Semester II 2026

MA135/MA160 - Algebra II



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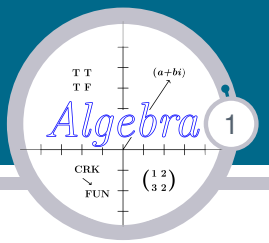
With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

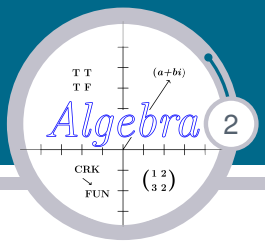
Lecture 5: Propositional Logic

Recall from last lecture



We introduced the connective $\Rightarrow$ called *implication*.
We showed that $P \Rightarrow Q \equiv (\neg P) \vee Q$.
The contrapositive of $P \Rightarrow Q$ is $(\neg Q \Rightarrow \neg P)$.

What is the negation of $S \Rightarrow T$?



Since

$$S \Rightarrow T \equiv \neg S \vee T,$$

it follows that

$$\neg(S \Rightarrow T) \equiv \neg(\neg S \vee T).$$

De Morgan's laws imply

$$\neg(\neg S \vee T) \equiv \neg(\neg S) \wedge \neg T \equiv S \wedge \neg T.$$

So the negation of 'If I am hungry, then I eat an apple' is

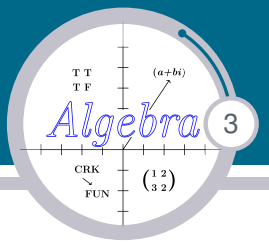
'I am hungry and I do not eat an apple'.

(What is S here and what is T?)

(Note, we do not always have to use P and Q?)

(Can you also convince yourself of this geometrically?)

Another connective $\Leftrightarrow$



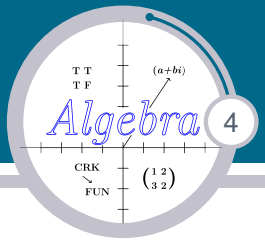
Another useful connective is $\Leftrightarrow$, shorthand for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. We write $P \Leftrightarrow Q$ and say P if and only if Q. The related truth table is

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Exercise

Confirm that $P \Leftrightarrow Q$ is indeed shorthand for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ i.e. that $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

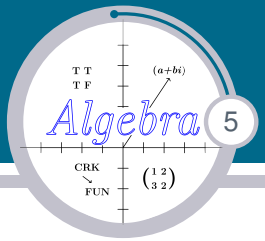
Summary



We have introduced 5 connectives:

- not $\neg$: $\neg P$ is true only if the statement P is false.
- and $\wedge$: $P \wedge Q$ is true only if both P and Q are true.
- or $\vee$: $P \vee Q$ is true if either one or both statements are true.
- implies $\Rightarrow$: $P \Rightarrow Q$ is true unless P is true and Q is false, and
- if and only if (iff) $\Leftrightarrow$: True only if both statements are true or both statements are false.

Practice



Example

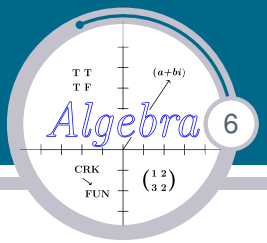
Decide whether or not $(\neg P) \vee Q$ is logically equivalent to $(\neg Q) \vee P$.

Solution

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee Q$	$(\neg Q) \vee P$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

We can see that $(\neg P) \vee Q \not\equiv (\neg Q) \vee P$.

Practice



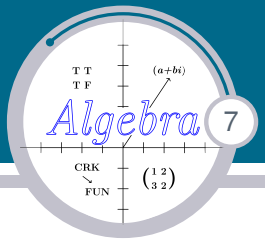
Example

Decide wheather $\neg(P \Leftrightarrow Q)$ is logically equivalent to $P \Leftrightarrow (\neg Q)$.

Solution

P	Q	$\neg Q$	$P \Leftrightarrow Q$	$\neg(P \Leftrightarrow Q)$	$P \Leftrightarrow (\neg Q)$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	F

Tautology



Definition

A logical formula (sometimes called a truth function) is called a tautology if it is always true.

That is, if it is true under all truth values of its propositions.

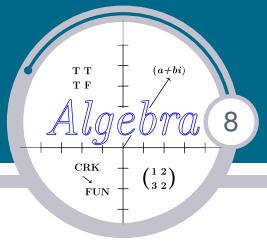
Geometrically it means that the whole Venn-diagram is covered.

Example: $P \vee \neg P$

This is the simplest tautology, as we can see from the table:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Logically valid



Definition

An argument is logically valid if the corresponding truth function is a tautology.

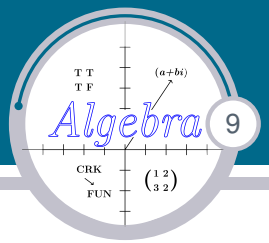
So, we need to 'decompose' a given argument to obtain its corresponding truth function. Let's have a look at an example:

Question

Is the following argument logically valid?

If it is Friday then John is going home. John is going home. Therefore it is Friday.

Logically valid argument (Example)



Question

Is the following argument logically valid?

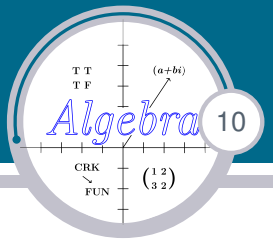
If it is Friday then John is going home. John is going home. Therefore it is Friday.

How do we approach this? First, let's replace its constituent propositions by letters; as many letters as the argument requires:

- $P :=$ It is Friday.
- $Q :=$ John is going home.

Translate the given argument into the corresponding truth function

Logically valid argument (Example)



Question

Is the following argument logically valid?

If it is Friday then John is going home. John is going home. Therefore it is Friday.

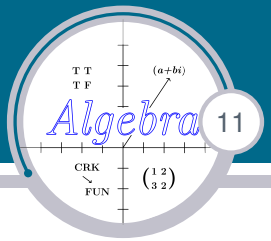
- $P :=$ It is Friday.
- $Q :=$ John is going home.

((If $\underbrace{\text{it is Friday}}_{P}$ then $\underbrace{\text{John is going home}}_{Q}$) and $\underbrace{\text{John is going home}}_{Q}$) implies $\underbrace{\text{it is Friday}}_{P}$.

((It is Friday implies John is going home) and John is going home) implies it is Friday.

$((P \Rightarrow Q) \wedge Q) \Rightarrow P$.

Logically valid argument (Example)



Summary

We have to check that $((P \Rightarrow Q) \wedge Q) \Rightarrow P)$ is a tautology, to see if the original argument is logically valid.

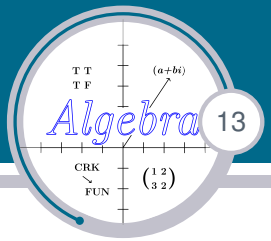
P	Q	$\underbrace{P \Rightarrow Q}_{A}$	$\underbrace{A \wedge Q}_{B}$	$B \Rightarrow P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The truth function is not a tautology, therefore the original argument is not logically valid. Note: that when P is F (It is not Friday) and Q is T (John is going home) the truth function is F.

Is $P \wedge Q \Rightarrow P \vee R$ a tautology?

P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\underbrace{P \vee R}_{B}$	$A \Rightarrow B$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Is $P \wedge Q \Rightarrow P \vee R$ a tautology?

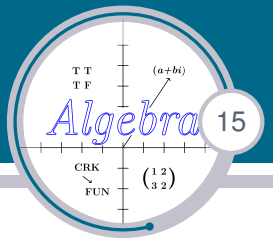


P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\underbrace{P \vee R}_{B}$	$A \Rightarrow B$
T	T	T	T		
T	T	F	T		
T	F	T	F		
T	F	F	F		
F	T	T	F		
F	T	F	F		
F	F	T	F		
F	F	F	F		

Is $P \wedge Q \Rightarrow P \vee R$ a tautology?

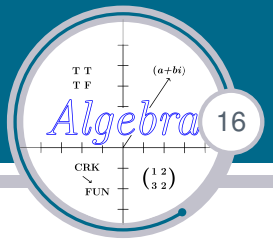
P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\underbrace{P \vee R}_{B}$	$A \Rightarrow B$
T	T	T	T	T	
T	T	F	T	T	
T	F	T	F	T	
T	F	F	F	T	
F	T	T	F		
F	T	F	F		
F	F	T	F		
F	F	F	F		

Is $P \wedge Q \Rightarrow P \vee R$ a tautology?



P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\underbrace{P \vee R}_{B}$	$A \Rightarrow B$
T	T	T	T	T	
T	T	F	T	T	
T	F	T	F	T	
T	F	F	F	T	
F	T	T	F	T	
F	T	F	F		
F	F	T	F	T	
F	F	F	F		

Is $P \wedge Q \Rightarrow P \vee R$ a tautology?

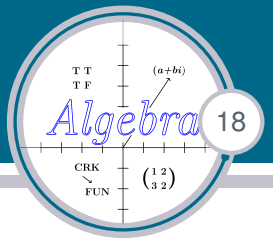


P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\underbrace{P \vee R}_{B}$	$A \Rightarrow B$
T	T	T	T	T	
T	T	F	T	T	
T	F	T	F	T	
T	F	F	F	T	
F	T	T	F	T	
F	T	F	F	F	
F	F	T	F	T	
F	F	F	F	F	

Is $P \wedge Q \Rightarrow P \vee R$ a tautology?

P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\underbrace{P \vee R}_{B}$	$\underbrace{A \Rightarrow B}_{(\neg A) \vee B}$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	
F	F	T	F	T	T
F	F	F	F	F	

Is $P \wedge Q \Rightarrow P \vee R$ a tautology?



P	Q	R	$\underbrace{P \wedge Q}_{A}$	$\underbrace{P \vee R}_{B}$	$\underbrace{A \Rightarrow B}_{(\neg A) \vee B}$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	F	T	T
F	F	F	F	F	T

So $(P \wedge Q) \Rightarrow (P \vee R)$ is a tautology.

Tautology?

Exercise

Is $(P \wedge \neg P) \Rightarrow Q$ a tautology?

P	Q	$\neg P$	$P \wedge \neg P$	$(P \wedge \neg P) \Rightarrow Q$
T	T	F		
T	F	F		
F	T	T		
F	F	T		