Semester II 2026

MA135/MA160 - Algebra II



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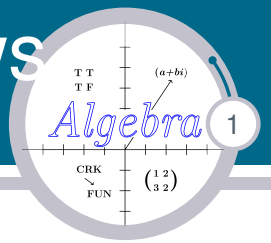
Lecture 13: Complex numbers

Modulus and Argument: Product/Division laws



Recall that for $z, w \in \mathbb{C}$ complex numbers, we have

$$|zw| = |z||w|, \qquad Arg(zw) = Arg(z) + Arg(w). \tag{1}$$

Let $z^{-1} := \frac{1}{z}$. (Note that $z^{-1} = \frac{1}{z}$ is just notation). By definition, we have

$$z \cdot z^{-1} = 1$$

which immediately implies

$$|z \cdot z^{-1}| = |1|, \qquad Arg(z \cdot z^{-1}) = Arg(1).$$

(If two things are equal, all their properties are equal, too.)
Clearly, $\boxed{|1| = 1}$ and $\boxed{\text{Arg}(1) = 0}$ (for the second, observe that $1 = 1 + 0i$ or $1 = 1 \cdot e^{0 \cdot i}$ is an other way of writing 1.

The equations (1) give us

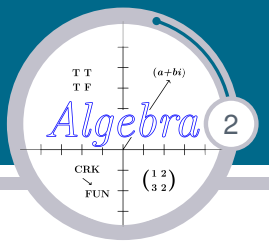
$$|z \cdot z^{-1}| = |z||z^{-1}| = 1$$

$$|z^{-1}| = \frac{1}{|z|}$$

$$Arg(z \cdot z^{-1}) = Arg(z) + Arg(z^{-1}) = 0$$

$$Arg(z^{-1}) = -Arg(z)$$

Modulus and Argument: Division



Clearly, we also have

$$|z^2| = |z||z| = |z|^2$$
$$|z^3| = |z^2 z| = |z^2||z| = |z|^2|z| = |z|^3$$
$$\dots$$

$$|z^n| = |z|^n.$$

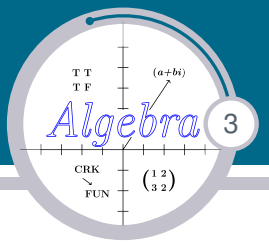
and

$$Arg(z^2) = Arg(z) + Arg(z) = 2Arg(z)$$
$$Arg(z^3) = Arg(z^2) + Arg(z) = 2Arg(z) + Arg(z) = 3Arg(z)$$
$$\dots$$

$$Arg(z^n) = nArg(z)$$

By our argument about z^{-1}, this is also true for negative n!

Modulus and Argument when taking powers



To sum up our discussion:

De Moivre's theorem

For any $n \in \mathbb{Z}$ (any integer, positive or negative), we have

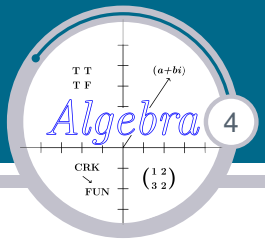
$$|z^n| = |z|^n, \qquad Arg(z^n) = n \cdot Arg(z)$$

In particular, what happens if $|z| = 1$?

We have $|z^n| = |z|^n = 1^n = 1$!

Why is this important?

The unit circle $S1$



For any real number $r \in \mathbb{R}$, the collection (set) of all complex numbers z with $|z| = r$ is clearly a circle with center $(0, 0)$ and radius r. (This is the definition of a circle: All points are equally far from the origin).

If we consider two points z, w with norm/modulus 2, than zw has norm/modulus 4!
That is to say, if two points on the complex plane lie on the circle with origin $(0, 0)$ and radius 2, their product is on the circle with origin $(0, 0)$ but radius 4.

However, if two points z, w lie on the circle with origin $(0, 0)$ and radius 1, their product lies on the same circle.

This circle is denoted by S_1 and is called *the unit circle*.

Example

Evaluate

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}.$$

Approach 1: First, let

$$z = \frac{1+\sqrt{3}i}{1-\sqrt{3}i},$$

and convert z into the form $z = a + bi$.

$$z = \frac{(1+\sqrt{3}i)}{(1-\sqrt{3}i)}\frac{(1+\sqrt{3}i)}{(1+\sqrt{3}i)} = \frac{(1+\sqrt{3}i)(1+\sqrt{3}i)}{(1^2+\sqrt{3}^2)}$$

$$= \frac{1(1+\sqrt{3}i)+\sqrt{3}i(1+\sqrt{3}i)}{4} = \frac{1+\sqrt{3}i+\sqrt{3}i+3i^2}{4}$$

$$= \frac{1+2\sqrt{3}i-3}{4} = \frac{-2+2\sqrt{3}i}{4} = \frac{-1+\sqrt{3}i}{2} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

Example, approach 1 (Arg)

Evaluate

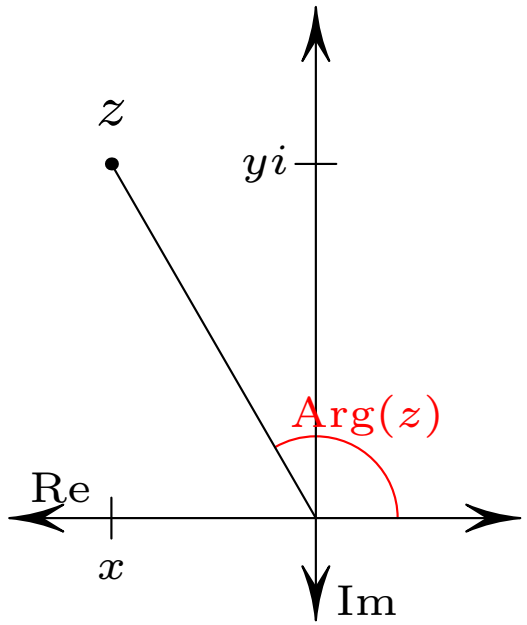
$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}.$$

$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is in quadrant 2 so,

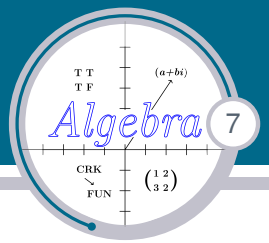
$$\begin{aligned}\text{Arg}(z) &= \pi - \tan^{-1}\left(\left|\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right|\right)\\ &= \pi - \tan^{-1}\left(\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}\end{aligned}$$

$$\begin{aligned}\text{Arg}(z^{10}) &= 10\text{Arg}(z) = 10\left(\frac{2\pi}{3}\right)\\ &= \frac{20\pi}{3} = 6\pi + \frac{2}{3}\pi = \frac{2}{3}\pi\end{aligned}$$

since $2\pi \equiv 0$, and subsequently $6\pi + \frac{2}{3}\pi \equiv \frac{2}{3}\pi$.



Example, approach 1 (modulus)



Since $|z^{10}| = |z|^{10}$, we only need calculate $|z|$ and take it to power 10. We have

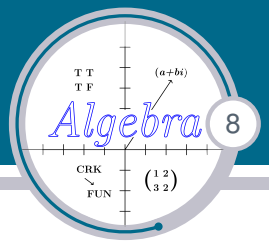
$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{(1+3)}{4}} = \sqrt{1} = 1.$$

Hence, $|z^{10}| = 1^{10} = 1$. It follows that

$$z^{10} = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Or simpler, since both $|z|^{10} = |z|$ and $Arg(z^{10}) = Arg(z)$, it must be that $z^{10} = z$.

Example: approach 2 (w)



Let $w = 1 + \sqrt{3}i$ and $u = 1 - \sqrt{3}i$ and $z = \dfrac{w}{u}$. We have

$$z^{10} = \left(\frac{w}{u}\right)^{10} = \left(\frac{w}{u}\right)^{10} = w^{10}u^{-10}.$$

$$|w| = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = 2$$

$\Rightarrow$

$$|w^{10}| = |w|^{10} = 2^{10} = 1024$$

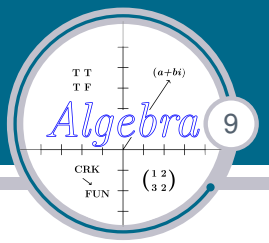
and

$$\text{Arg}(w) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$\Rightarrow$

$$\text{Arg}(w^{10}) = 10\text{Arg}(w) = \frac{10\pi}{3} = \frac{6\pi}{3} + \frac{4\pi}{3} = 2\pi + \frac{4\pi}{3} = \frac{4\pi}{3}$$

Example: approach 2 (u)



$$|u| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = 2$$

hence

$$|u|^{-10} = 2^{-10} \left(\text{i.e., } \frac{1}{2^{10}} = \frac{1}{1024}\right).$$

We also have

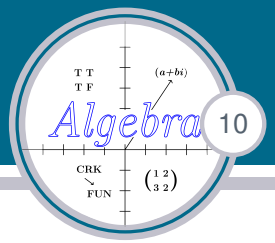
$$Arg(u) = 2\pi - \tan^{-1}\left(\left|\frac{\sqrt{3}}{1}\right|\right) = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

and hence

$$Arg(u^{-10}) = -10Arg(u) = -\frac{50\pi}{3} = -\left(\frac{48\pi}{3} + \frac{2\pi}{3}\right) = -16\pi - \frac{2\pi}{3}$$

$$= -\frac{2\pi}{3} = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}.$$

Example: approach 2 (z)



Summing it all up, we have

$$|w^{10}| = 2^{10} = 1024, \quad Arg(w^{10}) = \frac{4\pi}{3},$$

$$|u|^{-10} = 2^{-1}, \quad Arg(u^{-10}) = \frac{4\pi}{3}.$$

This gives us

$$|z^{10}| = |w^{10}||u^{-10}| = 2^{10}2^{-10} = 2^{10-10} = 2^0 = 1 \left(\text{i.e., } \frac{1024}{1024}\right)$$

$$Arg(z^{10}) = \text{Arg}(()w^{10}) + Arg(u^{-10}) = \frac{4\pi}{3} + \frac{4\pi}{3} = \frac{8\pi}{3} = 2\pi + \frac{2\pi}{3}$$

and hence

$$z^{10} = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$