Semester II 2026

MA135/MA160 - Algebra II



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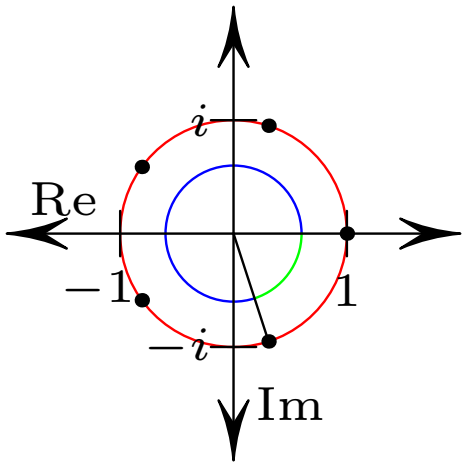
Lecture 15: Complex numbers

Factorizing $p(x) = x^5 - 1$ (recap)

The 5 fifth roots of unity are (details on yesterday's slides)

$$z_1 = \cos 0 + i \sin 0 = 1,$$
$$z_2 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5},$$
$$z_3 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5},$$
$$z_4 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}, \text{ and}$$
$$z_5 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}.$$



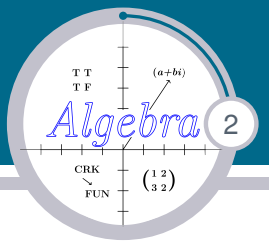
(i.e., we move 1/5 of the way around the circle to get each root)

$x^5 - 1 = (x - z_1)(x - z_2)(x - z_3)(x - z_4)(x - z_5)$

$x^5 - 1 = (x - z_1)((x - z_2)(x - \overline{z}_2))((x - z_3)(x - \overline{z}_3))$

since $z_5 = \overline{z}_2$ and $z_4 = \overline{z}_3$.

Factorizing $p(x) = x^5 - 1$ (recap)



$$z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \quad \text{so } z_3 + \overline{z_3} = 2\cos\frac{4\pi}{5}, \quad (2x)$$

$$\text{and } z_3\overline{z_3} = \left(\cos\frac{4\pi}{5}\right)^2 + \left(\sin\frac{4\pi}{5}\right)^2 = 1, \quad (x^2 + y^2)$$

$$(x - z_3)(x - \overline{z}_3) = (x^2 - (z_3 + \overline{z}_3)x + z_3\overline{z}_3)$$

Similarly

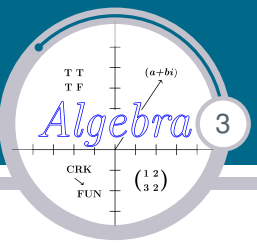
$$z_2 + \overline{z_2} = 2\cos\frac{2\pi}{5},$$

$$\text{and } z_2\overline{z_2} = \left(\cos\frac{2\pi}{5}\right)^2 + \left(\sin\frac{2\pi}{5}\right)^2 = 1.$$

Real roots of

$$x^5 - 1 = (x - 1)\left(x^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5} + 1\right).$$

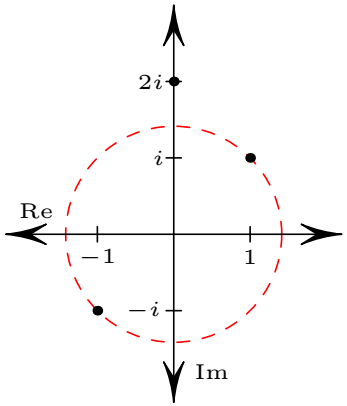
Example



Find the square roots of $2i$, i.e., solve $z^2 = 2i$.

$$|z^2| = |2i| = 2,$$

$$\text{so } |z|^2 = 2 \text{ and } |z| = \sqrt{2}.$$



$$\text{Arg}(z^2) = \text{Arg}(2i) = \frac{\pi}{2} \text{ or } 2\pi + \frac{\pi}{2} \text{ or } 4\pi + \frac{\pi}{2} \text{ or } \ldots.$$

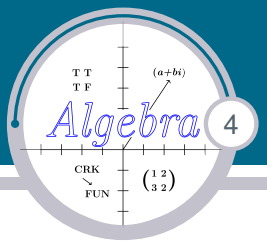
We usually write this as $\text{Arg}(z^2) = \dfrac{\pi}{2} + 2k\pi$ for $k = 0, 1, \ldots$.

$\text{Arg}(z^2) = 2\text{Arg}(z)$ so $\text{Arg}(z) = \dfrac{\pi}{4} + k\pi$ for $k = 0, 1, 2, \ldots$

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

Note: z_1, z_2 is not a conjugate pair.

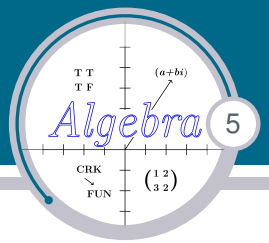
Steps to find roots of $z^n = w$



General approach to find the roots of $z^n = w$, for $n \in \mathbb{N}$ and $w \in \mathbb{C}$.

1. State w in its polar form, i.e., in terms of its modulus r and its argument θ.
2. Recognize that we can increase θ by 2π by adding it repeatedly, i.e., $\theta + 2k\pi$ for $k = 1, 2, 3, \ldots$.
   (Note: $r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$ for $k \in \mathbb{Z}$.
3. z will have the module $\sqrt[n]{r}$ and the argument $\frac{1}{n}(\theta + 2k\pi)$ for $k = 0, 1, \ldots, n-1$, i.e., we will have exactly n solutions $z_1, z_2, \ldots, z_n$. We stop at n solutions because $\frac{1}{n}(\theta + 2n\pi) = \frac{\theta}{n} + 2\pi = \frac{\theta}{n}$, i.e., the solutions will then repeat after $k = n - 1$.

Let's check it



Let's check this for the examples we have looked at. For $z^5 = 1$, we have $n = 5$, $r = 1$ and $\theta = 0$. Using this approach we set

- $w = 1$, $|w| = 1$ and $|z| = \sqrt[5]{1} = 1$.
- $\text{Arg}(w) = 0 + 2k\pi$ for $k = 0, 1, 2, 3, 4$.
- $\text{Arg}(z) = \frac{1}{5}(0 + 2k\pi)$ for $k = 0, 1, 2, 3, 4$, and
- $z_{k+1} = (\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5})$ for $k = 0, 1, 2, 3, 4$.

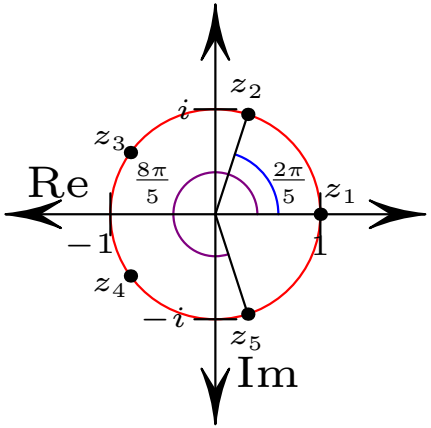
Therefore, we can easily solve $z^6 = 1$, where $n = 6$, $r = 1$ and $\theta = 0$, i.e.

$$z_{k+1} = \left(\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}\right) \text{ for } k = 0, 1, 2, 3, 4, 5.$$

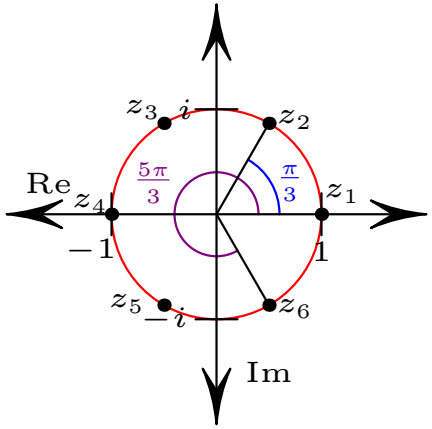
and similarly for $z^7 = 1$

$$z_{k+1} = \left(\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\right) \text{ for } k = 0, 1, 2, 3, 4, 5, 6.$$

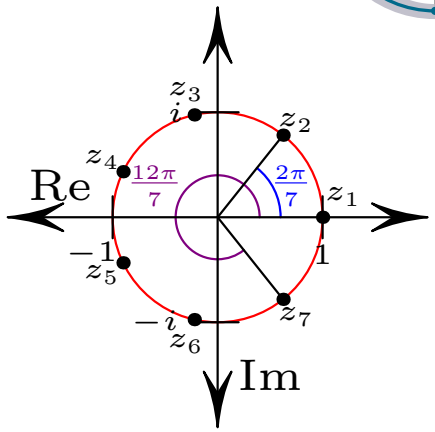
Conjugate pairs



$z^5 = 1$

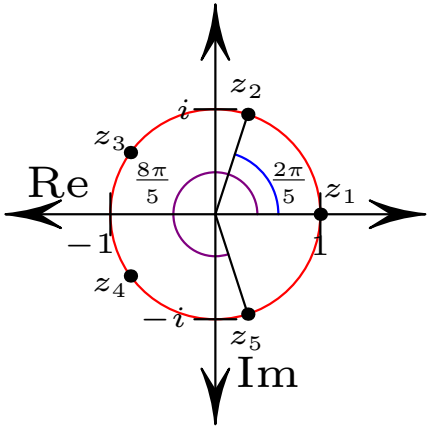


$z^6 = 1$

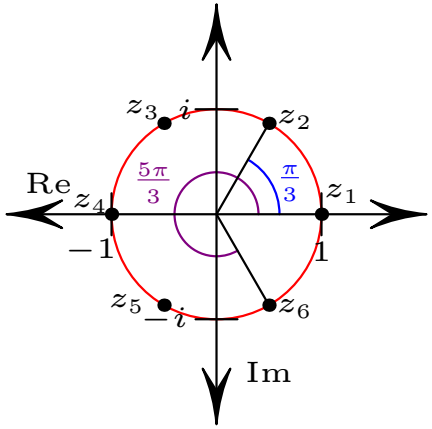


$z^7 = 1$

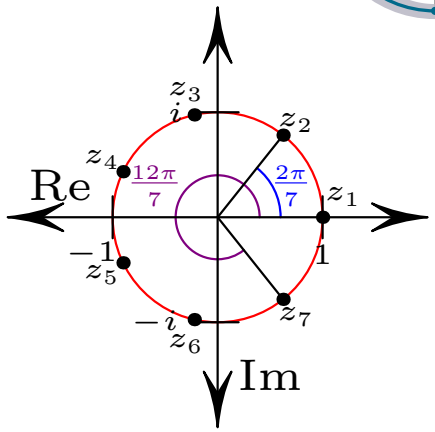
Conjugate pairs



$z^5 = 1$



$z^6 = 1$



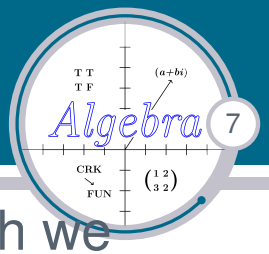
$z^7 = 1$

We can see that all the imaginary roots occur in conjugate pairs in all three graphs.

- For $z^5 = 1$, the pairs are $z_2 z_5$ and $z_3 z_4$, since $\underbrace{z_5 = \overline{z_2}}$ and $\underbrace{z_4 = \overline{z_3}}$.
- For $z^6 = 1$, the pairs are $z_2 z_6$ and $z_3 z_5$, since $\underbrace{z_6 = \overline{z_2}}$ and $\underbrace{z_5 = \overline{z_3}}$.
- For $z^7 = 1$, the pairs are $z_2 z_7$, $z_3 z_6$ and $z_4 z_5$, since $\underbrace{z_7 = \overline{z_2}}$, $\underbrace{z_6 = \overline{z_3}}$ and $\underbrace{z_5 = \overline{z_4}}$.

Note: this is only true for roots of real numbers.

Exercise: check if $w \neq 1$



For $z^2 = 2i$, we have $n = 2$, $r = 2$ and $\theta = \frac{\pi}{2}$. Using this approach we set

- $w = 2i$, $|w| = 2$ and $|z| = \sqrt{2}$.
- $\text{Arg}(w) = \frac{\pi}{2} + 2k\pi$ for $k = 0, 1$.
- $\text{Arg}(z) = \frac{1}{2}(\frac{\pi}{2} + 2k\pi)$ for $k = 0, 1$.

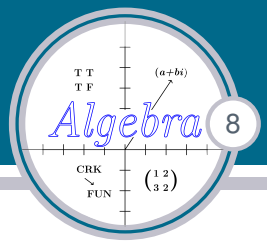
So,

$$z_{k+1} = \sqrt{2}\left(\cos\frac{1}{2}\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\frac{1}{2}\left(\frac{\pi}{2} + 2k\pi\right)\right) \text{ for } k = 0, 1.$$

$$\text{For } k = 0,\ z_1 = \sqrt{2}\left(\cos\frac{1}{2}\left(\frac{\pi}{2}\right) + i\sin\frac{1}{2}\left(\frac{\pi}{2}\right)\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

$$\begin{aligned}\text{For } k = 1,\ z_2 &= \sqrt{2}\left(\cos\frac{1}{2}\left(\frac{\pi}{2} + 2\pi\right) + i\sin\frac{1}{2}\left(\frac{\pi}{2} + 2\pi\right)\right)\\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \pi\right) + i\sin\left(\frac{\pi}{4} + \pi\right)\right)\\ &= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).\end{aligned}$$

Example



Find the square roots of $\dfrac{1+i}{\sqrt{2}}$. Let $z^2 = w = \dfrac{1+i}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$.

$$|w| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1 \text{ so } |z| = \sqrt{1} = 1$$

$$\tan^{-1}\left(\left|\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right|\right) = \tan^{-1} 1 = \frac{\pi}{4}.$$

$$\text{Arg}(w) = \frac{\pi}{4} + 2k\pi, \text{ for } k = 0, 1.$$

$$\text{Arg}(z) = \frac{1}{2}\left(\frac{\pi}{4} + 2k\pi\right) = \frac{\pi}{8} + k\pi, \text{ for } k = 0, 1.$$

Our two roots are $z_{k+1} = \cos\left(\frac{\pi}{8} + k\pi\right) + i\sin\left(\frac{\pi}{8} + k\pi\right)$ for $k = 0, 1$.

$$z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} \quad z_2 = \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}.$$

Note: z_1 and z_2 are not a conjugate pair.