Semester II 2026

MA135/MA160 - Algebra II



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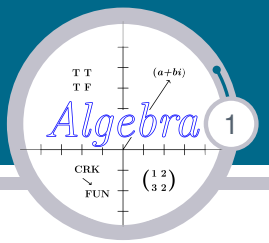
With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

Lecture 16: Systems of linear equations

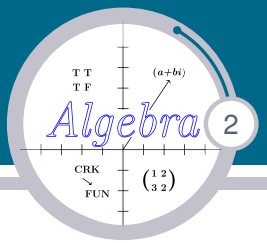
Summary of complex numbers



Let $z, w \in \mathbb{C}$, with $z = x + yi$ and $w = a + bi$, then

- $z + w = x + a + (y + b)i$
- $z - w = x - a + (y - b)i$
- $zw = xa - yb + (ya + xb)i$
- $\overline{z} = x - yi$, $z + \overline{z} = 2x \in \mathbb{R}$ and $z\overline{z} = x^2 + y^2 \in \mathbb{R}$.
- $\dfrac{z}{w} = \dfrac{z}{w}\dfrac{\overline{w}}{\overline{w}} = \dfrac{z\overline{w}}{a^2 + b^2}$.
- For $z = r(\cos\theta + i\sin\theta)$, r is its modulus, $|z|$ and θ is its argument, Arg (z).
- $|zw| = |z||w|$, and $|z^n| = |z|^n$.
- Arg $(zw) =$ Arg $(z)+$ Arg (w), and Arg $(z^n) = n$ Arg (z).

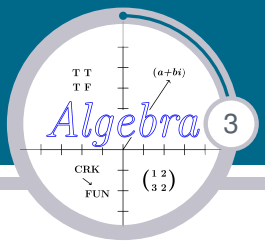
Summary of complex numbers (cont)



General approach to find the roots of $z^n = w$, for $n \in \mathbb{N}$ and $w \in \mathbb{C}$.

1. State w in its polar form, i.e., in terms of its modulus r and its argument θ.
2. Recognize that we can increase θ by 2π by adding it repeatedly, i.e., $\theta + 2k\pi$ for $k = 1, 2, 3, \ldots$.
   (Note: $r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$ for $k \in \mathbb{Z}$.
3. z will have the module $\sqrt[n]{r}$ and the argument $\frac{1}{n}(\theta + 2k\pi)$ for $k = 0, 1, \ldots, n-1$, i.e., we will have exactly n solutions $z_1, z_2, \ldots, z_n$. We stop at n solutions because $\frac{1}{n}(\theta + 2n\pi) = \frac{\theta}{n} + 2\pi = \frac{\theta}{n}$, i.e., the solutions will then repeat after $k = n - 1$.

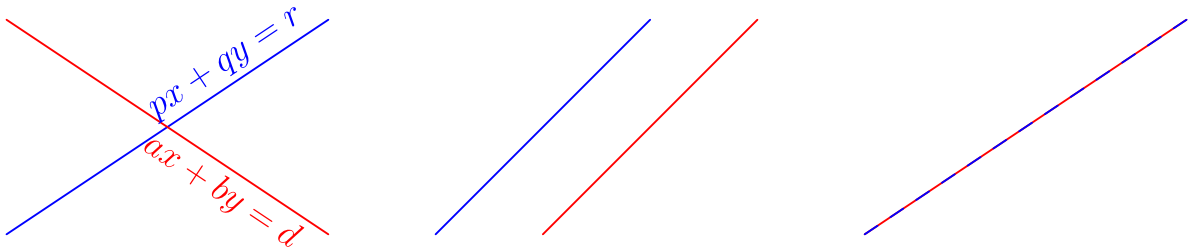
Systems of linear equations



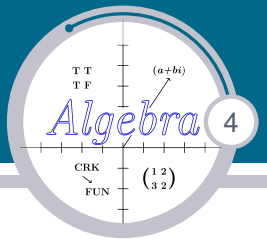
Firstly, what does $5x + y = 7$ describe geometrically?
It describes the line $y = 7 - 5x$.
In general, $ax + by = d$, where $a, b, d \in \mathbb{R}$, describes a straight line in $\mathbb{R}^2$. So when we are solving

$$ax + by = d.$$
$$px + qy = r,$$

we are finding out how the lines interact, i.e, do they intersect (one solution) are they parallel (no solution) or are they the same line (an infinite number of solutions)?



Example of system with 2 linear equations



Solve

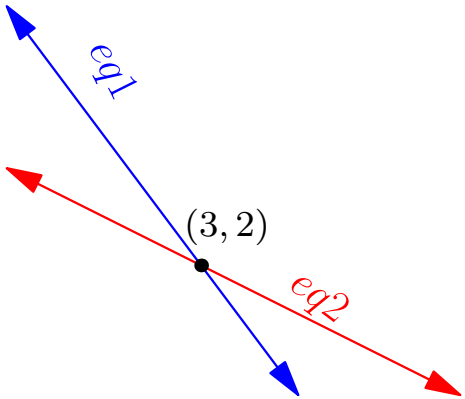
$$\begin{aligned} \text{R1:} \quad & 4x + 3y = 18, \\ \text{R2:} \quad & 2x + 4y = 14, \end{aligned}$$

Example of system with 2 linear equations

Solve

$$\begin{aligned} \text{R1:} \quad & 4x + 3y = 18, \\ \text{R2:} \quad & 2x + 4y = 14, \end{aligned}$$

$$\begin{aligned} \text{R1:} \quad & 4x + 3y = 18, \\ \text{R2:} \to 2\text{R2} \quad & 4x + 8y = 28, \end{aligned}$$

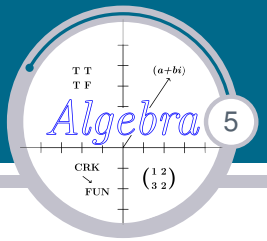


$$\begin{aligned} \text{R1:} \quad & 4x + 3y = 18, \\ \text{R2:} \to \text{R2-R1} \quad & 5y = 10, \end{aligned}$$

$$\begin{aligned} \text{R1:} \quad & 4x + 3y = 18, \\ \text{R2:} \to \text{R2/5} \quad & y = 2, \end{aligned}$$

and $4x + 3(2) = 18$, so $4x = 12$, and $x = 3$, the lines intersect at $(3, 2)$.

Exercise



Solve

$$\begin{aligned} \text{R1:} \quad & 3x - 5y = 29, \\ \text{R2:} \quad & 4x + 2y = 4, \end{aligned}$$

Exercise

Solve

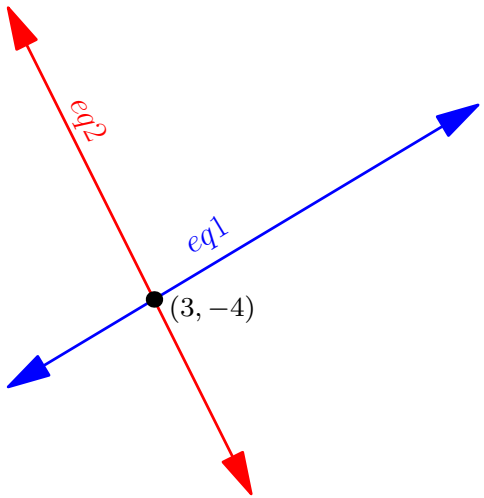
$$
\begin{aligned}
&\text{R1:} && 3x - 5y &&= 29,\\
&\text{R2:} && 4x + 2y &&= 4,
\end{aligned}
$$

$$
\begin{aligned}
&\text{R1:} && 3x - 5y &&= 29,\\
&\text{R2:} \to 3\text{R2} && 12x + 6y &&= 12,
\end{aligned}
$$

$$
\begin{aligned}
&\text{R1:} \to 4\text{R1} && 12x - 20y &&= 116,\\
&\text{R2:} && 12x + 6y &&= 12,
\end{aligned}
$$

$$
\begin{aligned}
&\text{R1:} && 12x - 20y &&= 116,\\
&\text{R2:} \to \text{R2-R1} && +26y &&= -104,
\end{aligned}
$$

$$
\begin{aligned}
&\text{R1:} && 12x - 20y &&= 116,\\
&\text{R2:} \to \text{R2/26} && y &&= -4,
\end{aligned}
$$



and $3x - 5(-4) = 29$, so $3x = 29 - 20$, $3x = 9$, and $x = 3$. So the lines intersect at $(3, -4)$.

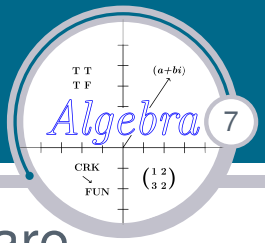
Example of a system of 3 equations and 3 variables

A factory requires energy, steel and labour to manufacture 3 machines, A, B, C. What production figures ensure all resources are used up?

Resource	units	A	B	C	weekly amount available
Energy	Mwh	2	3	2	100
Steel	tonnes	1	1	4	70
Labour	hrs	20	10	10	500

Let's suppose we make x units of machine A, y units of machine B, and z units of machine C.

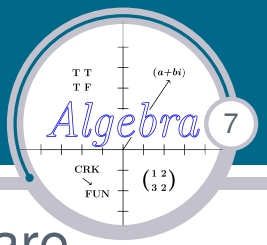
Example of a system of 3 equations (cont.)



If all resources are used up, then our system of linear equations are

$$\begin{aligned} \text{R1:} \quad & 2x + 3y + 2z = 100. \\ \text{R2:} \quad & x + y + 4z = 70. \\ \text{R3:} \quad & 20x + 10y + 10z = 500. \end{aligned}$$

Example of a system of 3 equations (cont.)



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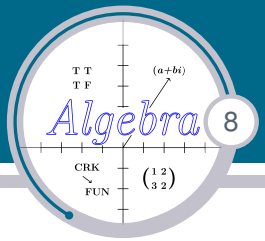
Let's manipulate the equations to find values for $x, y, \& z$.

$$\begin{aligned} \text{R1:} \quad & 2x + 3y + 2z = 100. \\ \text{R2:} \quad & x + y + 4z = 70. \\ \text{R3: R3/10} \quad & 2x + y + z = 50. \end{aligned}$$

$$\begin{aligned} \text{R1:} \quad & 2x + 3y + 2z = 100. \\ \text{R2:} \quad & x + y + 4z = 70. \\ \text{R3: R3-R1} \quad & 0 - 2y - z = -50. \end{aligned}$$

Solution (cont.)



$$\begin{aligned} \text{R1:} \to \text{R2:} \quad & x + y + 4z = 70. \\ \text{R2:} \to \text{R1:} \quad & 2x + 3y + 2z = 100. \\ \text{R3:} \quad & 0 - 2y - z = -50. \end{aligned}$$

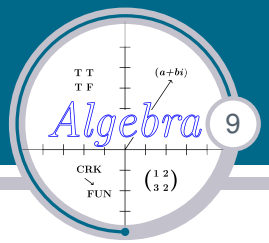
$$\begin{aligned} \text{R1:} \quad & x + y + 4z = 70. \\ \text{R2:} \to \text{R2-2R1} \quad & 0 + y - 6z = -40. \\ \text{R3:} \quad & 0 - 2y - z = -50. \end{aligned}$$

$$\begin{aligned} \text{R1:} \quad & x + y + 4z = 70. \\ \text{R2:} \quad & 0 + y - 6z = -40. \\ \text{R3:} \to \text{R3+2R2} \quad & 0 + 0 - 13z = -130. \end{aligned}$$

So, $z = 10$. Then, using R2: $y - 6(10) = -40$ so $y = 20$. Finally, using R1: $x + 20 + 4(10) = 70$, $x = 10$.

This process is known as Gaussian elimination or row operations.

Exercise



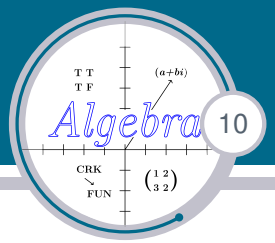
A brewery produces Brown Ale, Dark Brown Ale and Porter. The following table summarizes the amount of malt, hops and yeast used to produce one keg of beer together with the total amount of these resources available per day.

Resources	Quantities	Brown Ale	Dark Brown Ale	Porter	Available
Yeast	kg	3	2	6	250
Hops	kg	4	3	4	220
Malt	kg	4	2	3	170

(a) Let x, y, z be the number of kegs of Ale, Brown Ale, and Porter produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.

(b) Find the values of x, y, z which ensure that the daily supply of hops, malt and yeast are fully used.

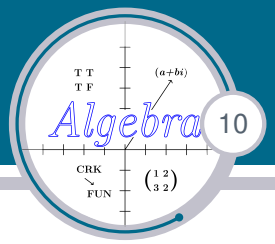
Solution



The system of linear equations, which hold precisely when all three resources are fully used, is

$$\begin{aligned} \text{R1:} \quad & 3x + 2y + 6z = 250. \\ \text{R2:} \quad & 4x + 3y + 4z = 220. \\ \text{R3:} \quad & 4x + 2y + 3z = 170. \end{aligned}$$

Solution



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Find the values of x, y, z which ensure that the daily supply of hops, malt and yeast are fully used.

$$\begin{aligned} & & \text{R1:} \quad & 3x + 2y + 6z = 250. \\ & & \text{R2:} \quad & 4x + 3y + 4z = 220. \\ \text{R3:} \quad & \to \ \text{R3-R2} \quad & & 0 - y - z = -50. \end{aligned}$$

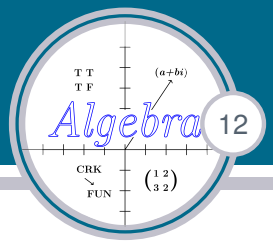
Solution (cont.)

$$\begin{aligned} \text{R1:} &\to 4\text{R1} & 12x + 8y + 24z &= 1000. \\ &\quad \text{R2:} & 4x + 3y + 4z &= 220. \\ &\quad \text{R3:} & 0 - y - z &= -50. \end{aligned}$$

$$\begin{aligned} &\quad \text{R1:} & 12x + 8y + 24z &= 1000. \\ \text{R2:} &\to 3\text{R2} & 12x + 9y + 12z &= 660. \\ &\quad \text{R3:} & 0 - y - z &= -50. \end{aligned}$$

$$\begin{aligned} &\quad \text{R1:} & 12x + 8y + 24z &= 1000. \\ \text{R2:} &\to \text{R2-R1} & 0 + y - 12z &= -340. \\ &\quad \text{R3:} & 0 - y - z &= -50. \end{aligned}$$

Solution (cont.)



$$\begin{aligned} \text{R1:} \quad & 12x + 8y + 24z = 1000. \\ \text{R2:} \quad & 0 + y - 12z = -340. \\ \text{R3:} \ \to \text{R3+R2} \quad & 0 - 0 - 13z = -390. \end{aligned}$$

So, $z = 30$. Using R2:, $y - 12(30) = -340$ so $y = 20$.

Finally, using R1: $12x + 8(20) + 24(30) = 1000$, then $12x = 1000 - 160 - 720 = 120$ so $x = 10$.