Semester II 2026

MA135/MA160 - Algebra II



OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Ilia Pirashvili, PhD

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

Lecture 6: Propositional Logic

Reminder of some recent definitions:

A logical expression (or truth function) is called a tautology if it is always true.

An argument is logically valid if the corresponding truth function is a tautology, e.g., $P \vee \neg P$.

Reminder of some recent definitions:

A logical expression (or truth function) is called a tautology if it is always true.

An argument is logically valid if the corresponding truth function is a tautology, e.g., $P \vee \neg P$.

Exercise: Is $(P \wedge \neg P) \Rightarrow Q$ a tautology?

P	Q	$\neg P$	$P \wedge \neg P$	$(P \wedge \neg P) \Rightarrow Q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Reminder of some recent definitions:

A logical expression (or truth function) is called a tautology if it is always true.

An argument is logically valid if the corresponding truth function is a tautology, e.g., $P \vee \neg P$.

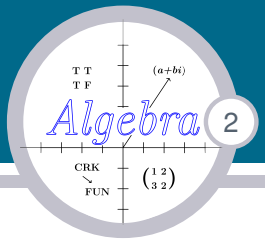
Exercise: Is $(P \wedge \neg P) \Rightarrow Q$ a tautology?

P	Q	$\neg P$	$P \wedge \neg P$	$(P \wedge \neg P) \Rightarrow Q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Two things to think about here,

- $A \Rightarrow B$ is only false if ***A* is true** and ***B* is false**, and
- $P \wedge \neg P$ is always **false**, (a contradiction).

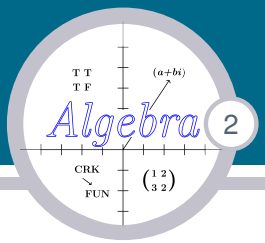
Contradiction



Definition

A logical formula is called a contradicton if it is always false.

Contradiction



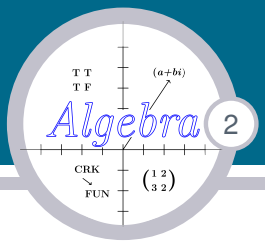
Definition

A logical formula is called a contradicton if it is always false.

An other way of saying that its negation is a tautology.

For example, $P \wedge \neg P$ is a contradiction - we verify by checking its truth table:

Contradiction



Definition

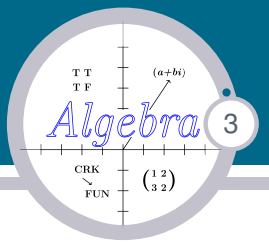
A logical formula is called a contradicton if it is always false.

An other way of saying that its negation is a tautology.

For example, $P \wedge \neg P$ is a contradiction - we verify by checking its truth table:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Examples



Write out the truth tables for the following boolean functions. Decide whether each is a tautology, a contradiction or neither.

(1) $(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$,

(2) $P \wedge Q \Rightarrow P \vee R$ (done yesterday),

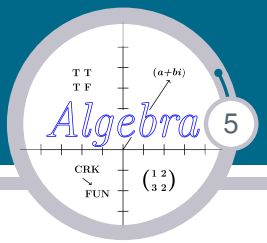
(3) $(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$.

Example 1

$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R				
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

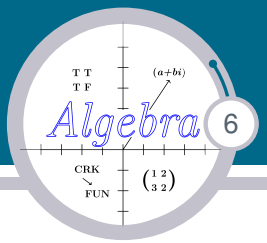
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$			
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

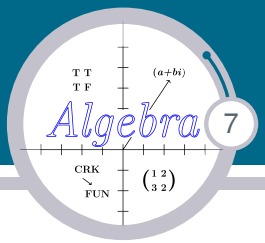
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$		
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

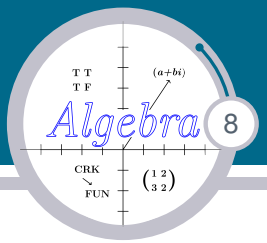
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

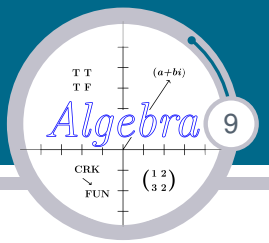
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

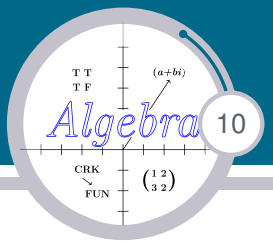
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T			
T	T	F	T			
T	F	T				
T	F	F				
F	T	T	T			
F	T	F	T			
F	F	T				
F	F	F				

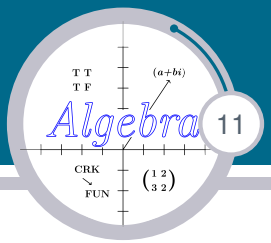
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T			
T	T	F	T			
T	F	T	T			
T	F	F				
F	T	T	T			
F	T	F	T			
F	F	T	T			
F	F	F				

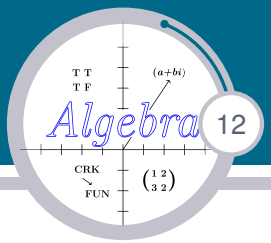
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T			
T	T	F	T			
T	F	T	T			
T	F	F	F			
F	T	T	T			
F	T	F	T			
F	F	T	T			
F	F	F	F			

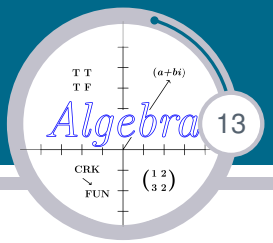
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T			
T	T	F	T			
T	F	T	T			
T	F	F	F	F		
F	T	T	T			
F	T	F	T			
F	F	T	T			
F	F	F	F			

Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T	T		
T	T	F	T	T		
T	F	T	T	T		
T	F	F	F	F		
F	T	T	T	T		
F	T	F	T	T		
F	F	T	T	T		
F	F	F	F	T		

Example 1

$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T	T		
T	T	F	T	T		
T	F	T	T	T	F	
T	F	F	F	F	F	
F	T	T	T	T		
F	T	F	T	T		
F	F	T	T	T		
F	F	F	F	T		

Example 1

$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

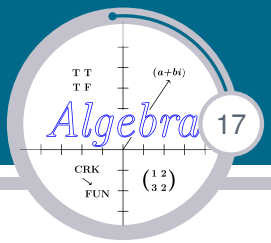
P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T	T	T	
T	T	F	T	T	T	
T	F	T	T	T	F	
T	F	F	F	F	F	
F	T	T	T	T	T	
F	T	F	T	T	T	
F	F	T	T	T	T	
F	F	F	F	T	T	

Example 1

$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	F	
T	F	F	F	F	F	
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	F	T	T	T

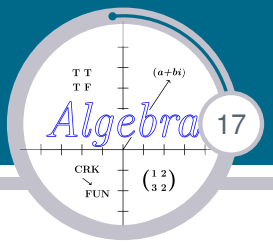
Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	F	T	T	T

Example 1



$(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$

P	Q	R	$\underbrace{Q \vee R}_{A}$	$\underbrace{P \Rightarrow A}_{B}$	$\underbrace{P \Rightarrow Q}_{C}$	$B \wedge C$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	F	T	T	T

So $(P \Rightarrow (Q \vee R)) \wedge (P \Rightarrow Q)$ is neither a tautology or a contradiction.

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R						
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$					
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$				
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

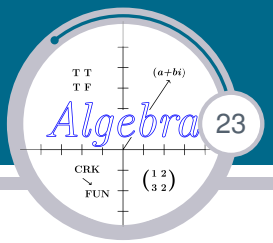
P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$			
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$		
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Example 3



$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

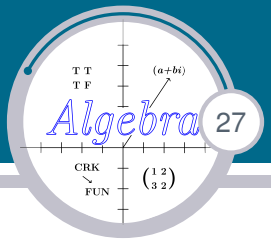
P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F					
T	T	F	F					
T	F	T	T					
T	F	F	T					
F	T	T	F					
F	T	F	F					
F	F	T	T					
F	F	F	T					

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F					
T	T	F	F					
T	F	T	T	T				
T	F	F	T	T				
F	T	T	F					
F	T	F	F					
F	F	T	T	T				
F	F	F	T	T				

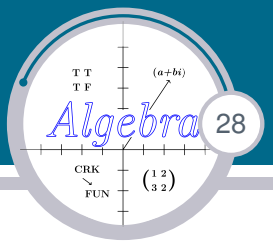
Example 3



$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T				
T	T	F	F					
T	F	T	T	T				
T	F	F	T	T				
F	T	T	F	T				
F	T	F	F					
F	F	T	T	T				
F	F	F	T	T				

Example 3



$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

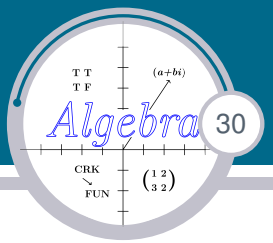
P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T				
T	T	F	F	F				
T	F	T	T	T				
T	F	F	T	T				
F	T	T	F	T				
F	T	F	F	F				
F	F	T	T	T				
F	F	F	T	T				

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T	T			
T	T	F	F	F				
T	F	T	T	T	T			
T	F	F	T	T	T			
F	T	T	F	T				
F	T	F	F	F	T			
F	F	T	T	T				
F	F	F	T	T				

Example 3



$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T	T			
T	T	F	F	F	F			
T	F	T	T	T	T			
T	F	F	T	T	T			
F	T	T	F	T	F			
F	T	F	F	F	T			
F	F	T	T	T	F			
F	F	F	T	T	F			

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T	T	F		
T	T	F	F	F	F	F		
T	F	T	T	T	T	F		
T	F	F	T	T	T	F		
F	T	T	F	T	F	T		
F	T	F	F	F	T	T		
F	F	T	T	T	F	T		
F	F	F	T	T	F	T		

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T	T	F		
T	T	F	F	F	F	F		
T	F	T	T	T	T	F		
T	F	F	T	T	T	F		
F	T	T	F	T	F	T		
F	T	F	F	F	T	T		
F	F	T	T	T	F	T	F	
F	F	F	T	T	F	T	F	

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

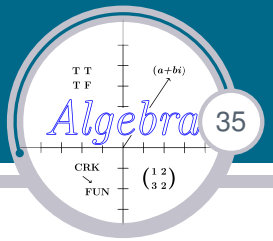
P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T	T	F	T	
T	T	F	F	F	F	F	T	
T	F	T	T	T	T	F	T	
T	F	F	T	T	T	F	T	
F	T	T	F	T	F	T	T	
F	T	F	F	F	T	T	T	
F	F	T	T	T	F	T	F	
F	F	F	T	T	F	T	F	

Example 3

$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T	T	F	T	T
T	T	F	F	F	F	F	T	T
T	F	T	T	T	T	F	T	T
T	F	F	T	T	T	F	T	T
F	T	T	F	T	F	T	T	T
F	T	F	F	F	T	T	T	T
F	F	T	T	T	F	T	F	T
F	F	F	T	T	F	T	F	T

Example 3



$(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$

P	Q	R	$\neg Q$	$\underbrace{(\neg Q) \vee R}_{A}$	$\underbrace{P \Leftrightarrow A}_{B}$	$\neg P$	$\underbrace{(\neg P) \Rightarrow Q}_{C}$	$B \Rightarrow C$
T	T	T	F	T	T	F	T	T
T	T	F	F	F	F	F	T	T
T	F	T	T	T	T	F	T	T
T	F	F	T	T	T	F	T	T
F	T	T	F	T	F	T	T	T
F	T	F	F	F	T	T	T	T
F	F	T	T	T	F	T	F	T
F	F	F	T	T	F	T	F	T

So $(P \Leftrightarrow ((\neg Q) \vee R)) \Rightarrow ((\neg P) \Rightarrow Q)$ is a tautology.

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

P .

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

Q .

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$P \wedge Q$.

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$$R \qquad .$$

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$$\neg R \qquad .$$

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$(P \wedge Q) \Rightarrow \neg R$.

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$$P \qquad .$$

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$((P \wedge Q) \Rightarrow \neg R) \wedge P$.

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$$R \qquad .$$

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$$Q\ .$$

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$$\neg Q \ .$$

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$$R \Rightarrow \neg Q \ .$$

Determining whether an argument is logically valid.

Represent the following argument by a truth function. Then decide if the argument is logically valid.

If the temperature and air pressure remain constant, there is no rain. The temperature remains constant. Therefore, if there is rain, then the air pressure does not remain constant.

Solution

Let

- P:= The temperature remains constant
- Q:= The air pressure remains constant
- R:= There is rain.

$(((P \wedge Q) \Rightarrow \neg R) \wedge P) \Rightarrow (R \Rightarrow \neg Q)$.

Determining whether an argument is logically valid.

$$(((P \wedge Q) \Rightarrow \neg R) \wedge P) \Rightarrow (R \Rightarrow \neg Q)$$

P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\neg R$	$\underbrace{A \Rightarrow \neg R}_{B}$	$\underbrace{B \wedge P}_{C}$	$\neg Q$	$\underbrace{R \Rightarrow \neg Q}_{D}$	$C \Rightarrow D$
T	T	T							
T	T	F							
T	F	T							
T	F	F							
F	T	T							
F	T	F							
F	F	T							
F	F	F							

Determining whether an argument is logically valid.

$$(((P \wedge Q) \Rightarrow \neg R) \wedge P) \Rightarrow (R \Rightarrow \neg Q)$$

P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\neg R$	$\underbrace{A \Rightarrow \neg R}_{B}$	$\underbrace{B \wedge P}_{C}$	$\neg Q$	$\underbrace{R \Rightarrow \neg Q}_{D}$	$C \Rightarrow D$
T	T	T	T	F	F	F	F	F	T
T	T	F	T	T	T	F	F	T	T
T	F	T	F	F	T	T	T	T	T
T	F	F	F	T	T	F	T	T	T
F	T	T	F	F	T	F	F	F	T
F	T	F	F	T	T	F	F	T	T
F	F	T	F	F	T	F	T	T	T
F	F	F	F	T	T	F	T	T	T

So $((((T \wedge P) \Rightarrow \neg R) \wedge T) \Rightarrow (R \Rightarrow \neg P)$ is a tautology, and the argument is logically valid.

Determining whether an argument is logically valid.

Alternative method:

We want to see if there are any circumstances that this statement can be false. $\underbrace{(((P \wedge Q) \Rightarrow \neg R) \wedge P)}_{A} \Rightarrow \underbrace{(R \Rightarrow \neg Q)}_{B}$

Determining whether an argument is logically valid.

Alternative method:

We want to see if there are any circumstances that this statement can be false. $\underbrace{(((P \wedge Q) \Rightarrow \neg R) \wedge P)}_{A} \Rightarrow \underbrace{(R \Rightarrow \neg Q)}_{B}$

1. When is this false? Only if A is true and B is false.

Determining whether an argument is logically valid.

Alternative method:

We want to see if there are any circumstances that this statement can be false. $\underbrace{(((P \wedge Q) \Rightarrow \neg R) \wedge P)}_{A} \Rightarrow \underbrace{(R \Rightarrow \neg Q)}_{B}$

1. When is this false? Only if A is true and B is false.
2. When can B be false? Only if R is true and $\neg Q$ is false (i.e. Q is true.)

Determining whether an argument is logically valid.

Alternative method:

We want to see if there are any circumstances that this statement can be false. $\underbrace{(((P \wedge Q) \Rightarrow \neg R) \wedge P)}_{A} \Rightarrow \underbrace{(R \Rightarrow \neg Q)}_{B}$

1. When is this false? Only if A is true and B is false.
2. When can B be false? Only if R is true and $\neg Q$ is false (i.e. Q is true.)
3. When is A true? Only if both $((P \wedge Q) \Rightarrow \neg R)$ and P are true.

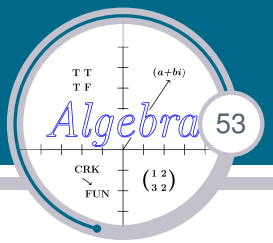
Determining whether an argument is logically valid.

Alternative method:

We want to see if there are any circumstances that this statement can be false. $\underbrace{(((P \wedge Q) \Rightarrow \neg R) \wedge P)}_{A} \Rightarrow \underbrace{(R \Rightarrow \neg Q)}_{B}$

1. When is this false? Only if A is true and B is false.
2. When can B be false? Only if R is true and $\neg Q$ is false (i.e. Q is true.)
3. When is A true? Only if both $((P \wedge Q) \Rightarrow \neg R)$ and P are true.
4. Gathering these facts for the argument not to be logically valid, we need P, Q and R to be true. Yet these values in turn will make A false - which contradicts the fact that A is true! So it is **not** possible for the truth function to be false and we conclude that the argument *is* logically valid.

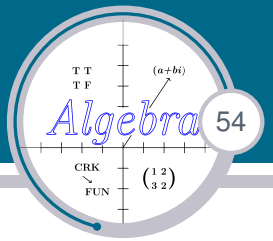
Logical validity example 2 (page 1)



Represent the following argument by a truth function. Then decide if the argument is logically valid.

If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security. If other countries feel endangered, they may start a preventative war. If our people get a false sense of security, they will put less effort into preserving peace. If fallout shelters are not built, we run the risk of tremendous losses in the event of war. Hence, either other countries may start a preventative war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war.

Logical validity example 2 (page 2)



Let's break it into sentences and look at each separately.

If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security.

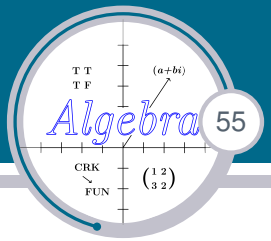
If other countries feel endangered, they may start a preventative war.

If our people get a false sense of security, they will put less effort into preserving peace.

If fallout shelters are not built, we run the risk of tremendous losses in the event of war.

Hence, either other countries may start a preventative war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war.

Logical validity example 2 (page 2)

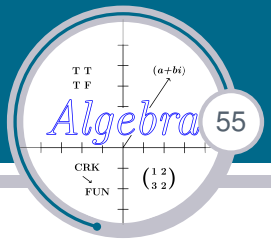


First sentence

If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security.

- B:= Fallout shelters are Built.
- E:= Other countries will feel Endangered.
- S:= Our people will get a false sense of Security.

Logical validity example 2 (page 2)



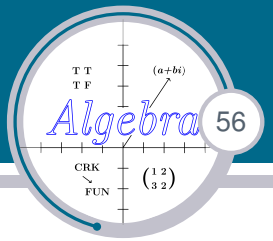
First sentence

If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security.

- B:= Fallout shelters are Built.
- E:= Other countries will feel Endangered.
- S:= Our people will get a false sense of Security.

$(B \Rightarrow (E \wedge S))$

Logical validity example 2 (page 3)

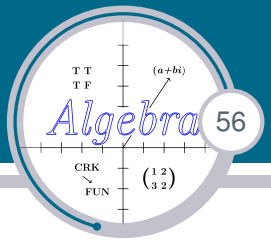


Second sentence

If other countries feel endangered, they may start a preventative war.

- E:= Other countries will feel endangered.
- P:= Other countries (they) may start a Preventative war.

Logical validity example 2 (page 3)



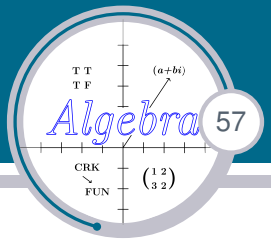
Second sentence

If other countries feel endangered, they may start a preventative war.

- E:= Other countries will feel endangered.
- P:= Other countries (they) may start a Preventative war.

$(E \Rightarrow P)$

Logical validity example 2 (page 4)

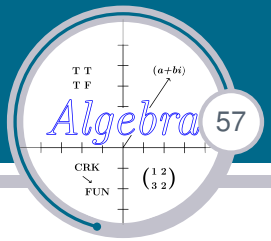


Third sentence

If our people will get a false sense of security, they will put less effort into preserving peace.

- S:= Our people will get a false sense of security.
- L:= Our people (they) will put Less effort into preserving peace.

Logical validity example 2 (page 4)



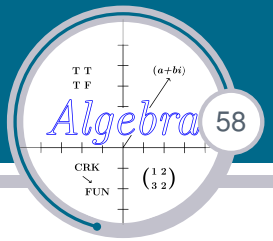
Third sentence

If our people will get a false sense of security, they will put less effort into preserving peace.

- S:= Our people will get a false sense of security.
- L:= Our people (they) will put Less effort into preserving peace.

$(S \Rightarrow L)$

Logical validity example 2 (page 5)

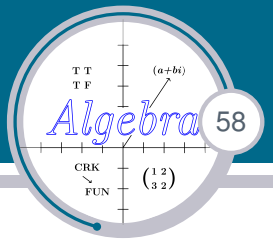


Fourth sentence

If fallout shelters are not built, we run the risk of tremendous losses in the event of war.

- B:= Fallout shelters are built.
- R:= We run the Risk of tremendous losses in the event of war.

Logical validity example 2 (page 5)



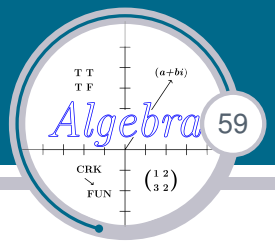
Fourth sentence

If fallout shelters are not built, we run the risk of tremendous losses in the event of war.

- B:= Fallout shelters are built.
- R:= We run the Risk of tremendous losses in the event of war.

$(\neg B \Rightarrow R)$

Logical validity example 2 (page 6)



Fifth and final sentence

Hence, either other countries may start a preventative war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war.

- P:= Other countries (they) may start a preventative war.
- L:= Our people (they) will put less effort into preserving peace.
- R:= We run the risk of tremendous losses in the event of war.

Logical validity example 2 (page 6)

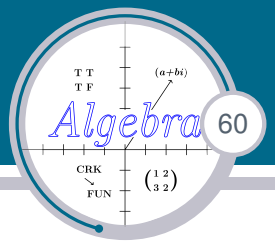
Fifth and final sentence

Hence, either other countries may start a preventative war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war.

- P:= Other countries (they) may start a preventative war.
- L:= Our people (they) will put less effort into preserving peace.
- R:= We run the risk of tremendous losses in the event of war.

$((P \wedge L) \vee R)$

Logical validity example 2 (page 7)



If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security. $(B \Rightarrow (E \wedge S))$

If other countries feel endangered, they may start a preventative war. $(E \Rightarrow P)$

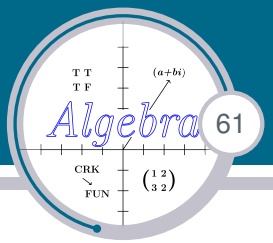
If our people will get a false sense of security, they will put less effort into preserving peace. $(S \Rightarrow L)$

If fallout shelters are not built, we run the risk of tremendous losses in the event of war. $(\neg B \Rightarrow R)$

Hence, either other countries may start a preventative war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war. $((P \wedge L) \vee R)$

$$((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R) \Rightarrow ((P \wedge L) \vee R)$$

Logical validity example 2 (page 8)



$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R)}_{C}$$

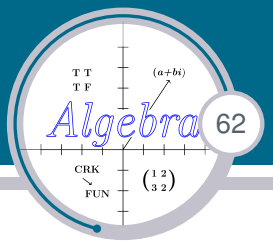
We want to show if this argument is logically valid, i.e., is the truth statement a tautology.

(How many rows would there be in the corresponding truth table?)

When is $A \Rightarrow C$ not true?

Only when A is true and C is false. Can this happen? Remember we only need one instance.

Logical validity example 2 (page 9)



$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R))}_{C}$$

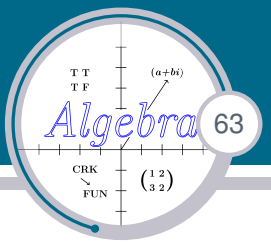
When is C false?

C is false only when both $(P \wedge L)$ and R are are both false.

When is $(P \wedge L)$ false?

When either P or L is false or both of them are false i.e.

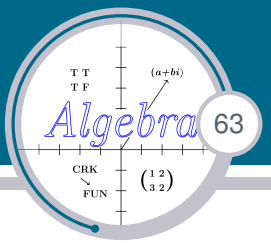
Logical validity example 2 (page 10)



$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R))}_{C}$$

For A to be true then $(B \Rightarrow (E \wedge S))$ and $(E \Rightarrow P)$ and $(S \Rightarrow L)$ and $(\neg B \Rightarrow R)$ all need to be true.

Logical validity example 2 (page 10)

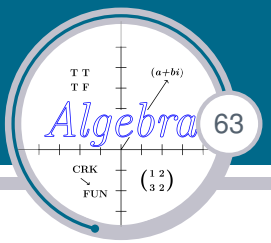


$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R))}_{C}$$

For A to be true then $(B \Rightarrow (E \wedge S))$ and $(E \Rightarrow P)$ and $(S \Rightarrow L)$ and $(\neg B \Rightarrow R)$ all need to be true.

Remember R is false so $(\neg B \Rightarrow R)$ is only true if $\neg B$ is false, i.e., B is true.

Logical validity example 2 (page 10)



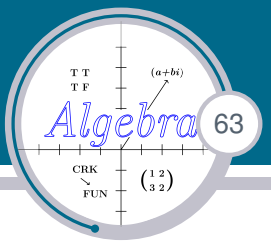
$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R))}_{C}$$

For A to be true then $(B \Rightarrow (E \wedge S))$ and $(E \Rightarrow P)$ and $(S \Rightarrow L)$ and $(\neg B \Rightarrow R)$ all need to be true.

Remember R is false so $(\neg B \Rightarrow R)$ is only true if $\neg B$ is false, i.e., B is true.

If B is true then $(E \wedge S)$ is also true (because $B \Rightarrow (E \wedge S)$). When can this happen?

Logical validity example 2 (page 10)



$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R))}_{C}$$

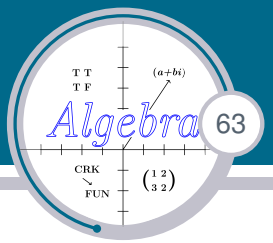
For A to be true then $(B \Rightarrow (E \wedge S))$ and $(E \Rightarrow P)$ and $(S \Rightarrow L)$ and $(\neg B \Rightarrow R)$ all need to be true.

Remember R is false so $(\neg B \Rightarrow R)$ is only true if $\neg B$ is false, i.e., B is true.

If B is true then $(E \wedge S)$ is also true (because $B \Rightarrow (E \wedge S)$). When can this happen?

Only if E and S are true.

Logical validity example 2 (page 10)



$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R))}_{C}$$

For A to be true then $(B \Rightarrow (E \wedge S))$ and $(E \Rightarrow P)$ and $(S \Rightarrow L)$ and $(\neg B \Rightarrow R)$ all need to be true.

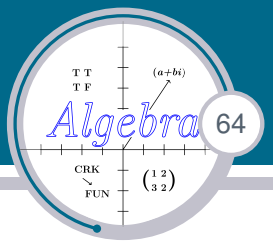
Remember R is false so $(\neg B \Rightarrow R)$ is only true if $\neg B$ is false, i.e., B is true.

If B is true then $(E \wedge S)$ is also true (because $B \Rightarrow (E \wedge S)$). When can this happen?

Only if E and S are true.

If E is true then P must be true (because (E $\Rightarrow$ P)).

Logical validity example 2 (page 11)



$$\underbrace{((B \Rightarrow (E \wedge S)) \wedge (E \Rightarrow P) \wedge (S \Rightarrow L) \wedge (\neg B \Rightarrow R))}_{A} \Rightarrow \underbrace{((P \wedge L) \vee R))}_{C}$$

So far what do we need for the argument not to be logically valid? That

- R is false,
- $P \wedge L$ is false,
- B is true,
- E is true,
- S is true,
- P is true,

If S is true then L is true (because $S \Rightarrow L$ is true). So both P and L are true. But for the argument not to be logically valid we need at least one of P or L to be false. So the argument is logically valid.