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MA135/MA160 - Algebra II



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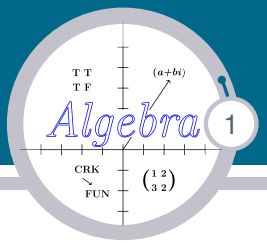
Lecture 14: Complex numbers

Recap



Remember, for any complex numbers z and w that

$$\overline{zw} = (\overline{z})\,(\overline{w})$$
$$|zw| = |z|\,|w|$$
$$\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w)$$

where for any $z = x + yi$, $\overline{z} = x - yi$ is the complex conjugate. Note that

$$z + \overline{z} = (x + yi) + (x - yi) = 2x \in \mathbb{R} \quad \text{and}$$
$$z\overline{z} = |z| = x^2 + y^2 \in \mathbb{R}_{\geq 0}$$

The sum of a complex number and its conjugate is always a real number.

The product of a complex number and its conjugate is always a non-negative real number, so,

$$\mathrm{Arg}(z\overline{z}) = \mathrm{Arg}(z) + \mathrm{Arg}(\overline{z}) = 0, \quad \text{and}$$
$$\mathrm{Arg}(z) = -\mathrm{Arg}(\overline{z}).$$

Example

For example, if $z = -3 - 2i$ then $\overline{z} = -3 + 2i$.

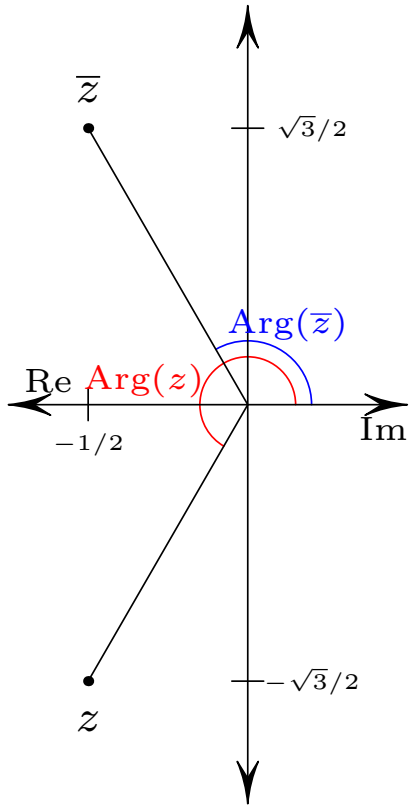
$$z + \overline{z} = (-3 - 2i) + (-3 + 2i) = -6$$

$$z\overline{z} = (-3)^2 + (-2)^2 = 9 + 4 = 13$$

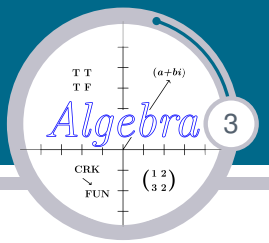
$$\text{Arg}(z) = \pi + \tan^{-1}\left(\left|\frac{-2}{-3}\right|\right) \ (3^{rd} \text{ quadrant})$$

$$\text{Arg}(\overline{z}) = \pi - \tan^{-1}\left(\left|\frac{-2}{-3}\right|\right) \ (2^{nd} \text{ quadrant})$$

$$\text{Arg}(z\overline{z}) = \pi + \tan^{-1}\left(\left|\frac{-2}{-3}\right|\right) + \pi - \tan^{-1}\left(\left|\frac{-2}{-3}\right|\right) = 2\pi = 0$$



The fundamental theorem of algebra



The fundamental theorem of algebra

Let $f(X) \in \mathbb{C}[X]$ be a polynomial of degree n. Than

$$f(X) = \prod_{i=0}^{n-1}(X - x_i) = (X - x_0)(X - x_1) \cdot \cdots \cdot (X - x_{n-1}).$$

Note that the roots x_i need not be distinct.

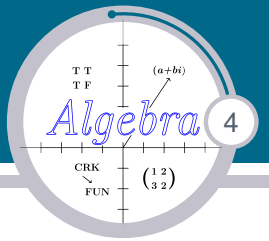
To put it in words, any function over the complex numbers splits in linear factors.

The linear factors, of course, need not be distinct. For example,

$$x^2 + 2x + 1 = (x + 1)(x + 1)$$

has a so called *double root*.

How far can we factorise over $\mathbb{R}$?



The fundamental theorem of algebra tells us that we can factorise any polynomial totally (down to linear factors) over the complex numbers.

What about the real numbers?

Factorising over $\mathbb{R}$

Any polynomial $f(x) \in \mathbb{R}[X]$ with real coefficients can be written as a product of linear and quadratic terms.

This is because the coefficients of $f(x)$ are real, and hence, the complex roots MUST appear as conjugate pairs.

That we can not factorise further is clear: E.g., $x^2 + 1 = 0$ has clearly no real-valued roots / can not be solved over $\mathbb{R}$ / can not be written as a product of linear factors with real coefficients.

The surprising thing, however, is that we can factorise everything down to quadrics.

But how do we do this?

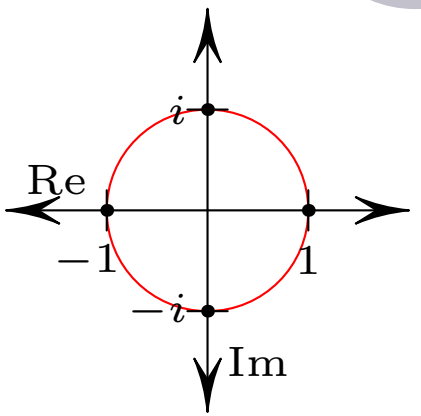
We will study polynomials of the form

$$x^n = r.$$

Factorizing polynomials

Find all complex numbers z such that $z^4 = 1$. Then factorize the polynomial $f(x) = x^4 - 1$ (find its zeros), as a

1. product of its linear (complex) factors,
2. product of its real linear/quadratic factors.



We know that

$$\begin{aligned} 1^4 &= 1 \\ (-1)^4 &= 1 \\ (i^2)^2 &= 1 \\ (-i^2)^2 &= 1 \end{aligned}$$

which implies that $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $z_4 = -i$ are all roots of $x^4 = 1$. As the polynomial is of degree 4 and we have found all 4 roots, it can have no other roots. It follows that

$$x^4 - 1 = (x-1)(x+1)(x-i)(x+i).$$

We have thus solved 1., that is, factorised $f(x)$ over $\mathbb{C}$ as

$$x^4 - 1 = (x-1)(x+1)(x-i)(x+i).$$

To solve solve it over $\mathbb{R}$, we need to:

1. Identify the complex factors;
2. Group them in conjugate pairs;
3. Multiply out the (conjugate) paired factors.

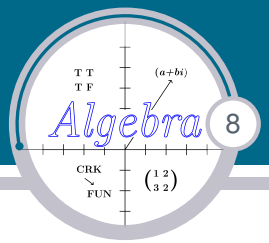
In our case:

1. The complex roots are $(x-i), (x+i)$;
2. The conjugate pairs are $(x-i)(x+i)$;
3. Their product is $x^2 + 1$.

This gives us the following factorisation over $\mathbb{R}$

$$x^4 - 1 = (x-1)(x+1)(x^2+1).$$

Factoring $x^5 = 1$



Find all complex numbers z such that $z^5 = 1$. Then (fully) factorize the polynomial $p(x) = x^5 - 1$

1. over $\mathbb{C}$ (that is, as a product of its linear factors),
2. over $\mathbb{R}$ (as a product of its real linear/quadratic factors).

Since $\left|z^5\right| = 1 = |z|^5$, $|z| = \sqrt[5]{1} = 1$ and z lies on $S1$, the unit circle (radius $r = 1$) with centre (0,0).

But what points on the circle satisfies $z^5 = 1$?

As

$$\text{Arg}(z^5) = 5\text{Arg}(z) = \text{Arg}(1) = 2n\pi \quad \text{for any } n \in \mathbb{Z},$$

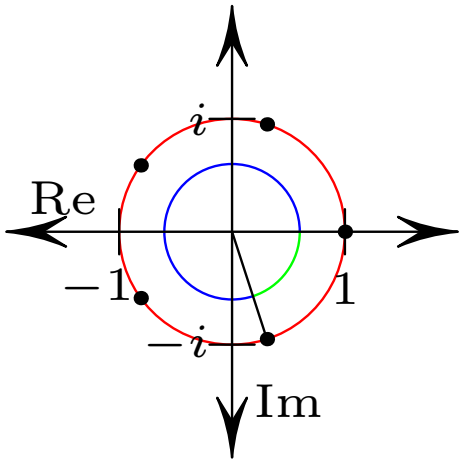
we have $\text{Arg}(z) = \dfrac{2\pi}{5}$ for any $n \in \mathbb{Z}$.

Written out, we have $5\text{Arg}(z) = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$ and hence $\text{Arg}(z) = \ldots, -\dfrac{4\pi}{5}, -\dfrac{2\pi}{5}, \dfrac{0,2}{5}\pi, \dfrac{4\pi}{5}, \ldots$.

Factorizing $p(x) = x^5 - 1$ (cont.)

So the 5 fifth roots of unity (solutions to $z^5 = 1$) are

$$z_1 = \cos 0 + i \sin 0 = 1,$$
$$z_2 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5},$$
$$z_3 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5},$$
$$z_4 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}, \text{ and}$$
$$z_5 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}.$$

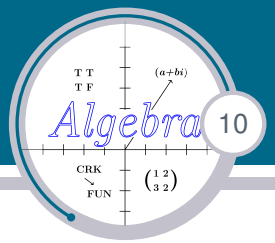


Note that, for example, $\frac{8\pi}{5} = 2\pi - \frac{2\pi}{5} = -\frac{2\pi}{5}$. So,

$$z_5 = \cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) = \cos\left(\frac{2\pi}{5}\right) - i \sin\left(\frac{2\pi}{5}\right) = \overline{z_2}$$

Similarly $z_4 = \overline{z_3}$. So, the 4 non-real roots occur in conjugate pairs.

Factorizing $z^5 = 1$ (cont.)



The 5 fifth roots of unity are z_1, z_2, z_3, z_4 and z_5, that is:

$$x^5 - 1 = (x - z_1)(x - z_2)(x - z_3)(x - z_4)(x - z_5).$$

Since we observed that $z_5 = \overline{z_2}$ and $z_4 = \overline{z_3}$, we group them together equation to obtain

$$\begin{aligned} x^5 - 1 &= (x - z_1)((x - z_2)(x - z_5))((x - z_3)(x - z_4)) \\ &= (x - z_1)((x - z_2)(x - \overline{z_2}))((x - z_3)(x - \overline{z_3})). \end{aligned}$$

We claim that this is a real-valued factorisation in irreducible real parts (1 linear, 2 quadratic). Observe that

$$\begin{aligned} (x - z_2)(x - \overline{z_2}) = x(x - \overline{z_2}) - z_2(x - \overline{z_2}) &= x^2 - x\overline{z_2} - xz_2 + z_2\overline{z_2} \\ &= x^2 - (z_2 + \overline{z_2})x + \overline{z_2}^2 = x^2 - (z_2 + \overline{z_2})x + z_2\overline{z_2}. \end{aligned}$$

As mentioned beofore, the sum and product of complex conjugates are always real! Hence, this will indeed be a real-valued quadratic polynomial. (It is irreducible since its two linear factors are both complex.)
Likewise for the pair z_3, z_4.

So the real roots of $p(x) = x^5 - 1$ are

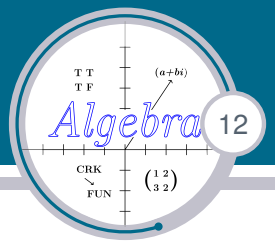
$$x^5 - 1 = (x - z_1)(x^2 - (z_2 + \overline{z_2})x + z_2\overline{z_2})(x^2 - (z_3 + \overline{z_3})x + z_3\overline{z_3}),$$

since, we know that both $(z + \overline{z})$ and $z\overline{z}$ are real numbers.

Recall the following identities:

- For $z = a + bi$, we have
  - $z + \overline{z} = 2a$,
  - $z\overline{z} = a^2 + b^2$
- for any α, we have

$$(\cos\alpha)^2 + (\sin\alpha)^2 = 1.$$



$$z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$$

and hence

$$z_2\overline{z_2} = \left(\cos\frac{2\pi}{5}\right)^2 + \left(\sin\frac{2\pi}{5}\right)^2 = 1.$$

Similarly, $z_3\overline{z_3} = 1$.

We can get there simpler by recalling that $z\overline{z} = |z|$ and we know that for all z_1, z_2, z_3, z_4, z_5, the norm is 1 (they are on the unit circle S_1).

We can also get there if we argue geometrically, since mutiplication multiplies the norm/modulu and adds the angles/Arguments.

We can also use the the polar coordinates/Eurler identity and see that

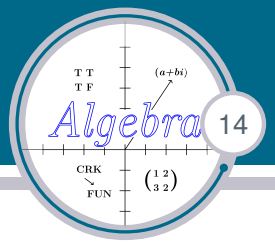
$$z\overline{z} = e^{\frac{2\pi i}{5}} e^{\frac{8\pi i}{5}} = e^{\frac{(2+8)\pi i}{5}} = e^{2i\pi} = 1$$

Factorize $p(x) = x^5 - 1$ (cont.)

Hence, the factorisaion of $x^5 - 1$ in irreducible components over $\mathbb{R}$ is

$$\begin{aligned} x^5 - 1 &= (x - z_1)(x^2 - (z_2 + \overline{z_2})x + z_1\overline{z_2})(x^2 - (z_3 + \overline{z_3})x + z_3\overline{z_3}) \\ &= (x - z_1)\left(x^2 - 2\cos\frac{2\pi}{5}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5}x + 1\right) \end{aligned}$$

Summary of how to find "real" roots of unity



Steps to find the real roots of unity for $z^n = 1$. We know

1. $|z^n| = 1$ so $|z| = 1$ (i.e., lie on the circle with centre $(0, 0)$ and radius $r = 1$),
2. n Arg $(1) = 0$ or $2\pi \ldots 2n\pi \ldots$, so
3. $Arg(z) = 0, \frac{2\pi}{n}$ or $\ldots or \frac{2n\pi}{n} \ldots$,
4. Find linear roots: $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots, \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}, 1,$
5. Identify the conjugate pairs (i.e., where their angles sum to 2π)
6. Multiply the roots that are conjugate pairs to get real roots.

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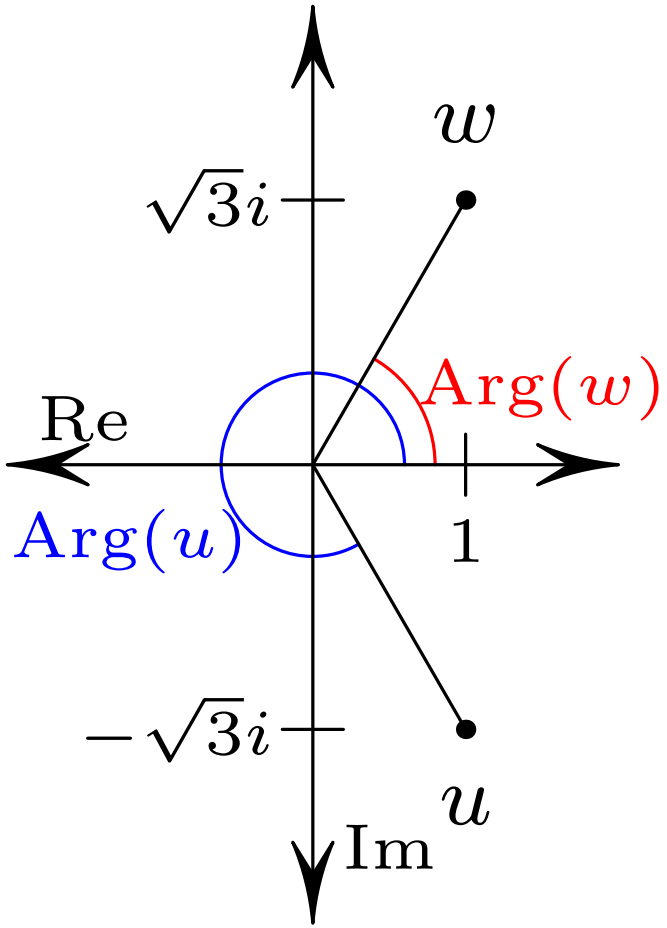
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