Semester II 2026

MA135/MA160 - Algebra II



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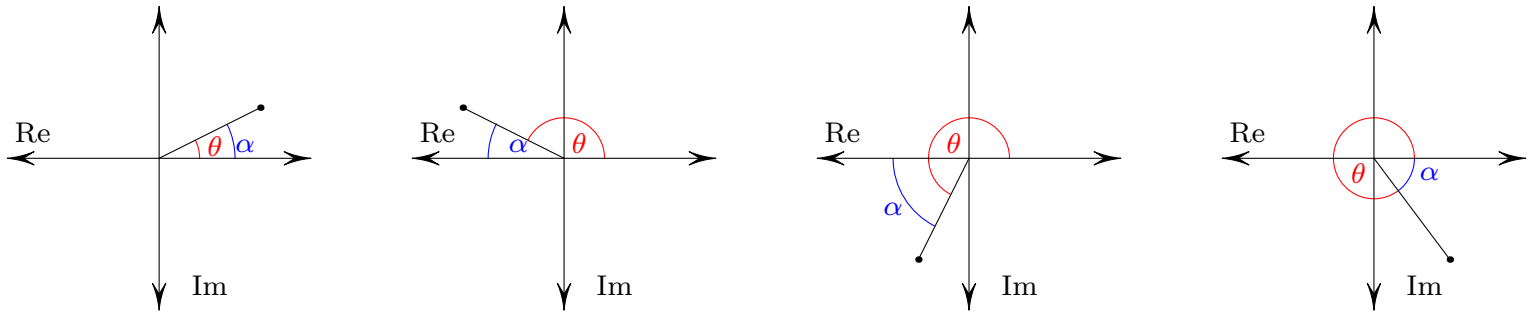
Lecture 12: Complex numbers

Recap: determine Arg(z)

Steps to determine Arg(z) for $z = x + yi$:

1. Locate/plot z on an Argand diagram.
2. Calculate $\alpha = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$.
3. Interpret the angle according to which quadrant z lies in.



Q1: Arg(z) = α. Q2: Arg(z) = $\pi-\alpha$. Q3: Arg(z) = $\pi+\alpha$. Q4: Arg(z) = $2\pi-\alpha$.

Example Arg(z)

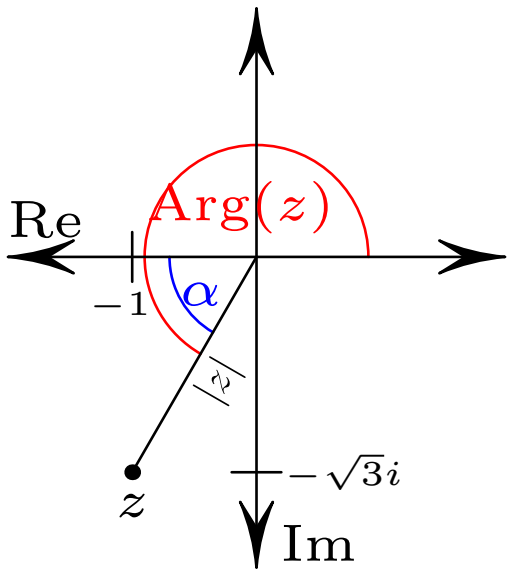
Let $z = -1 - \sqrt{3}i$, find Arg(z)

$$\text{Arg}(z) = \pi + \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \pi + \frac{\pi}{3} = \frac{4\pi}{3}.$$

Note that

$$\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \frac{\pi}{3},$$

which is in the first quadrant, this is why we need to check in which quadrant z lies to determine Arg(z).

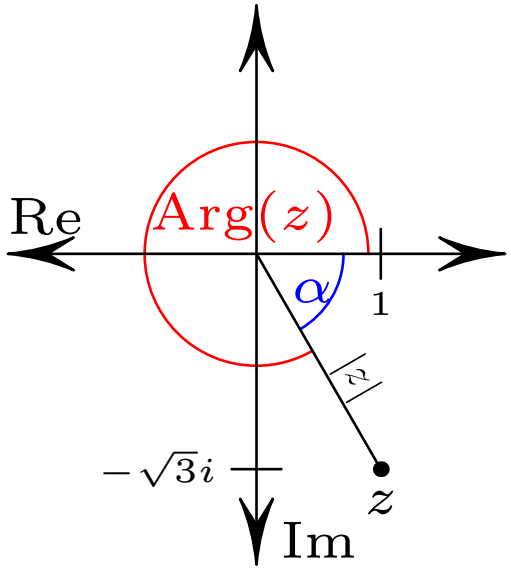


Exercise: Arg(z)

Let $z = 1 - \sqrt{3}i$, find $|z|$ and Arg(z).

$$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$$

$$\text{Arg}(z) = 2\pi - \tan^{-1}\left(\left|\frac{-\sqrt{3}}{1}\right|\right) = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$



Theorem:

Let $\theta = (z)$. Recall that any complex number can be written as

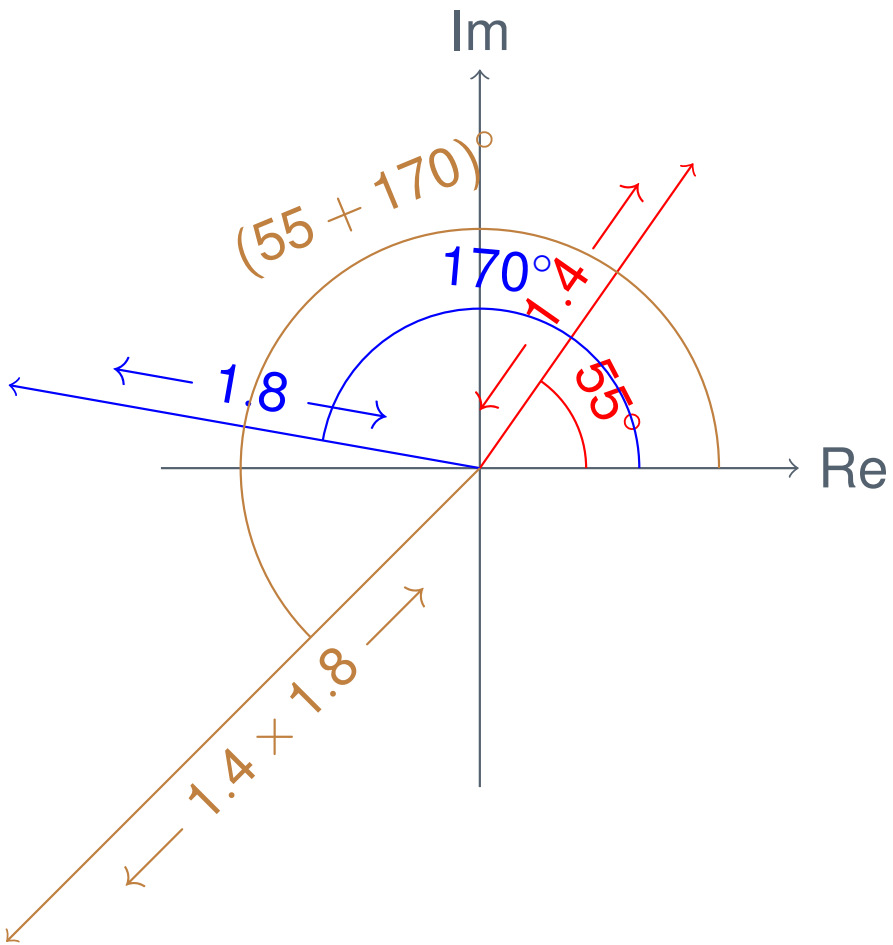
$$z = |z| (\cos\theta + i \sin\theta) .$$

We have:

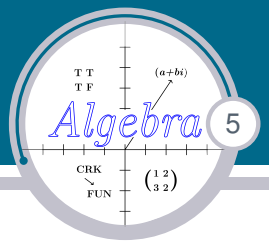
Theorem

For any complex number

$$|zw| = |z||w|$$
$$\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$$



Theorem: (proof)



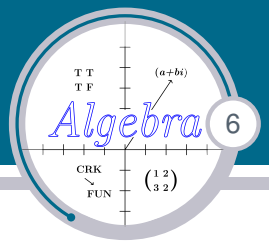
Let $|z| = r$, $\text{Arg}(z) = \theta$, $|w| = s$, and $\text{Arg}(w) = \gamma$.

$$\begin{aligned} zw &= r(\cos\theta + i\sin\theta)s(\cos\gamma + i\sin\gamma) \\ &= sr(\cos\theta + i\sin\theta)(\cos\gamma + i\sin\gamma) \\ &= sr(\cos\theta(\cos\gamma + i\sin\gamma) + i\sin\theta(\cos\gamma + i\sin\gamma)) \\ &= sr(\cos\theta\cos\gamma + i\cos\theta\sin\gamma + i\sin\theta\cos\gamma + i^2\sin\theta\sin\gamma) \\ &= sr(\cos\theta\cos\gamma - \sin\theta\sin\gamma + i(\cos\theta\sin\gamma + \sin\theta\cos\gamma)) \\ &= sr\left(\cos(\theta+\gamma) + i\sin(\theta+\gamma)\right)^{\star}. \end{aligned}$$

i.e., $|zw| = sr = |z||w|$ and $\text{Arg}(zw) = \theta + \gamma = \text{Arg}(z) + \text{Arg}(w)$.

* Compound angle formulae (F&T, pg.14).

Theorem: (alternative "proof")



Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The following is not a proof, since we assume Euler's identity without proving it. The proof of it is outside the bounds of this module.

Eulers identity immediatley gives us our desired result, since for

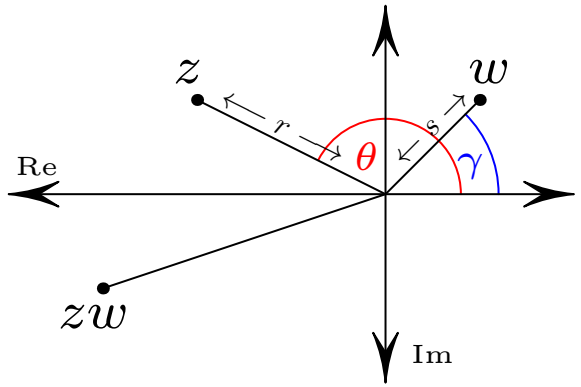
$$z = r(\cos\theta + i\sin\theta), \quad w = s(\cos\gamma + i\sin\gamma)$$

we get

$$\begin{aligned} zw &= re^{i\theta}se^{i\gamma} \\ &= rse^{i(\theta+\gamma)} \end{aligned}$$

as desired, since for any number $x \in R$, we have $x^{a+b} = x^a x^b$.

Modulus and Argument



$\text{Arg}(z) = \theta, |z| = r$.
$\text{Arg}(w) = \gamma, |w| = s$.
Where is zw?
We need $|zw|$ and $\text{Arg}(zw)$.

We can use our theorem $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w) = \theta + \gamma$, and $|zw| = |z||w| = rs$.

In particular, $\text{Arg}(z^2) = \text{Arg}(zz) = \text{Arg}(z) + \text{Arg}(z) = 2\text{Arg}(z)$.
$|z^2| = |zz| = |z||z| = |z|^2 = r^2$.

In general, if $z = r(\cos\theta + i\sin\theta)$, then

$$\text{Arg}(z^2) = 2\theta,$$
$$\text{Arg}(z^3) = \text{Arg}(z^2 z) = \text{Arg}(z^2) + \text{Arg}(z) = 2\theta + \theta = 3\theta,$$

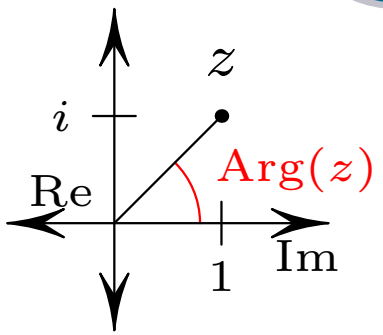
and $\text{Arg}(z^n) = n\theta$ i.e., $\text{Arg}(z^n) = n\text{Arg}(z)$.

Example: modulus and argument



Find $|z|$ and $\text{Arg}(z)$ for

$$z = \frac{3i^{30} - i^{19}}{2i - 1}.$$



Let's simplify the numerator. We know $i^2 = -1$. So $i^{30} = (i^2)^{15} = (-1)^{15} = -1$, and $i^{19} = i^{18}i = (i^2)^9 i = (-1)^9 i = -i$. This gives us

$$\begin{aligned} z &= \frac{3(-1) - (-i)}{-1 + 2i} = \frac{-3 + i}{-1 + 2i} = \frac{(-3 + i)}{(-1 + 2i)} \frac{(-1 - 2i)}{(-1 - 2i)} \\ &= \frac{(-3 + i)(-1 - 2i)}{((-1)^2 + 2^2)} = \frac{-3(-1 - 2i) + i(-1 - 2i)}{1 + 4} \\ &= \frac{3 + 6i - i - 2i^2}{5} = \frac{3 + 5i + 2}{5} = \frac{5 + 5i}{5} = 1 + i. \end{aligned}$$

Hence, $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Since z lies in the 1st quadrant $\text{Arg(z)} = \tan^{-1}(|1/1|) = \pi/4$.

Alternatively, we can use our new method and say:

$$z = \frac{-3+i}{-1+2i},$$

with $w = -3+i$, $\quad v = -1+2i \quad$ and $\quad z = \frac{w}{v}$.
Use

$$|z| = \frac{|w|}{|v|}, \quad \text{Arg}(z) = \text{Arg}(w) - \text{Arg}(v).$$

We have

$$|w| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}, \quad |v| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

$$\text{Arg}(w) = \tan^{-1}\left(\frac{|1|}{|-3|}\right), \quad \text{Arg}(v) = \tan^{-1}\left(\frac{|2|}{|-1|}\right)$$

$$\tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}\left(\tfrac{a-b}{1+ab}\right)$$

to deduce our answer.