# Semester II 2026

# MA135/MA160 - Algebra II



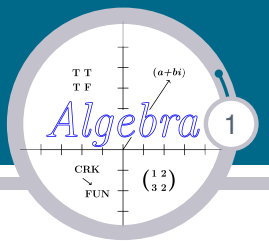

OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

**Ilia Pirashvili, PhD**

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

# Lecture 2: Propositional logic

# Recap

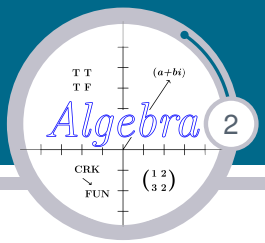



Yesterday we introduced three connectives,

- negation(not): $\neg$, For example: If $P :=$ UoG is in Galway, then $\neg P :=$ UoG is "not" in Galway,
- and: $\wedge$, remember $P \wedge Q$ is true only if both $P$ and $Q$ are true,
- or: $\vee$, remember $P \vee Q$ is true if either $P$ or $Q$ is true.

We generated truth tables for $\neg P$, $\neg Q$, $P \wedge Q$ and $P \vee Q$.

# Recap




Before we start today, let's regenerate truth tables for $\neg Q$, $Q \wedge P$ and $Q \vee P$ (without looking back at what we did yesterday).

| $P$ | $Q$ | $\neg Q$ | $Q \wedge P$ | $Q \vee P$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | F | T |
| F | T | F | F | T |
| F | F | T | F | F |

# Comparison

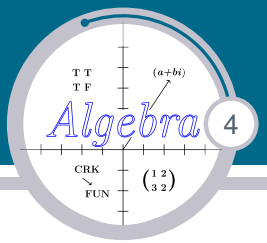
How does $Q \wedge P$ and $Q \vee P$ compare with $P \wedge Q$ and $P \wedge Q$ respectively?

| $P$ | $Q$ | $Q \wedge P$ | $Q \vee P$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | F |

| $P$ | $Q$ | $P \wedge Q$ | $P \vee Q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | F |

So we can write $P \wedge Q$ or $Q \wedge P$ and $P \vee Q$ or $Q \vee P$.

# Modular arithmetic

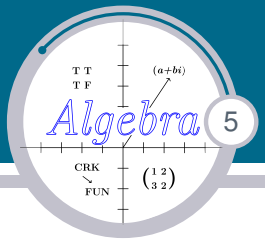



Examples in modular arithmetic to replicate the truth table for $\neg$, $\wedge$ and $\vee$.

| $x$ | $y$ | $1 + x \mod 2$ | $xy \mod 2$ | $xy + x + y \mod 2$ |
|---|---|---|---|---|
| 1 | 1 | | | |
| 1 | 0 | | | |
| 0 | 1 | | | |
| 0 | 0 | | | |

These functions are known as Boolean functions as they only have two possible outcomes, 0 and 1.

# Modular arithmetic

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|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 |

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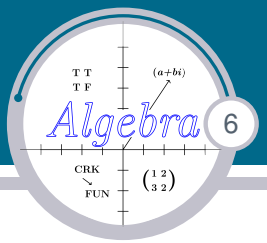
# Modular arithmetic



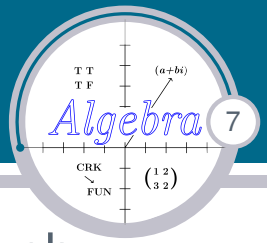

By replacing $x$ with $P$, $y$ with $Q$, 1 with $T$ and 0 with $F$, we see that this indeed agrees with:

| $P$ | $Q$ | $\neq P$ | $P \wedge Q$ | $P \vee Q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | F |

## Remark

Indeed, every boolean function can be written in terms of $\neg$, $\wedge$ and $\vee$. Likewise, every expression that can be obtained with $\neg$, $\wedge$ and $\vee$ has a corresponding boolean function.

# More complicated expressions

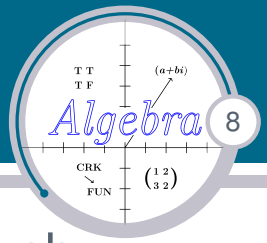



We can use $\neg$, $\wedge$ and $\vee$ to build more complicated expressions such as

$$((\neg P) \wedge Q) \vee (P \wedge R)$$

| $P$ | $Q$ | $R$ | $\neg$ P | $\underbrace{\neg P \wedge Q}_{A}$ | $\underbrace{P \wedge R}_{B}$ | $A \vee B$ |
|---|---|---|---|---|---|---|
| T | T | T | | | | |
| T | T | F | | | | |
| T | F | T | | | | |
| T | F | F | | | | |
| F | T | T | | | | |
| F | T | F | | | | |
| F | F | T | | | | |
| F | F | F | | | | |

# More complicated expressions



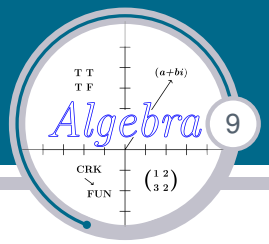

We can use $\neg$, $\wedge$ and $\vee$ to build more complicated expressions such as

$$((\neg P) \wedge Q) \vee (P \wedge R)$$

| $P$ | $Q$ | $R$ | $\neg$ P | $\underbrace{\neg P \wedge Q}_{A}$ | $\underbrace{P \wedge R}_{B}$ | $A \vee B$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | T | F | F | F | F | F |
| T | F | T | F | F | T | T |
| T | F | F | F | F | F | F |
| F | T | T | T | T | F | T |
| F | T | F | T | T | F | T |
| F | F | T | T | F | F | F |
| F | F | F | T | F | F | F |

# $P \vee Q \wedge P$ ($P$ or $Q$ and $P$)

**Exercise:**

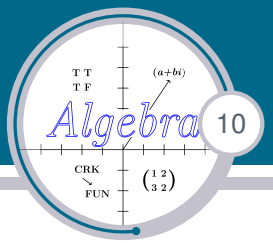



Does it matter if we solve $P \vee Q \wedge P$ as $(P \vee Q) \wedge P$ or $P \vee (Q \wedge P)$?

| $P$ | $Q$ | $P \vee Q$ | $Q \wedge P$ | $(P \vee Q) \wedge P$ | $P \vee (Q \wedge P)$ |
|---|---|---|---|---|---|
| T | T | | | | |
| T | F | | | | |
| F | T | | | | |
| F | F | | | | |

# $P \vee Q \wedge P$ ($P$ or $Q$ and $P$)

**Exercise:**




Does it matter if we solve $P \vee Q \wedge P$ as $(P \vee Q) \wedge P$ or $P \vee (Q \wedge P)$?

| $P$ | $Q$ | $P \vee Q$ | $Q \wedge P$ | $(P \vee Q) \wedge P$ | $P \vee (Q \wedge P)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | T | F | T | T |
| F | T | T | F | F | F |
| F | F | F | F | F | F |

The relevant columns are identical, so we can write that

$$(P \vee Q) \wedge P \equiv P \vee (Q \wedge P),$$

and say that $(P \vee Q) \wedge P$ is logically equivalent to $P \vee (Q \wedge P)$.

## Be aware!

We can not simply ignore brackets by default.

Consisder $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$. We have

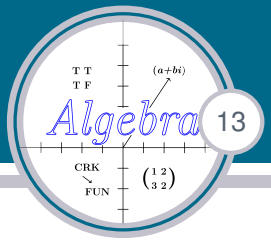
| $P$ | $Q$ | $R$ | $(P \wedge Q) \vee R$ | $P \wedge (Q \vee R)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | T | F | T | T |
| T | F | T | T | T |
| T | F | F | F | F |
| F | T | T | T | F |
| F | T | F | F | F |
| F | F | T | T | F |
| F | F | F | F | F |

Different looking logical expressions can be logically equivalent. Likewise, boolean functions can look different, but agree with each other.

We have already seen $(P \vee Q) \wedge P \equiv P \vee (Q \wedge P)$. From the same table, we can also see that $(P \vee Q) \wedge P \equiv P$, hence its not just about brackets.

For boolean functions, observe that $f(x, y) = x + y$ and $g(x, y) = (x + y)^2$ agree.
Try this out! Try every combination of $x$ and $y$ (they can only be 0 or 1). Basically, show that $f(1, 1) = g(1, 1)$, $f(1, 0) = g(1, 0)$, $f(0, 1) = g(0, 1)$ and $f(0, 0) = g(0, 0)$.
Do you see a simpler way of proving that $f \equiv g$?

# Exercise




Write out the truth table for $( \neg P \vee (P \wedge Q))$. We'll go through the solution to this next week.