Semester II 2026

MA135/MA160 - Algebra II



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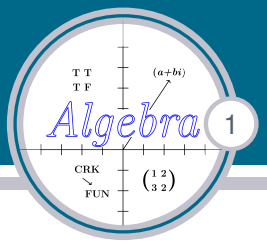
Lecture 18: Systems of linear equations

Recap



Planes in $\mathbb{R}^3$ can be represented in their algebraic form as

$$ax + by + cz = d, \qquad \text{for } a, b, c, d \in \mathbb{R},$$

and in their vector form as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P + s\mathbf{u} + t\mathbf{v}, \qquad \text{for } s, t \in \mathbb{R}.$$

where P is any point in the plane and $\mathbf{u}$, $\mathbf{v}$ are two direction vectors that span the plane.

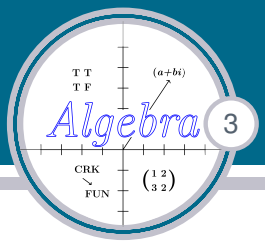
Example

Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 3x + 5y + 7z = 15 \right\} \subseteq \mathbb{R}^3$.

Recall how we expressed S, i.e., the solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that satisfy $3x + 5y + 7z = 15$, in the form $P + s\mathbf{u} + t\mathbf{v}$ where we took P to be $\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$ in S and direction vectors $\mathbf{u}$ to be $\begin{pmatrix} -\frac{5}{3} \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}$ to be $\begin{pmatrix} -\frac{7}{3} \\ 0 \\ 1 \end{pmatrix}$.
Recall that P could be any choice of point on the plane S.

This demonstrated how an equation of the form $ax + by + cz = d$ can be expressed as $P + s\mathbf{u} + t\mathbf{v}$, for some $P, \mathbf{u}$ and $\mathbf{v}$.

Recap: how can two planes interact?



A linear equation of the form

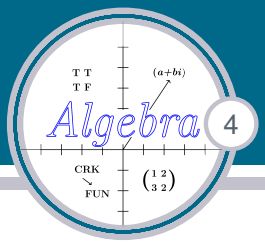
$$ax + by + cz = d,$$

determines a plane in $\mathbb{R}^3$. Observation: A system of two linear equations

$$\begin{aligned} ax + by + cz &= d, \quad \text{and,} \\ a'x + b'y + c'z &= d', \end{aligned}$$

describe two planes in $\mathbb{R}^3$. They can intersect along a line, i.e., the system has infinitely many solutions determined by one parameter (as we see in the examples below.)

Other possibilities



Other possibilities are that

- the planes are parallel, i.e., the system has no solution For example, if the two planes are

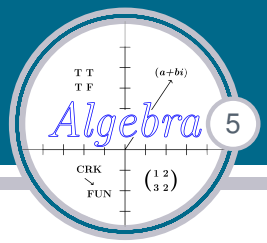
$$3x + 2y + 3z = 0,$$
$$3x + 2y + 3z = 2,$$

  the two planes are parallel and will never intersect.
- or the planes are identical, the system has infinitely many solutions For example, if the two planes are

$$3x + 2y + 3z = 4,$$
$$6x + 4y + 6z = 8.$$

  Then their solutions are identical.

Example



Solve the linear system

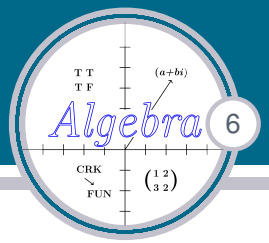
$$\begin{aligned} \text{R1:} \quad & 3x + 5y + 7z = 15 \\ \text{R2:} \quad & x + y + z = 1 \end{aligned}$$

$$\begin{aligned} \text{R1:} \to \text{R2:} \quad & x + y + z = 1 \\ \text{R2:} \to \text{R1:} \quad & 3x + 5y + 7z = 15 \end{aligned}$$

$$\begin{aligned} \text{R1:} \quad & x + y + z = 1 \\ \text{R2:} \to \text{R2- 3R1:} \quad & 2y + 4z = 12 \end{aligned}$$

$$\begin{aligned} \text{R1:} \quad & x + y + z = 1 \\ \text{R2:} \to \text{R2/2:} \quad & y + 2z = 6 \end{aligned}$$

Example (cont)



Let $z = t$, then

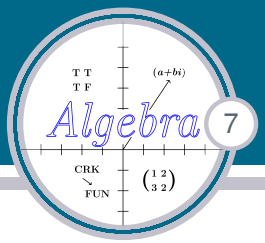
$$\begin{aligned} y + 2t &= 6 \quad \text{so } y = 6 - 2t, \quad \text{and} \\ x + (6 - 2t) + t &= 1 \quad \text{so } x = 1 - 6 + 2t - t, \textit{i.e.}, x = -5 + t. \end{aligned}$$

Any solution $Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the system has the form

$$\begin{pmatrix} -5 + t \\ 6 - 2t \\ t \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} -5 \\ 6 \\ 0 \end{pmatrix}}_{P} + t \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{\mathbf{v}}$$

i.e., the system has as its solutions all points on the line determined by $P = \begin{pmatrix} -5 \\ 6 \\ 0 \end{pmatrix}$ and the direction vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Exercise



Solve the linear system

$$R1 : 2x + 3y + 4z = 10$$
$$R2 : \ x + \ y + \ z = \ 4$$

$$R2 \to R1 : \ x + \ y + \ z = \ 4$$
$$R1 \to R2 : 2x + 3y + 4z = 10$$

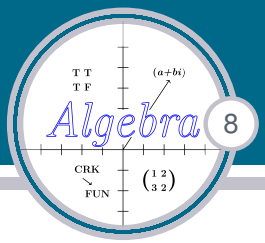
$$R2 \to R1 : \ x + y + \ z = 4$$
$$R2 \to R2 - 2R1 : \quad y + 2z = 2$$

Let $z = t$, then $y = 2 - 2t$ and $x = 4 - (2 - 2t) - t = 2 + t$.

Any solution $Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the system has the form

$$\begin{pmatrix} 2+t \\ 2-2t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}}_{P} + t \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{\mathbf{v}}$$

Matrices and linear equations



The linear system

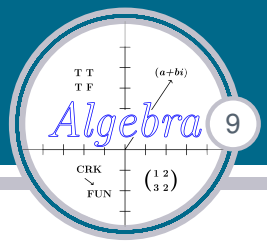
$$\begin{aligned} x + 2y + 3z &= 10 \\ 2x + 5y + 5z &= 21 \\ 3x + 8y + 6z &= 31 \end{aligned}$$

can be expressed as

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \underbrace{\begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix}}_{\mathbf{b}},$$

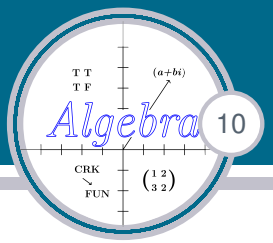
i.e., $\mathbf{Av} = \mathbf{b}$.
$\mathbf{A}$ is a 3×3 matrix (3 rows, 3 columns)
$\mathbf{v}$ is a 3×1 matrix (vector)
$\mathbf{b}$ is a 3×1 matrix (vector)

Review of Matrices



- Review: An $m \times n$ matrix is an array of numbers with m rows and n columns.
- An $m \times 1$ matrix is called a column vector.
  For example $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ is a column vector;
- A $1 \times n$ matrix is called a row vector.
  $\begin{pmatrix} 1 & 7 & 5 \end{pmatrix}$ is a row vector, as is $\begin{pmatrix} 1 & 5 & 3 & -2 & 7 & 5 \end{pmatrix}$.
- Finally recall we multiply matrices by multiplying each row vector **r** by a column vector **c** (of the same length).For example
  $\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 2(1) + 5(4) = 22.$
- Note matrices are denoted by an uppercase letter and a vector by a lowercase letter.

Review of matrices (cont)



Another example,

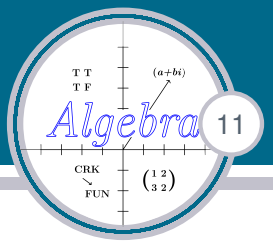
$$\underbrace{\begin{pmatrix} 2 & 5 \end{pmatrix}}_{1\times 2 \text{ matrix}} \underbrace{\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}}_{2\times 2 \text{ matrix}} = \begin{pmatrix} 2(1)+5(4) & 2(2)+5(3) \end{pmatrix} = \underbrace{\begin{pmatrix} 22 & 19 \end{pmatrix}}_{1\times 2 \text{ matrix}}.$$

Note: the highlighted numbers (2) must be the same, the blue numbers (1and 2) determine the size of the resulting matrix. More

generally, if $\mathbf{r} = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix}$ (an $1 \times n$ vector) and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ (a

$n \times 1$ vector), then $\mathbf{rc} = \left(r_1 c_1 + r_2 c_2 + \cdots + r_n c_n \right)$.

Review of matrices (cont)



Two 3×3 matrices

$$\mathbf{A} = \begin{pmatrix} -R1- \\ -R2- \\ -R3- \end{pmatrix}, \mathbf{B} = \begin{pmatrix} | & | & | \\ C1 & C2 & C3 \\ | & | & | \end{pmatrix}$$

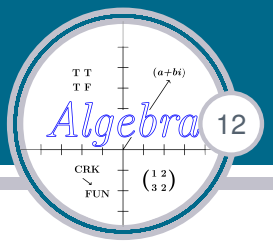
are then multiplied as follows:

$$\mathbf{AB} = \begin{pmatrix} R1(C1) & R1(C2) & R1(C3) \\ R2(C1) & R2(C2) & R2(C3) \\ R3(C1) & R3(C2) & R3(C3) \end{pmatrix}$$

For example if $R1 = \begin{pmatrix} r1_1 & r1_2 & r1_3 \end{pmatrix}$ and $C1 = \begin{pmatrix} c1_1 \\ c1_2 \\ c1_3 \end{pmatrix}$ then

$R1(C1) = r1_1(c1_1) + r1_2(c1_2) + r1_3(c1_3)$.

Matrices: example



Two 3×3 matrices

$$\mathbf{B} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix},$$

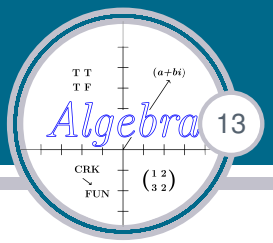
are then multiplied as follows:

$$\mathbf{BA} = \begin{pmatrix} 10(1) - 12(2) + 5(3) & 10(2) - 12(5) + 5(8) & 10(3) - 12(5) + 5(6) \\ -3(1) + 3(2) - 1(3) & -3(2) + 3(5) - 1(8) & -3(3) + 3(5) - 1(6) \\ -1(1) + 2(2) - 1(3) & -1(2) + 2(5) - 1(8) & -1(3) + 2(5) - 1(6) \end{pmatrix}$$

$$= \begin{pmatrix} 10 - 24 + 15 & 20 - 60 + 40 & 30 - 60 + 30 \\ -3 + 6 - 3 & -6 + 15 - 8 & -9 + 15 - 6 \\ -1 + 4 - 3 & -2 + 10 - 8 & -3 + 10 - 6 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{I}}$$

$\mathbf{I}$ – the identity matrix.

Matrices: example (cont)



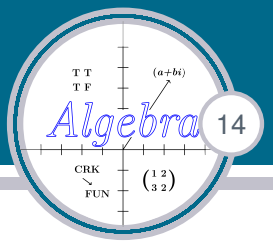
Note that if a matrix has the same number of rows and columns then it is called a square matrix ($n \times n$). Here (previous slide) $\mathbf{B}$, $\mathbf{A}$, and $\mathbf{BA} = \mathbf{I}$ are all square matrices.

Also, $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$ for all square matrices.

If $\mathbf{AB} = \mathbf{I}$, then $\mathbf{B}$ is said to be the inverse of $\mathbf{A}$ and we can denote $\mathbf{B}$ as $\mathbf{A}^{-1}$, and $\mathbf{A}$ is said to be the inverse of $\mathbf{B}$ and we can denote $\mathbf{A}$ as $\mathbf{B}^{-1}$.

Multiplication of a matrix and its inverse is commutative, so if $\mathbf{B}$ is the inverse of $\mathbf{A}$ then $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, i.e., $\mathbf{MM}^{-1} = \mathbf{I}$ and $\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$, for any square matrix $\mathbf{M}$. (However, in general matrix multiplication is not commutative.)

Linear system and matrices



Remember our linear system, $\mathbf{Av} = \mathbf{b}$,

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \underbrace{\begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix}}_{\mathbf{b}}.$$

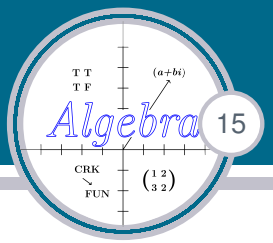
This can be written as $\mathbf{A}^{-1}\mathbf{Av} = \mathbf{A}^{-1}\mathbf{b}$,

$$\underbrace{\mathbf{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \underbrace{\begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix}}_{\mathbf{b}},$$

since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, then $\mathbf{A}^{-1}\mathbf{Av} = \mathbf{Iv} = \mathbf{v}$, which is easier to solve.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10(10) - 12(21) + 5(31) \\ -3(10) + 3(21) - 1(31) \\ -1(10) + 2(21) - 1(31) \end{pmatrix} = \begin{pmatrix} 100 - 252 + 155 \\ -30 + 63 - 31 \\ -10 + 42 - 31 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Summary



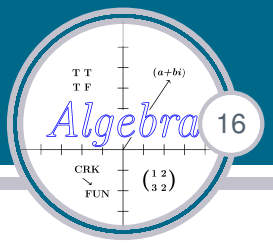
In summary, given $\mathbf{Av} = \mathbf{b}$, where $\mathbf{A}$ is a 3×3 matrix, $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ say, and we can find the inverse matrix $\mathbf{A}^{-1}$, then we can solve $\mathbf{Av} = \mathbf{b}$ easily as follows:

$$\begin{aligned} \mathbf{Av} &= \mathbf{b} \quad \text{then} \\ \mathbf{A}^{-1}(\mathbf{Av}) &= \mathbf{A}^{-1}\mathbf{b} \\ \text{i.e., } \mathbf{Iv} &= \mathbf{v} = \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

How do we find $\mathbf{A}^{-1}$?

- Use the cofactor method (from Semester 1).
- Use Gauss-Jordan method (also Gaussian elimination or row operations).

Find $\mathbf{A}^{-1}$



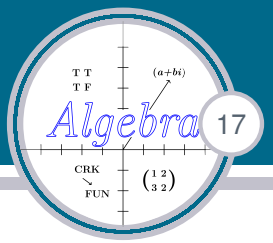
The Gauss-Jordan method for finding the inverse for an $n \times n$ matrix $\mathbf{A}$ is

$$\underbrace{\left(A \quad : \quad I\right)}_{3\times 6 \text{ matrix}} \to \left(I \quad : \quad B\right),$$

and we will have $B = A^{-1}$. There are three kinds of row operation:

1. Row swap : $\mathrm{R}_i \leftrightarrow \mathrm{R}_j$,
2. Divide/multiply a row by a non-zero scalar $\mathrm{R}_i \to \lambda \mathrm{R}_i$ $(\lambda \in \mathbb{R}, \lambda \neq 0)$,
3. Add/subtract a multiple of one row from another: $\mathrm{R}_i \to \mathrm{R}_i + \lambda \mathrm{R}_j$ $(\lambda \in \mathbb{R}, i \neq j)$.

Example



Let A be

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix},$$

and find A^{-1} using the Gauss-Jordan method.
Solution:

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{pmatrix}}_{A:I} \text{R3:} \longrightarrow \text{R3-3R1} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{pmatrix}$$

$$\text{R2:} \longrightarrow \text{R2-2R1} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{pmatrix} \text{R3:} \longrightarrow \text{R3-2R2} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{pmatrix}$$

Example (cont)

$$\text{R3:} \longrightarrow \text{-R3} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix} \text{R1:} \longrightarrow \text{R1-2R2} \begin{pmatrix} 1 & 0 & 3 & 7 & 6 & 2 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}$$

$$\text{R3:} \longrightarrow \text{-R3} \begin{pmatrix} 1 & 0 & 3 & 7 & 6 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix} \text{R2:} \longrightarrow \text{R2+R3} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 10 & -12 & 5 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}}_{\mathbf{I : B{=}A^{-1}}}$$

so

$$A^{-1} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$