Semester II 2026

MA135/MA160 - Algebra II



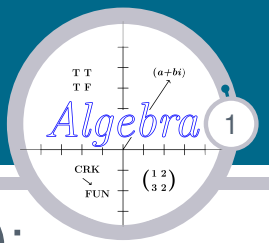
Ollscoil na Gaillimhe
University of Galway

Ilia Pirashvili, PhD

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

Lecture 4: Logically valid arguments

More connectives: implication



The next connective we consider is implies, denoted by $\Rightarrow$ (or $\rightarrow$):

Suppose $P :=$ 'Riley is from Galway', $Q :=$ 'Riley is from Ireland'.

Then $P \Rightarrow Q$ is the Proposition (synonymous with Statement):
'If Riley is from Galway, then Riley is from Ireland. ' T.

However $Q \Rightarrow P$ is the proposition
'If Riley is from Ireland, then Riley is from Galway.' F

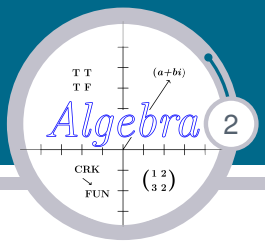
$$x = 2 \Rightarrow x^2 = 4, \quad T.$$

However, does

$$x^2 = 4 \Rightarrow x = 2? \quad F.$$

For propositions P and Q, we write $P \Rightarrow Q$ and say P implies Q, or if P then Q.

The truth table for implies $\Rightarrow$



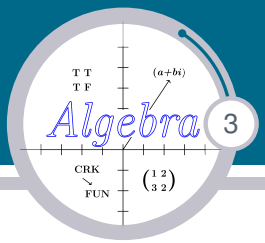
Consider the sentences:
P: I have a lecture today.
Q: I will go in to university.

If (I have a lecture today) then (I will go in to university).

Under what circumstances do we regard $P \Rightarrow Q$ as being false?
Thinking this through will help motivate the definition of $\Rightarrow$
I.e. it will help us understand the truth table defining $\Rightarrow$.

P	Q	$P \Rightarrow Q$
T	T	
T	F	
F	T	
F	F	

The truth table for implies $\Rightarrow$



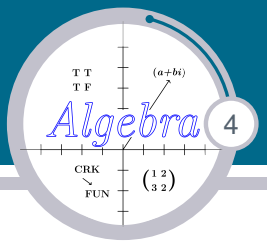
Consider the propositions:
P: I have a lecture today.
Q: I will go in to the university.

If (I have a lecture today) then (I will go in to the university).

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Compare and contrast! See how $P \Rightarrow Q$ is logically equivalent to $(\neg P) \vee Q$ (look back at the truth table developed earlier!)

Example



Suppose P denotes $\boldsymbol{x \leq 3}$ and $\boldsymbol{Q}$ denotes $\boldsymbol{y > 12}$.

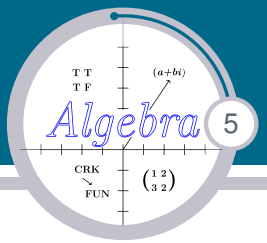
Then $\boldsymbol{P \Rightarrow Q}$ means 'If $\boldsymbol{x \leq 3}$ then $\boldsymbol{y > 12}$'.

As an aside for now but repeated for emphasis, we know that $\boldsymbol{P \Rightarrow Q}$ is logically equivalent to $\boldsymbol{(\neg P) \vee Q}$ - which in this case is $\boldsymbol{x > 3}$ **or** $\boldsymbol{y > 12}$.

Now suppose we are told that $\boldsymbol{P \Rightarrow Q}$ is true.

Then we know that it can NEVER be the case that $\boldsymbol{P}$ is true and $\boldsymbol{Q}$ is false i.e. it cannot be the case that $\boldsymbol{x \leq 3}$ is true whilst $\boldsymbol{y \leq 12}$ is also true.
That is, we cannot have both that $\boldsymbol{x \leq 3}$ and $\boldsymbol{y \leq 12}$ are each true.
This is all we can deduce from the given information.

Suppose that 'If $x \leq 3$ then $y > 12$' is true.



Suppose also that $y = 13$. Which of the following can we deduce?

(a) $x \leq 3$ (b) $x > 3$ (c) $y \leq 12$ (d) Nothing

Suppose instead that $y = 10$. Which of the following can we deduce?

(a) $x \leq 3$ (b) $x > 3$ (c) $y > 12$ (d) Nothing

Suppose now that $x = 2$ is all we know (as well as the implication being true). Which of the following can we deduce?

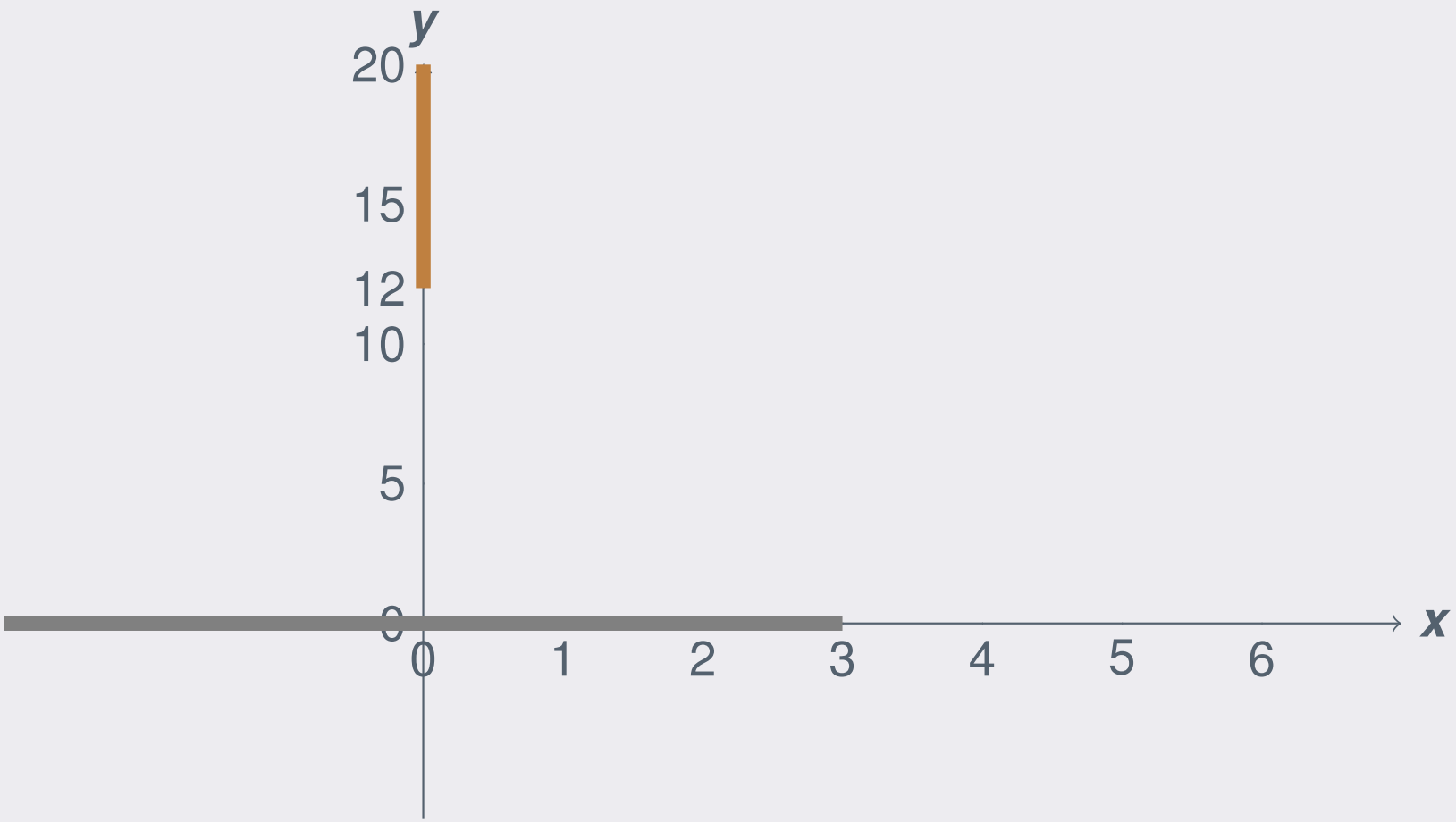
(a) $x > 3$ (b) $y > 12$ (c) $y \leq 12$ (d) Nothing

Finally, suppose that $x = 5$. Which of the following can we deduce?

(a) $x \leq 3$ (b) $y > 12$ (c) $y \leq 12$ (d) Nothing

Example: $P = x \leq 3$, $Q = y > 12$

What does $P = x \leq 3$ and $Q = y > 12$ mean geometrically?
Perhaps the following?



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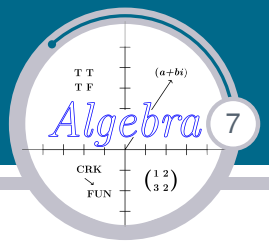
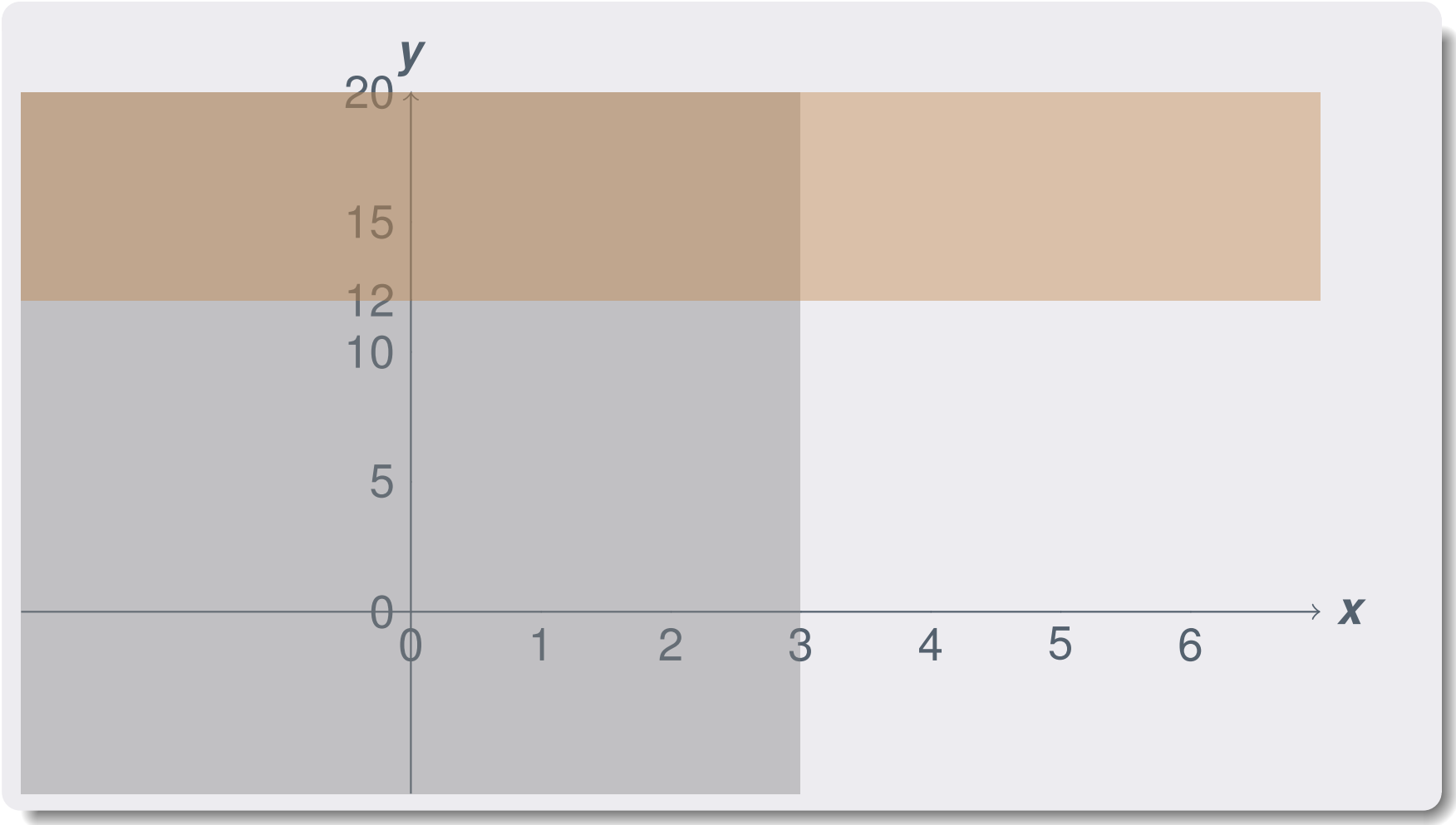
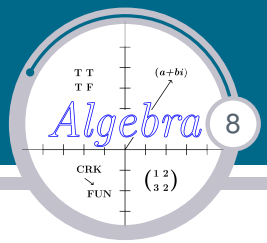


What does $P = x \leq 3$ and $Q = y > 12$ mean geometrically?
No, its actually:



Example: $P = x \leq 3$, $Q = y > 12$

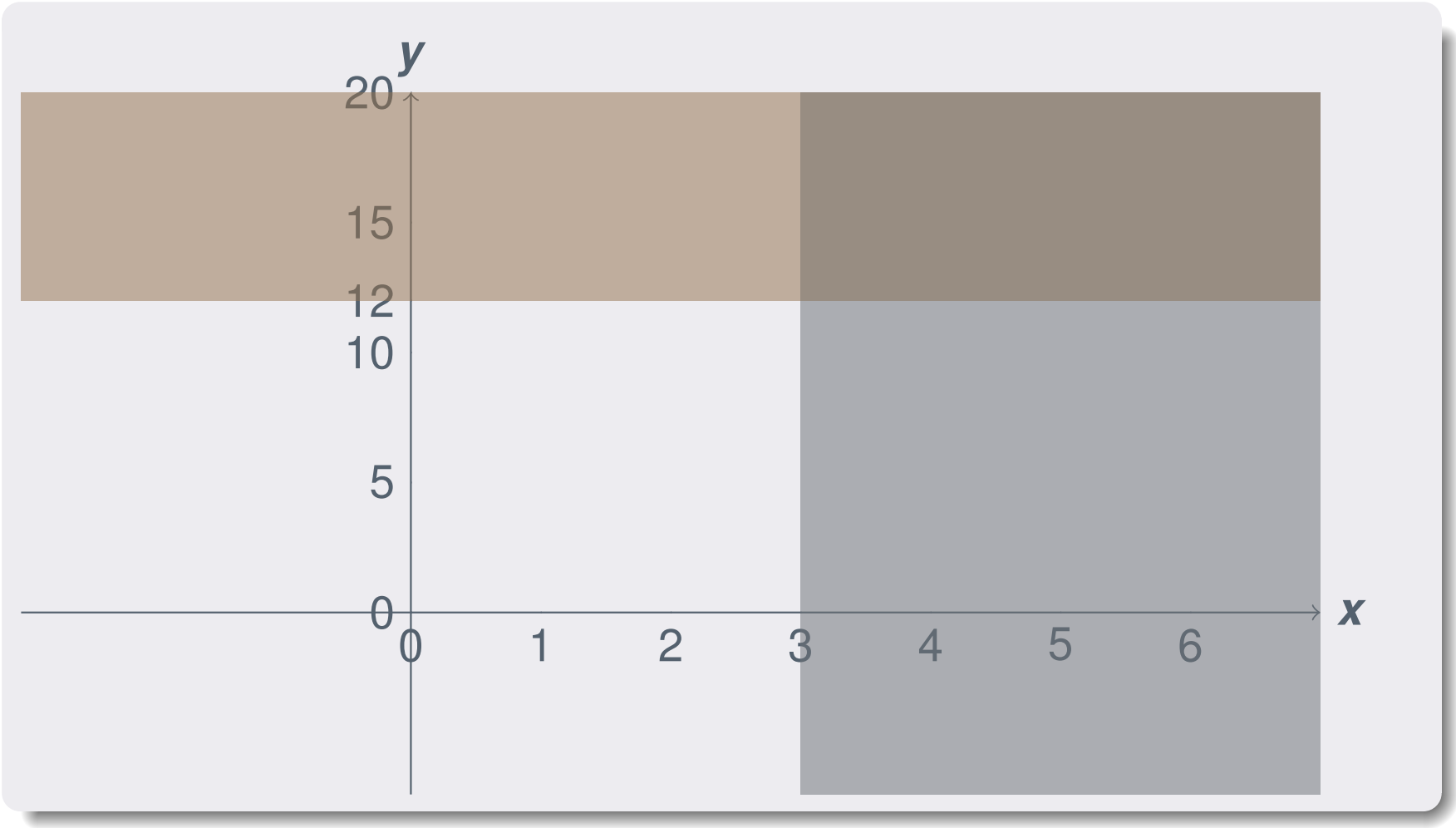


What does $x \leq 3 \Rightarrow y > 12$ mean geometrically?



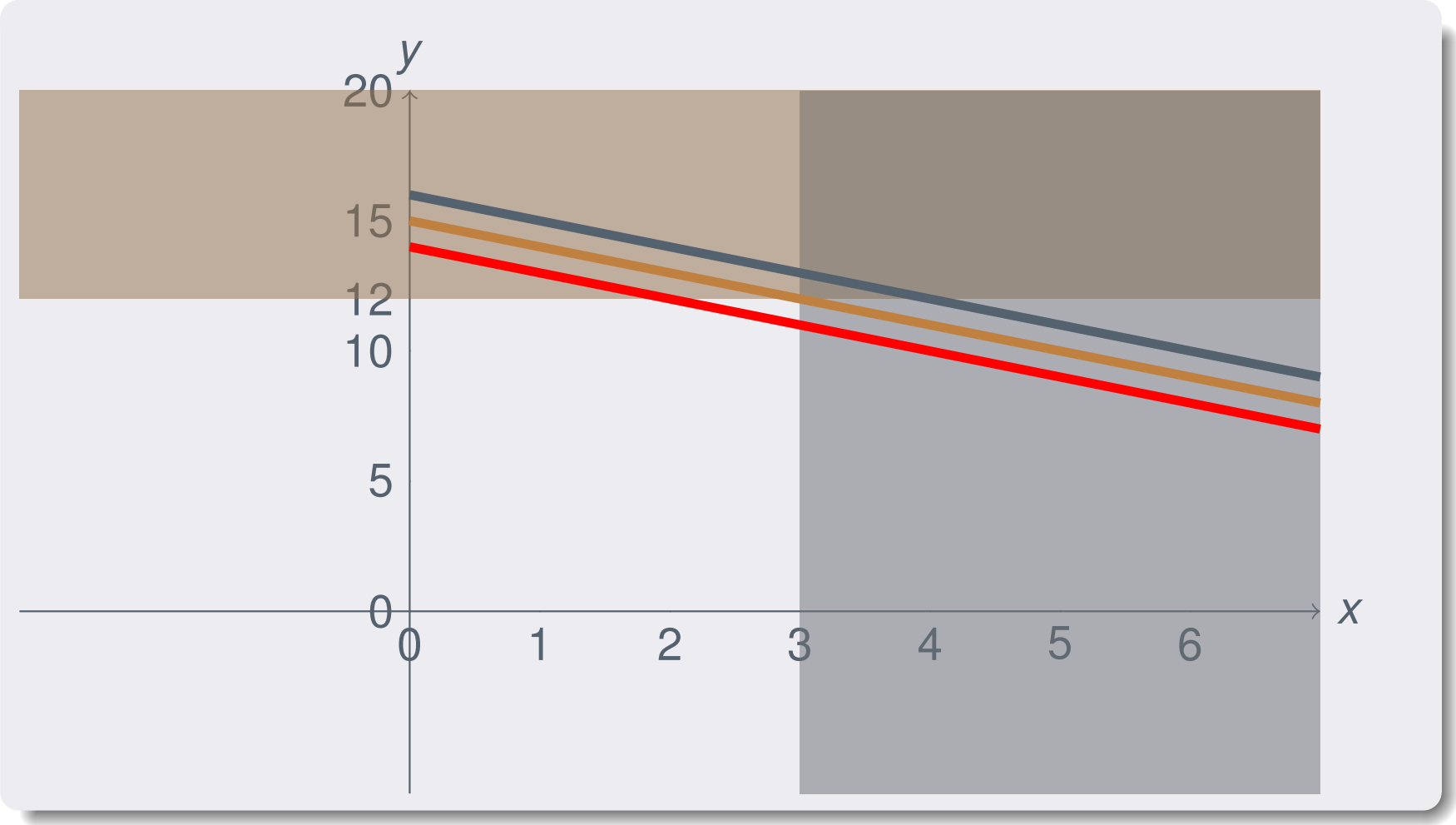
Example: $P = x \leq 3$, $Q = y > 12$

What does $x \leq 3 \Rightarrow y > 12$ mean geometrically?
Using $\neg P \vee Q$ tells us thaat



Example: $P = x \leq 3$, $Q = y > 12$

For $y = -x + 16$ and $y = -x + 15$ (grey) and $P \Rightarrow Q$ (brown).
But for $y = -x + 14$ (red), this is not true ($x = 3 \Rightarrow y = 11$).



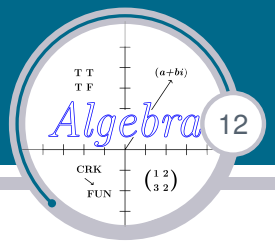
Boolean function equivalent for $\Rightarrow$

Example: Find a boolean function $f(x, y)$ such that

x	y	$f(x, y)$
1	1	1
1	0	0
0	1	1
0	0	1

Method - trial and error.
As discussed the solution is $1 + x + xy \bmod 2$.

The converse and contrapositive of $P \Rightarrow Q$

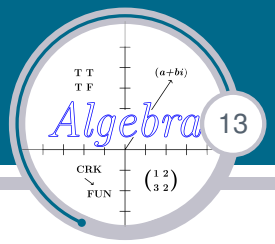


What do we expect to learn about $Q \Rightarrow P$ from $P \Rightarrow Q$?

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

(We can immediately write down $Q \Rightarrow P$ by symmetry since we know $P \Rightarrow Q$.)
Clearly, from $P \Rightarrow Q$ we not say $Q \Rightarrow P$.
This is also clear when think about it "logically":

The converse and contrapositive of $P \Rightarrow Q$

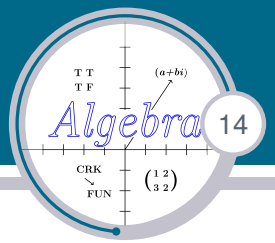


What do we expect to learn about $Q \Rightarrow P$ from $P \Rightarrow Q$?

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

The statement 'If John is from Galway, then John is from Ireland' is certainly true.
The **converse** 'If John is from Ireland, then John is from Galway' is clearly false.
However, the **contrapositive** 'if John is not from Ireland, then John is not from Galway' is clearly true (as expected from truth table above!)

The converse and contrapositive of $P \Rightarrow Q$



What do we expect to learn about $Q \Rightarrow P$ from $P \Rightarrow Q$?

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
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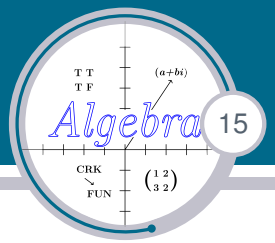
However, the **contrapositive** 'if John is not from Ireland, then John is not from Galway' is clearly true.
This can also be seen from the truth table.

The converse and contrapositive of $P \Rightarrow Q$

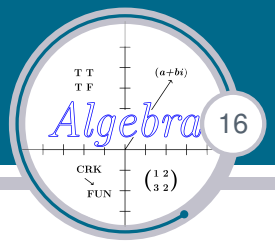


What do we expect to learn about $Q \Rightarrow P$ from $P \Rightarrow Q$?

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P$	$\neg Q$
T	T	T	T	F	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

However, the **contrapositive** 'if John is not from Ireland, then John is not from Galway' is clearly true.
This can also be seen from the truth table.

The converse and contrapositive of $P \Rightarrow Q$



What do we expect to learn about $Q \Rightarrow P$ from $P \Rightarrow Q$?

$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T	F	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	T

However, the **contrapositive** 'if John is not from Ireland, then John is not from Galway' is clearly true.
This can also be seen from the truth table.

What is the negation of $S \Rightarrow T$?

Recall again that $S \Rightarrow T \equiv \neg S \vee T$.

So the negation of $S \Rightarrow T$ is logically equivalent to the negation of $\neg S \vee T$.

That is, $\neg(S \Rightarrow T)$ is logically equivalent to $\neg(\neg S \vee T)$

De Morgan's laws imply

$\neg(\neg S \vee T) \equiv \neg(\neg S) \wedge \neg T \equiv S \wedge \neg T$.

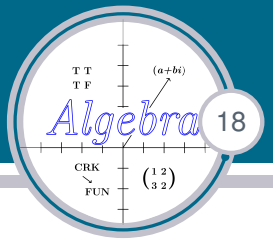
So the negation of 'If I am hungry, then I eat an apple' is

'I am hungry and I do not eat an apple'.

(What is S here and what is T?)

(Can you also convince yourself of this geometrically?)

Summary

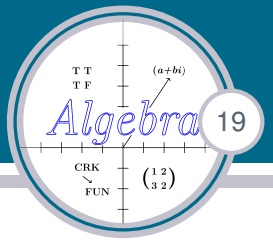


We have defined the logical connectives $\neg$, $\wedge$, $\vee$ and $\Rightarrow$. Note we can express $\Rightarrow$ in terms of $\neg$ and $\vee$.

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the last two columns agree for every option we say that $(\neg P) \vee Q \equiv P \Rightarrow Q$ i.e. $P \Rightarrow Q$ is logically equivalent to $(\neg P) \vee Q$.

Another connective $\Leftrightarrow$



Another useful connective is $\Leftrightarrow$, shorthand for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. We write $P \Leftrightarrow Q$ and say P if and only if Q. The related truth table is

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Let's confirm that $P \Leftrightarrow Q$ is indeed shorthand for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ i.e. that $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.