Semester II 2026

MA135/MA160 - Algebra II



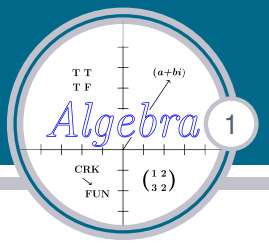
OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Ilia Pirashvili, PhD

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

Lecture 19: Systems of linear equations

Recap: example



Let A be

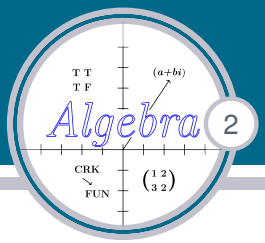
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix},$$

and find A^{-1} using the Gauss-Jordan method.
Solution:

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{pmatrix}}_{A:I} R_3 :\to R_3\text{-}3R_1 \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{pmatrix}$$

$$R_2 :\to R_2\text{-}2R_1 \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{pmatrix} R_3 :\to R_3\text{-}2R_2 \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{pmatrix}$$

Example (cont)



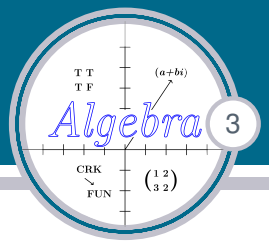
$$R_3 \colon \longrightarrow \text{-}R_3 \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix} R_1 \colon \longrightarrow R_1\text{-}2R_2 \begin{pmatrix} 1 & 0 & 3 & 7 & 6 & 2 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}$$

$$R_2 \colon \longrightarrow R_2{+}R_3 \begin{pmatrix} 1 & 0 & 3 & 7 & 6 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix} R_2 \colon \longrightarrow R_2{+}R_3 \underbrace{\begin{pmatrix} 1 & 0 & 0 & 10 & -12 & 5 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}}_{\mathbf{I : B{=}A^{-1}}}$$

so

$$A^{-1} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

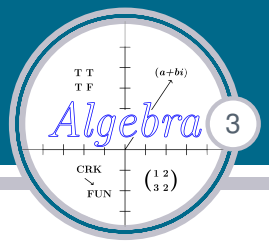
Exercise Gauss-Jordan example



(Exam 2021 Q6(a)) Use row operations to find the inverse of

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}.$$

Exercise Gauss-Jordan example



(Exam 2021 Q6(a)) Use row operations to find the inverse of

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}.$$

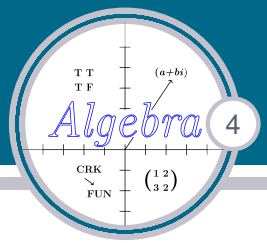
Solution:

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{pmatrix} R_3 \to R_3\text{-}5R_1 \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{pmatrix}$$

$$R_3 \to R_3\text{+}4R_2 \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{pmatrix} R_2 \to R_2\text{-}4R_3 \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{pmatrix}$$

$$R_1 \to R_1\text{-}2R_2 \begin{pmatrix} 1 & 0 & 3 & -39 & 30 & 8 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{pmatrix} R_1 \to R_1\text{-}3R_3 \underbrace{\begin{pmatrix} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{pmatrix}}_{I \;:\; \text{inverse}}$$

Example of a linear system with 4 variables



Solve the system:

$$\begin{aligned} R_1: &\quad w + 3x + 3y + 2z = 1 \\ R_2: &\quad 2w + 6x + 9y + 5z = 5 \\ R_3: &\quad -w - 3x + 3y \qquad = 5 \end{aligned}$$

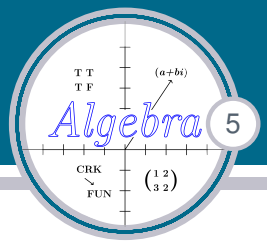
$$\begin{aligned} R_1: &\quad w + 3x + 3y + 2z = 1 \\ R_2: &\quad 2w + 6x + 9y + 5z = 5 \\ R_3: \to R_3{+}R_1 &\quad 0 + 0 + 6y + 2z \qquad = 6 \end{aligned}$$

$$\begin{aligned} R_1: &\quad w + 3x + 3y + 2z = 1 \\ R_2: \to R_2{-}2R_1 &\quad 0 + 0 + 3y + z = 3 \\ R_3: &\quad 0 + 0 + 6y + 2z = 6 \end{aligned}$$

$$\begin{aligned} R_1: &\quad w + 3x + 3y + 2z = 1 \\ R_2: &\quad 0 + 0 + 3y + z = 3 \\ R_3: \to R_3{-}2R_2 &\quad 0 + 0 + 0 + 0 = 0 \end{aligned}$$

R_3 is redundant as it is a linear combination of R_1 and R_2 (i,e,. $R_3 + R_1 - 2(R_2 - 2R_1) = 0$, so $R_3 = -5R_1 + 2R_2$). and adds nothing to the system.

Example of a linear system (cont)



Our simplified system is now

$$\begin{aligned} R_1: &\quad w + 3x + 3y + 2z = 1 \\ R_2: &\qquad\qquad 3y + z = 3, \end{aligned}$$

and its solution is the same as to the original system. We have 2 equations and 4 variables so two of the variables depend on the others, let w and y be the dependent variables (we pick the first variables in each equation of the simplified system) and x and z be the independent variables.
Let's take $x = s$ and $z = t$. Then $3y = 3 - t$ so $y = 1 - t/3$, and

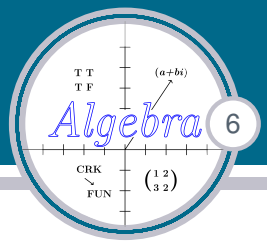
$$\begin{aligned} w &= 1 + 3x - 3y - 2z \\ &= 1 - 3s - 3(1 - t/3) - 2t = 1 - 3s - 3 + t - 2t = -2 - 3s - t \end{aligned}$$

The solutions are

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 - 3s - t \\ s \\ 1 - \frac{t}{3} \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{P} + s \underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{u} + t \underbrace{\begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}}_{v},$$

for all $s, t \in \mathbb{R}$. These solutions form a plane in $\mathbb{R}^3$. (We have a 2-D solution space.)

Another example of a linear system



Solve

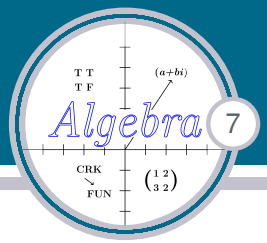
$$\begin{aligned} R_1: &\quad w - 2x + y - z = 4, \\ R_2: &\quad 2w - 3x + 2y - 3z = -1, \\ R_3: &\quad 3w - 5x + 3y - 4z = 3, \\ R_4: &\quad -w + x - y + 2z = 5. \end{aligned}$$

Solution:

$$\begin{aligned} R_1: &\quad w - 2x + y - z = 4, \\ R_2: &\quad 2w - 3x + 2y - 3z = -1, \\ R_3: &\quad 3w - 5x + 3y - 4z = 3, \\ R_4: \to R_4 + R_1 &\quad 0 - x - 0 + z = 9. \end{aligned}$$

$$\begin{aligned} R_1: &\quad w - 2x + y - z = 4, \\ R_2: &\quad 2w - 3x + 2y - 3z = -1, \\ R_3: \to R_3 \text{:-}3R_1 &\quad 0 + x + 0 - z = -9, \\ R_4: &\quad 0 - x - 0 + z = 9. \end{aligned}$$

Another example of a linear system (cont)

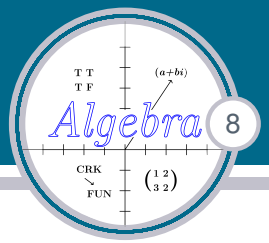


$$
\begin{aligned}
R_1: &\quad w - 2x + y - z = 4,\\
R_2: \to R_2\text{-}2R_1 &\quad 0 + x + 0 - z = -9,\\
R_3: &\quad 0 + x + 0 - z = -9,\\
R_4: &\quad 0 - x - 0 + z = 9.
\end{aligned}
$$

$$
\begin{aligned}
R_1: &\quad w - 2x + y - z = 4,\\
R_2: &\quad 0 + x + 0 - z = -9,\\
R_3: &\quad 0 + x + 0 - z = -9,\\
R_4: \to R_4\text{+}R_3 &\quad 0 + 0 + 0 + 0 = 0.
\end{aligned}
$$

$$
\begin{aligned}
R_1: &\quad w - 2x + y - z = 4,\\
R_2: &\quad 0 + x + 0 - z = -9,\\
R_3: \to R_3\text{-}R_2 &\quad 0 + 0 + 0 + 0 = 0,\\
R_4: &\quad 0 + 0 + 0 + 0 = 0.
\end{aligned}
$$

Another example of a linear system (cont)



Again, we have a system of 2 equations in 4 unknowns. Here our dependent variables are w and x and the independent variables are y and z. Our simplified system is

$$\begin{aligned} R_1: \quad & w - 2x + y - z = 4, \\ R_2: \quad & x + \quad -z = -9. \end{aligned}$$

Let $y = s$ and $z = t$:
We get $x = -9 + t$, and $w = 4 + 2x - y + z$, so $w = 4 + 2(-9 + t) - s + t = 4 - 18 + 2t - s + t = -14 - s + 3t$. Our solutions are

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -14 - s + 3t \\ -9 + t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -14 \\ -9 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -s \\ 0s \\ s \\ 0s \end{pmatrix} + \begin{pmatrix} 3t \\ 1t \\ 0t \\ 1t \end{pmatrix} = \underbrace{\begin{pmatrix} -14 \\ -9 \\ 0 \\ 0 \end{pmatrix}}_{P} + s \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{u} + t \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{v}$$

for all $s, t \in \mathbb{R}$. Again, we have a 2-D solution space.