Semester II 2026

MA135/MA160 - Algebra II



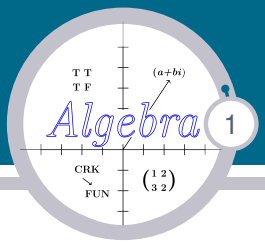
OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Ilia Pirashvili, PhD

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

Lecture 3: Predicates and Venn diagrams

Recap



Last week we introduced three connectives,

- negation(not), $\neg$. For example: If $P :=$ Galway is in Connacht, then $\neg P :=$ Galway is not in Connacht,
- and, $\wedge$. Remember: $P \wedge Q$ is true only if both P and Q are true,.
- or, $\vee$. Remember: $P \vee Q$ is true if either P or Q (or both) is true.

Solution: Write out the truth table for $(\neg P) \vee (P \wedge Q)$.

P	Q	$\neg P$	$P \wedge Q$	$(\neg P) \vee (P \wedge Q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

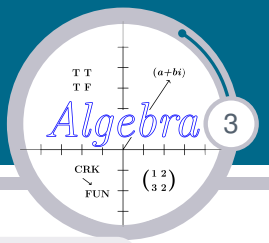
Exercise: Write out the truth table for $(\neg P) \vee Q$

P	Q	$\neg P$	$(\neg P) \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Compare this with $(\neg P) \vee (P \wedge Q)$.
Recall, different expressions can be equivalent.

Next let's explore the effect of negation ($\neg$) on each of the compound statements $P \wedge Q$ and $P \vee Q$.

De Morgan's Laws



P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Thus, we see the first part of

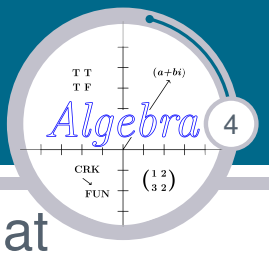
De Morgan's Laws

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q.$$

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q.$$

(Exercise: prove the second equality yourself). These two equalities are called **de Morgan's Laws** and give us a recipe for negating compound statements.

Predicates



Predicates are logical expressions (sentences) with variables, that become statements (simple sentences) once the variables are given values or bounds by quantifiers.

Predicates (Informal, but our definition)

A predicate is a statement whose truth is dependent on one or more variables.

Example

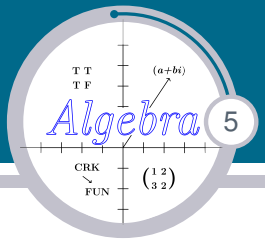
$x \leq 2$ is a predicate, since depending on x, its either true or false.

For example, if $x = 1$, this is true. If $x = 5$ it is false. Likewise, if x is a car, its false. (You could argue that its makes no sense.)

Note

To avoid undue technicality, we will write $x \leq 2$ and understand that x has to be a number of sorts: $x \in \mathbb{R}$ unless otherwise specified.

Connectives of Predicates: $\neg$



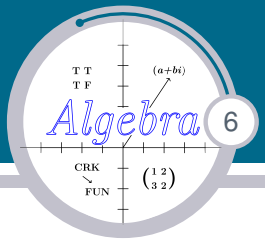
Keeping things a little informal, we will use $P(x)$ to denote predicates with one variable. E.g., $x \leq 2$. When x is clear, we will even use use P.
We will keep using $\neg, \vee, \wedge$ as with statements and write $\neg P(x)$ or even $\neg P$.

We apply a connective to a predicate by applying it for every possible variable.
That is, the negation $\neg P$ of a predicate P is a predicate that is true whenever P is false and false whenever P is true.

Example

E.g, if $P = P(x) = (x \leq 2)$, then $\neg P$ is "x is not less-than or-equal to 2". Clearly this is equivalent to saying $\neg P = (x > 2)$.

Connectives of Predicates: $\wedge$ & $\vee$



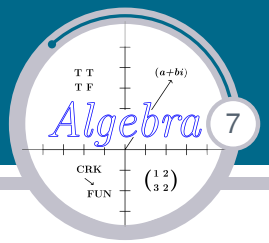
Let P be $x \leq 2$ and Q be $x \geq 0$.
Then $P \wedge Q$ means P *and* Q, hence, x has to be *both* less-than-or-equal to 2 *and* greater-or-equal to 0.
Clearly, this becomes a true statement for $x = 0, \frac{1}{17}, 1, \frac{5}{4}, 2$, but false for $-1, -2, \pi, 3, 4$.

$P \vee Q$ means P *or* Q. We can see that this is true for *every* real number.

Note

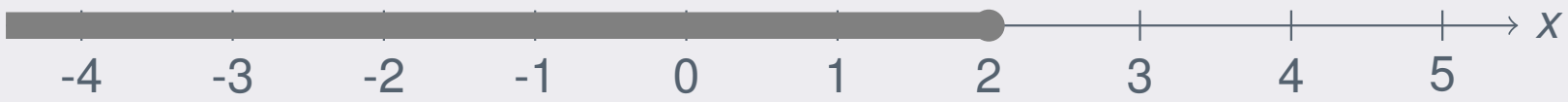
All our laws for connectives of statements apply to connectives of predicates.

Visual representation



This is much clearer, when represented visually:

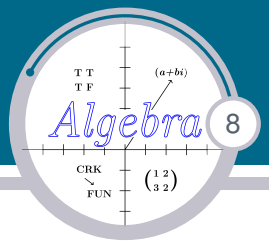
$P = (x \leq 2)$



$\neg P = (x > 2)$



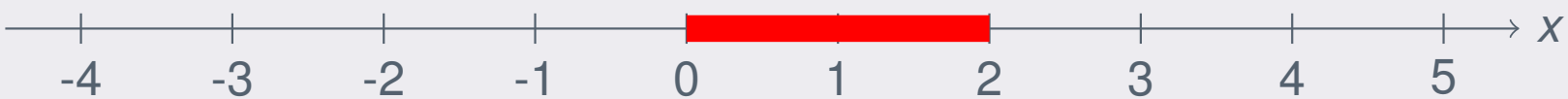
Visual representation



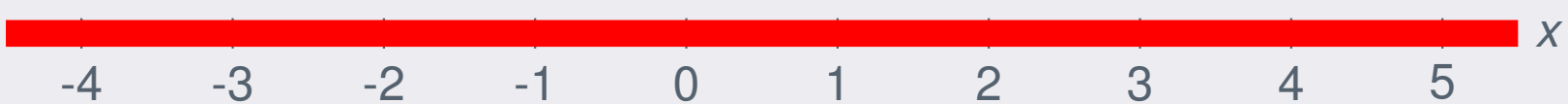
$P = (x \leq 2)$ & $Q = (x \geq 0)$



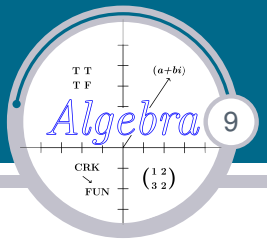
$P \wedge Q$



$P \vee Q$



Negating the predicate: $x > 2$ or $x < 0$.



Example

Let $P = (x > 2)$ and $Q = (x < 0)$. What is $\neg(P \vee Q)$?

By de Morgan's law, we know $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

so its negation is $\neg(x > 2) \wedge \neg(x < 0)$

I.e. the negation is

$$\boldsymbol{x \leq 2} \text{ and } \boldsymbol{x \geq 0}$$

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Visual representation

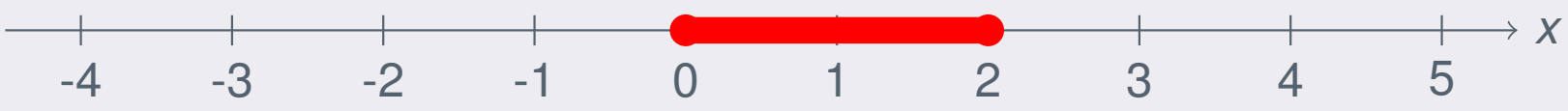
$P = (x > 2)$ & $Q = (x < 0)$



$P \vee Q$



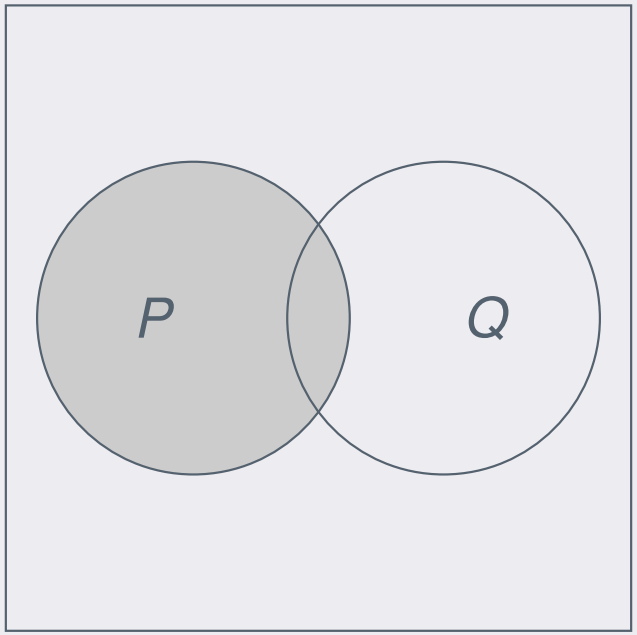
$\neg(P \vee Q)$



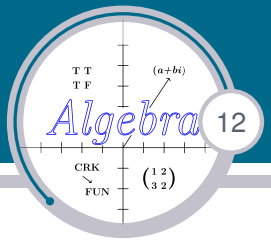
Venn Diagrams

A **Venn diagram** is a graphical representation of sets / predicates that shows their relationships using overlapping regions.

Venn diagram with 2 Sets: P

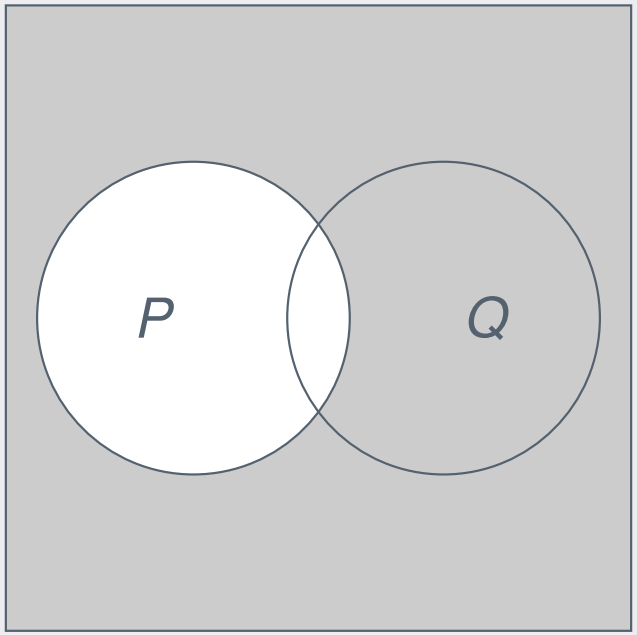


Venn Diagrams



A **Venn diagram** is a graphical representation of sets / predicates that shows their relationships using overlapping regions.

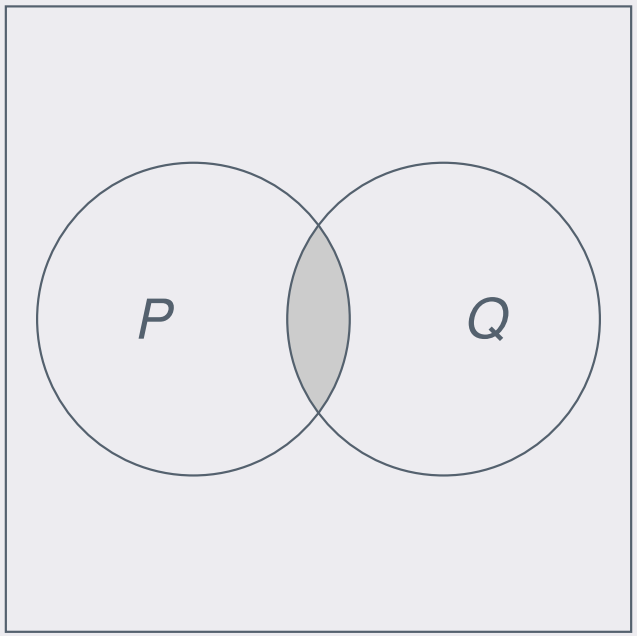
Venn diagram with 2 Sets: $\neg P$



Venn Diagrams

A **Venn diagram** is a graphical representation of sets / predicates that shows their relationships using overlapping regions.

Venn diagram with 2 Sets: $P \wedge Q$



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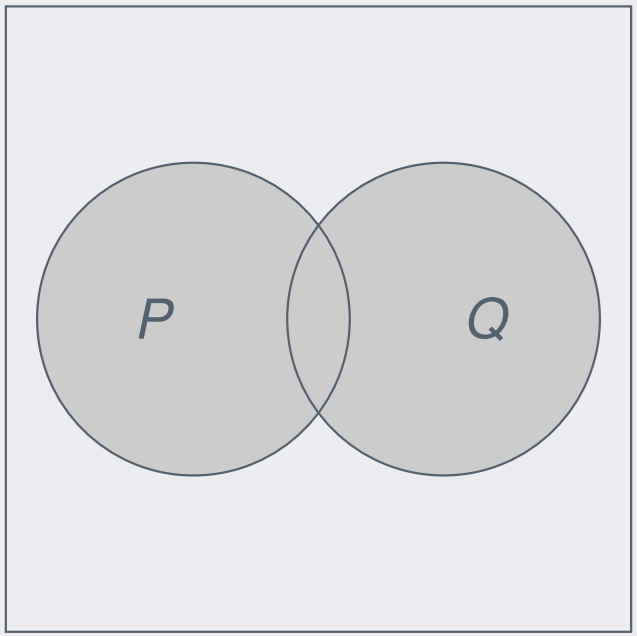
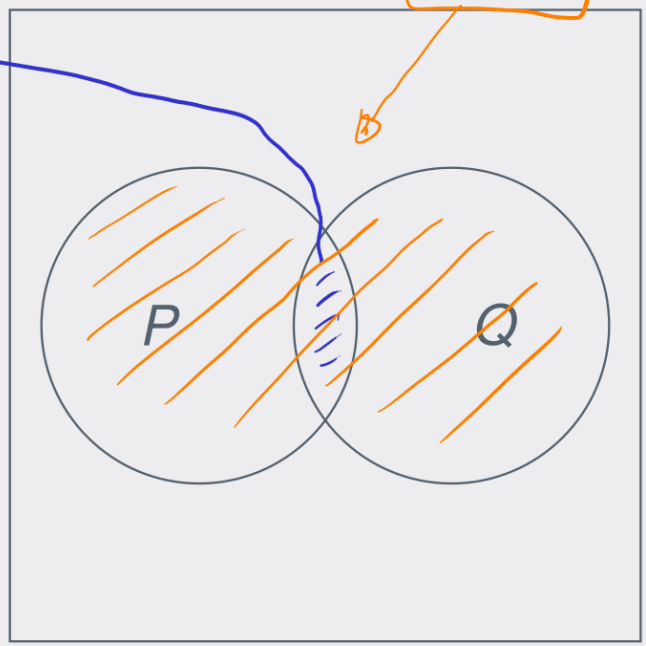
Venn diagram with 2 Sets: $P \vee Q$



Proofs with Venn Diagrams

Using Venn diagrams, it is much easier to prove logical equivalences with few (2-3) statements.
For example

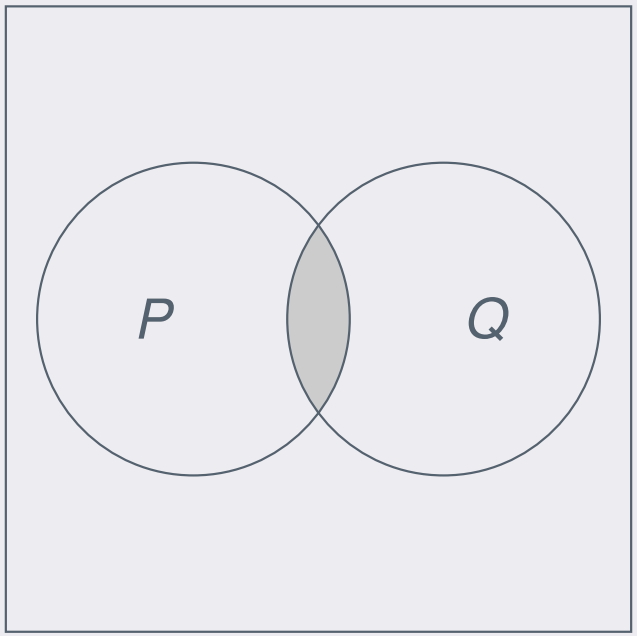
Proof: $(P \vee Q) \wedge P \equiv P \equiv P \vee (Q \wedge P)$



De Morgan's Laws with Venn Diagramms

$\neg(P \wedge Q) = \neg P \vee \neg Q$

$P \wedge Q$ is the small shaded area



De Morgan's Laws with Venn Diagramms

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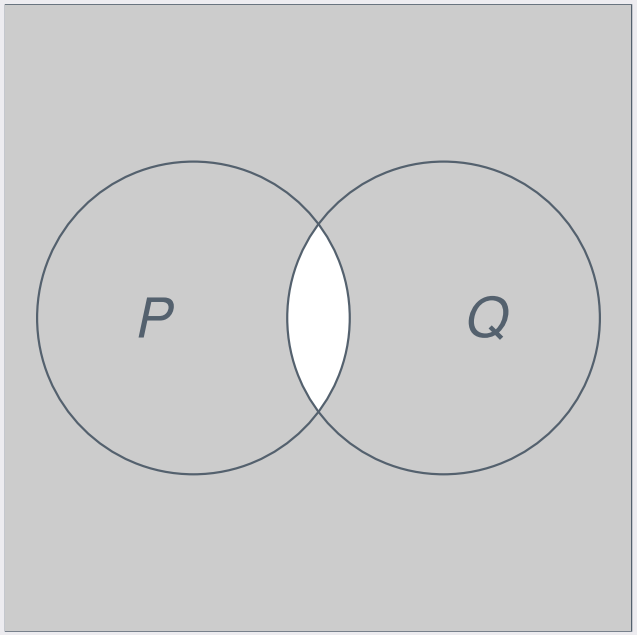
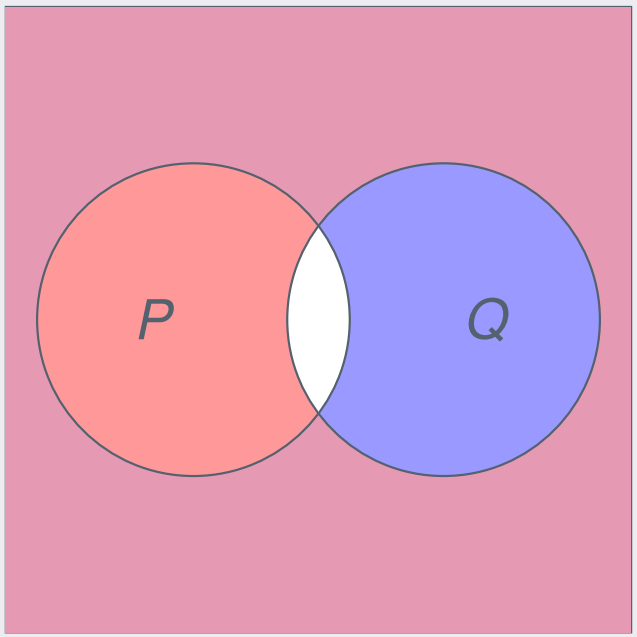
$\neg(P \wedge Q)$ becomes the large shaded area



De Morgan's Laws with Venn Diagramms

$\neg(P \wedge Q) = \neg P \vee \neg Q$

On the other hand, $\neg P$ is blue (and purple), $\neg Q$ is red (and purple), and hence, their union is all the non-white area, as desired.



De Morgan's Laws with Venn Diagramms

Exercise: Prove $\neg(P \vee Q) = \neg P \wedge \neg Q$

