+bc)x+bdx^2\end{aligned}$$

and recall that $x^2 = i^2$, by definition, satisfied $x^2+1=0$. That is to say, $x^2 = -$. Substituting it in the equation now gives exactly our desired result:

$$\begin{aligned}(a+bx)(c+dx) &= ac+(ad+bc)x+bdx^2\\ &= ac+(ad+bc)x+bd(-1)\\ &= (ac-bd)+(ad+bc)x.\end{aligned}$$

# Examples




Let $z = 2 - 5i$ and $w = 4 + 3i$. Calculate $z + w$.

$$\begin{aligned} z + w &= (2 - 5i) + (4 + 3i) \\ &= (2 + 4) + (-5 + 3)i \\ &= 6 - 2i. \end{aligned}$$

We call 6 the real part of $6 - 2i$;
we call -2 the imaginary part of $6 - 2i$.
More generally, if $z = x + yi$, then the real part of $z$ is $x$, and the imaginary part of $z$ is $y$.

# Examples



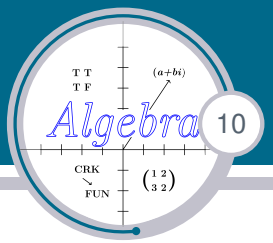
Let $z = 2 - 5i$ and $w = 4 + 3i$. Calculate $z - w$ and $zw$.

$$\begin{aligned} z - w &= (2 - 5i) - (4 + 3i) \\ &= (2 - 4) + (-5 - 3)i \\ &= -2 - 8i \end{aligned}$$

and

$$\begin{aligned} zw &= (2 - 5i)(4 + 3i) \\ &= 2(4 + 3i) - 5i(4 + 3i) \\ &= 8 + 6i - 20i - 15i^2 \\ &= 8 + (6 - 20)i - 15(-1) \\ &= 8 - 14i + 15 \\ &= 23 - 14i \end{aligned}$$

# Exercises




Let's try some exercises

$$\text{(i)}(4+2i)+(7+3i)$$
$$\text{(ii)}(6-3i)-(4-2i)$$
$$\text{(iii)}(2-4i)(-3+2i)$$

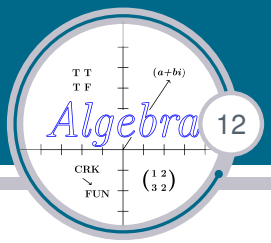
# Exercises - solution (i)



(i) $(4+2i)+(7+3i)$

$$\begin{aligned}(4+2i)+(7+3i) &= (4+7)+(2+3)i \\ &= 11+5i.\end{aligned}$$

# Exercises - solution (ii)




(ii) $(6 - 3i) - (4 - 2i)$

$$\begin{aligned}(6 - 3i) - (4 - 2i) &= (6 - 4) + (-3 - (-2))i \\ &= 2 - i.\end{aligned}$$

# Exercises - solution (iii)



(ii) $(2-4i)(-3+2i)$

$$\begin{aligned}(2-4i)(-3+2i) &= 2(-3+2i)-4i(-3+2i)\\ &= -6+4i+12i-8i^2\\ &= -6+16i+8\\ &= 2+16i.\end{aligned}$$