Semester II 2026

MA135/MA160 - Algebra II



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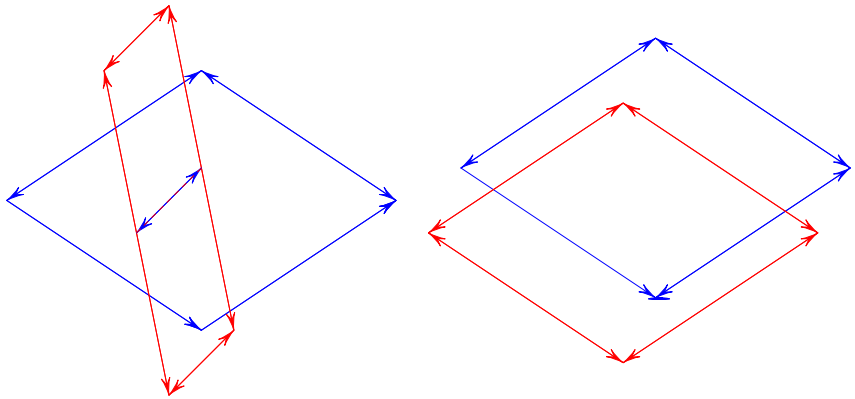
Lecture 17: systems of linear equations

Three variables in geometry

A linear equation in three variables i.e. a linear equation of the form $ax + by + cz = d$ determines a plane in $\mathbb{R}^3$. So if we have a system of two linear equations in x, y, z, we have two planes, how might they interact? I.e. how might they intersect?

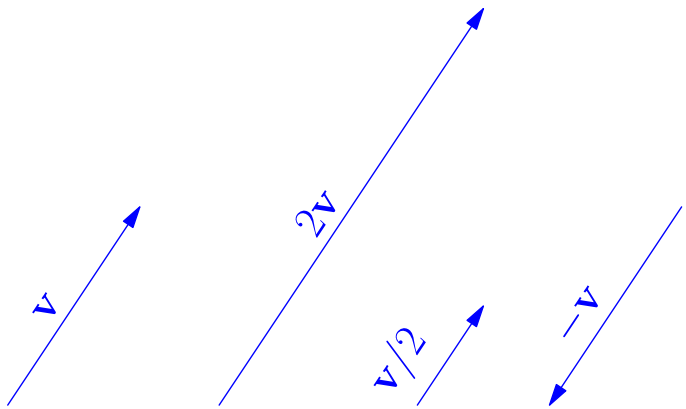


That is, do they intersect, are they parallel or are they identical?

A little detour into vectors

A vector is a directed line segment, i.e., a straight line of a given length in a given direction.

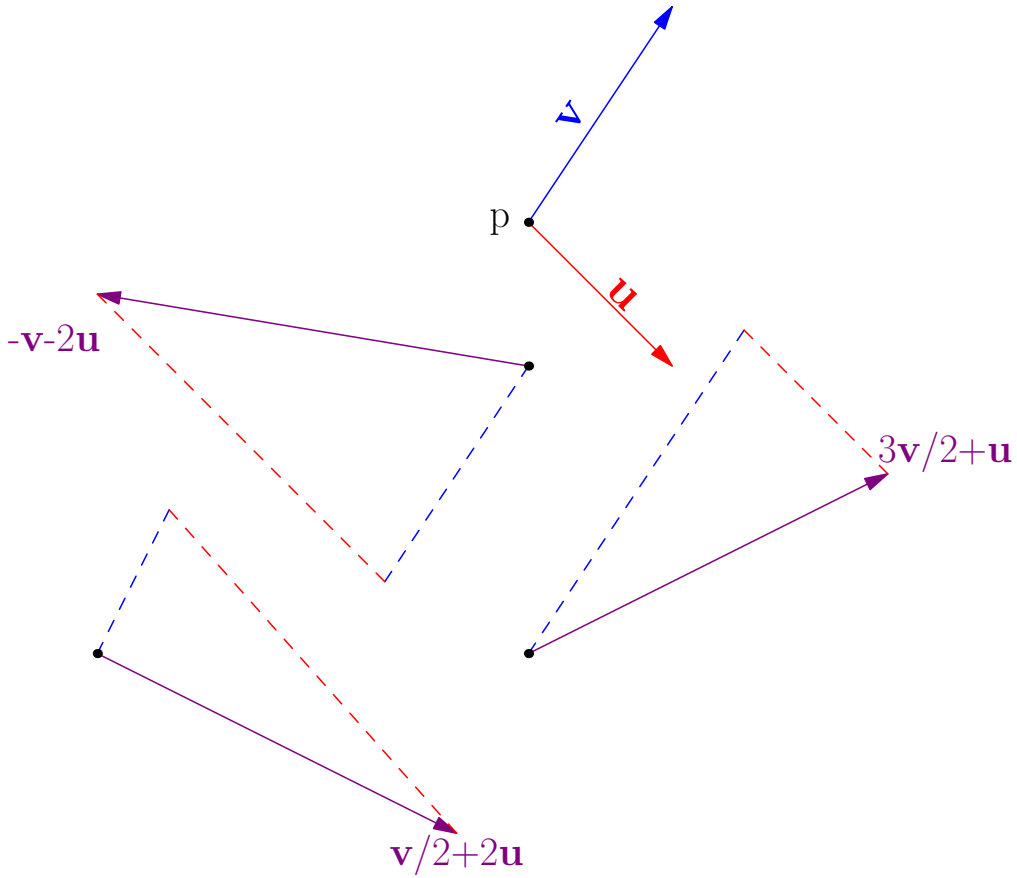
For example, the vector "v" is denoted as $\mathbf{v}$ (bold font, other options include $\vec{v}$). The arrow on the line segment describes its direction. We can scale vectors (i.e. multiply them by a real number.)



Adding/subtracting vectors

Note the (starting) position of a vector is irrelevant. A straight line in $\mathbb{R}^3$ is determined by a point p on the line and a direction vector $\mathbf{v}$. Let, $\mathbf{u}$ and $\mathbf{v}$ be vectors, and plot

$$\frac{3}{2}\mathbf{v} + \mathbf{u}$$
$$\frac{1}{2}\mathbf{v} + 2\mathbf{u}$$
$$-\mathbf{v} - 2\mathbf{u}$$

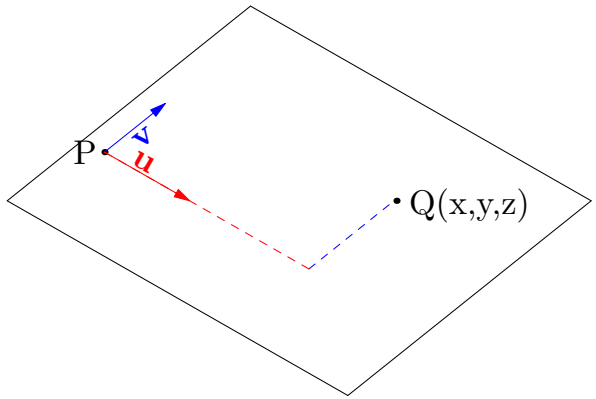


A plane

A plane in $\mathbb{R}^3$ is determined by a point P on the plane and two directional vectors. If Q is any point on the plane, then it can be described in terms of P, $\mathbf{u}$ and $\mathbf{v}$. In $\mathbb{R}^3$, $s\mathbf{u} + t\mathbf{v}$, spans a plane through the origin $(0, 0, 0)$ while $P + s\mathbf{u} + t\mathbf{v}$ spans a plane through P, for example,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}}_{P} + s \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{\mathbf{u}} + t \underbrace{\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}}_{\mathbf{v}}$$

is a plane through $P = (0, 0, 7)$, where $s, t \in \mathbb{R}$. The equation



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P + s\mathbf{u} + t\mathbf{v} \quad \text{for } s, t \in \mathbb{R},$$

is known as the vector equation for a plane (where P is a point on the plane, and $\mathbf{u}$ and $\mathbf{v}$ are two direction vectors in the plane).
Note that P can be any point on the plane and $\mathbf{u}$ and $\mathbf{v}$ can be any two directional vectors that lie on the plane that are not parallel.

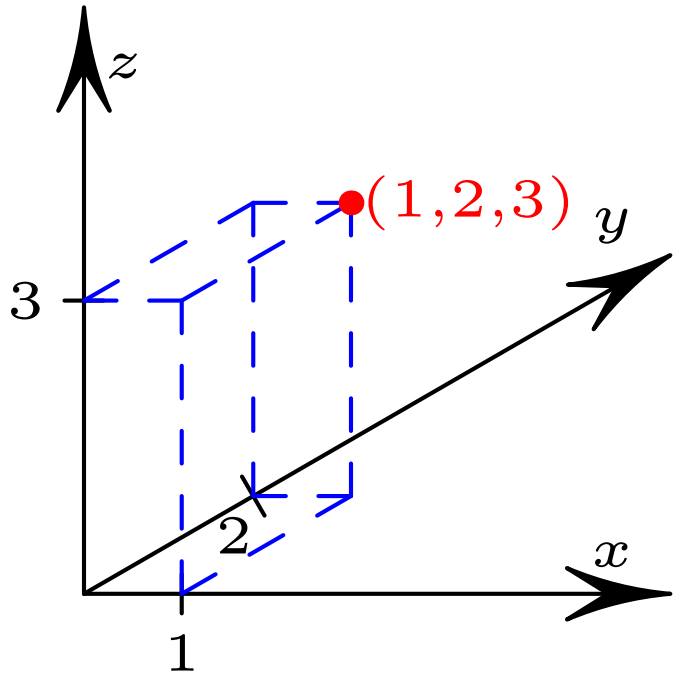
Algebraic form for a plane

From the vector equation for a plane we can derive the algebraic form

$$ax + by + cz = d, \quad \text{where } a, b, c, d \in \mathbb{R}$$

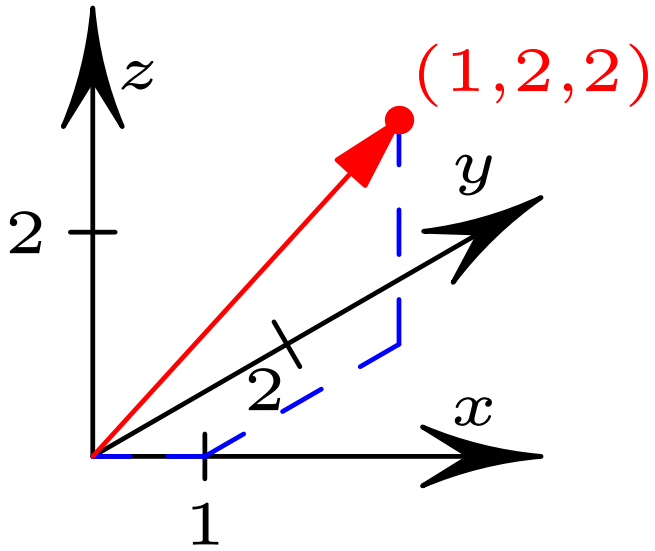
for a plane in $\mathbb{R}^3$.

In $\mathbb{R}^3$, a point P is a triple, i.e., $P = (x, y, z)$ for some real numbers x, y and z. For example, $(1, 2, 3)$ is a point in the $\mathbb{R}^3$.

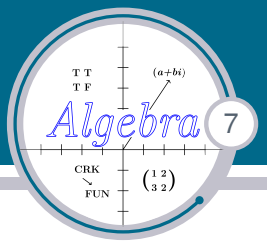


Algebraic form for a plane

A direction vector $\mathbf{v}$ in $\mathbb{R}$ is also a triple for example $\mathbf{v} = (1, 2, 2)$



Example



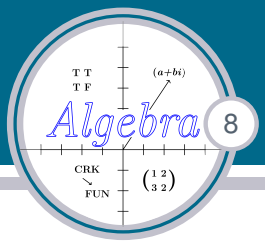
Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 3x + 5y + 7z = 15 \right\} \subseteq \mathbb{R}^3$.

We shall express S, i.e., the solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that satisfy $3x + 5y + 7z = 15$, in the form $P + s\mathbf{u} + t\mathbf{v}$ for some point P in S and direction vectors $\mathbf{u}$, $\mathbf{v}$.

First find a point P in S - any point will do. For example, let $y, z = 0$ to get $3x = 15$, i.e., $x = 5$, i.e., take $P = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \in S$.

$\left(\text{We could let } x, z = 0 \text{ to get } P = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \in S \ldots \right)$

Algebraic form for a plane (cont)



If we set $y = s$ and $z = t$, then

$$\begin{aligned} 3x &= 15 - 5y - 7z, \\ x &= \frac{15}{3} - \frac{5}{3}y - \frac{7}{3}z, \\ x &= 5 - \frac{5}{3}s - \frac{7}{3}t. \end{aligned}$$

Thus an arbitrary point $Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$, has the form

$$Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 - \frac{5}{3}s - \frac{7}{3}t \\ s \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}}_{P} + s \underbrace{\begin{pmatrix} -\frac{5}{3} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{7}{3} \\ 0 \\ 1 \end{pmatrix}}_{\text{direction vectors}}.$$

Now Q takes the form $P + s\mathbf{u} + t\mathbf{v}$, where $P, \mathbf{u}$ and $\mathbf{v}$ are as given. This demonstrated how an equation of the form $ax + by + cz = d$ can be expressed as $P + s\mathbf{u} + t\mathbf{v}$, for some $P, \mathbf{u}$ and $\mathbf{v}$.