Semester II 2026

MA135/MA160 - Algebra II



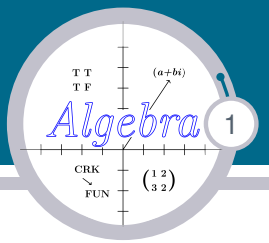
OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Ilia Pirashvili, PhD

With special thanks to Prof. Aisling McCluskey & Dr. Róisín Hill

Lecture 11: Complex numbers

Recap



If $z = a + bi$, then its conjugate is $\overline{z} = a - bi$, so,

$$z + \overline{z} = 2a$$
$$z\overline{z} = a^2 + b^2.$$

We had also seen that $z\overline{z} = |z|^2$, where

$$|z| = \sqrt{a^2 + b^2},$$

Equivalently, $|z| = \sqrt{z\overline{z}}$ and subsequently deduced that for all complex numbers z, w, we have

$$|zw| = |z| \cdot |w|$$

and

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

Recap – Examples

Example

Let $z = 2 + 3i$. Then

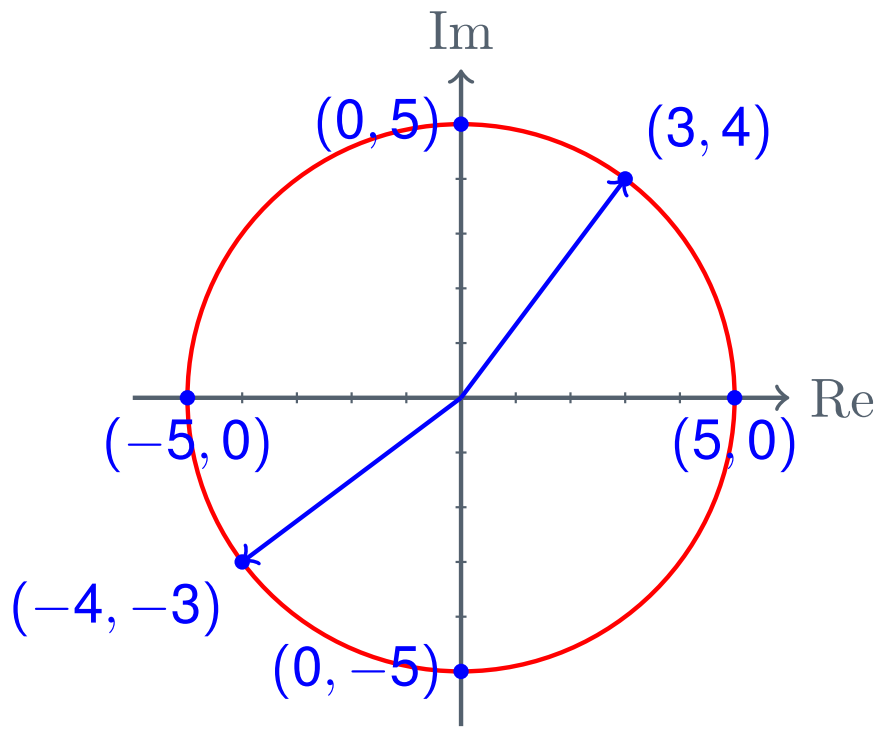
$$\begin{aligned} \overline{z} &= 2 - 3i, \\ z + \overline{z} &= 2(2) = 4, \\ z\overline{z} &= 2^2 + (-3)^2 = 4 + 9 = 13, \\ |z| &= \sqrt{13}. \end{aligned}$$

If $w = 1 + i$, then

$$\begin{aligned} |w| &= \sqrt{2} \\ zw &= -1 + 5i \\ |zw| &= \sqrt{1 + 25} = \sqrt{26} = \sqrt{2} \cdot \sqrt{13} \\ \frac{z}{w} &= \frac{5}{2} + \frac{1}{2}i \\ \left|\frac{z}{w}\right| &= \sqrt{\frac{25}{4} + \frac{1}{4}} = \frac{\sqrt{26}}{\sqrt{2}} = \frac{|z|}{|w|} \end{aligned}$$

The insufficiency of the modulus

Recall that $|z|$ is the length of the straight line from $O = (0, 0)$ to z on the complex plane. The distance, however, does not fully recover the point.

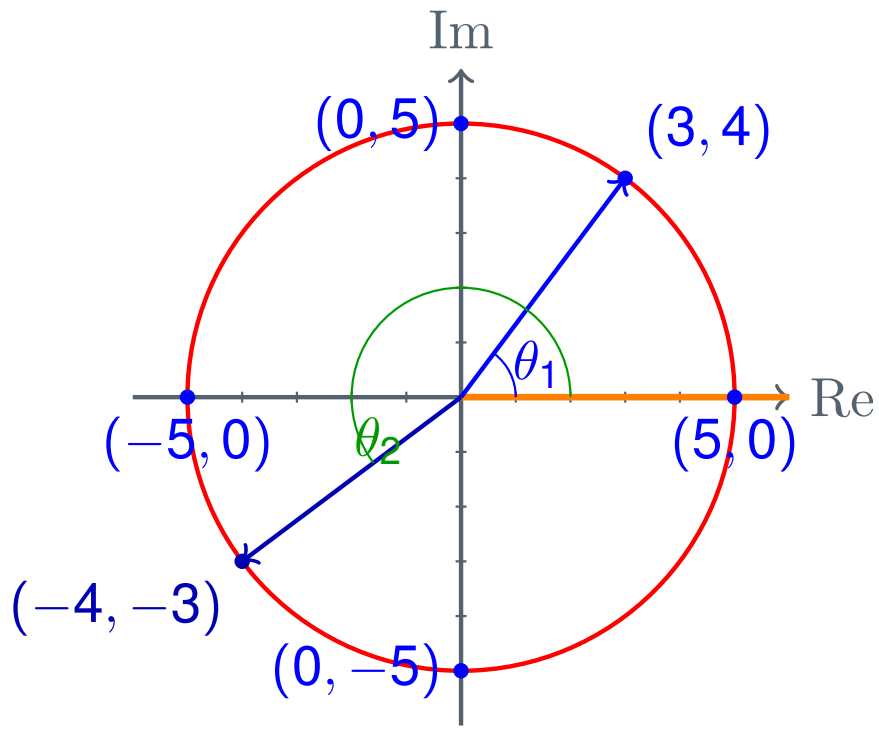


All the points of the circle have the same modulus / distance form the origin. But no other.

The insufficiency of the modulus

Recall that $|z|$ is the length of the straight line from $O = (0, 0)$ to z on the complex plane. The distance, however, does not fully recover the point.

Points of the same circle, however, can be distinguished with their *angular-distance*, that is to say, the angle they make with the positive part of the real line $[0, \inf]$

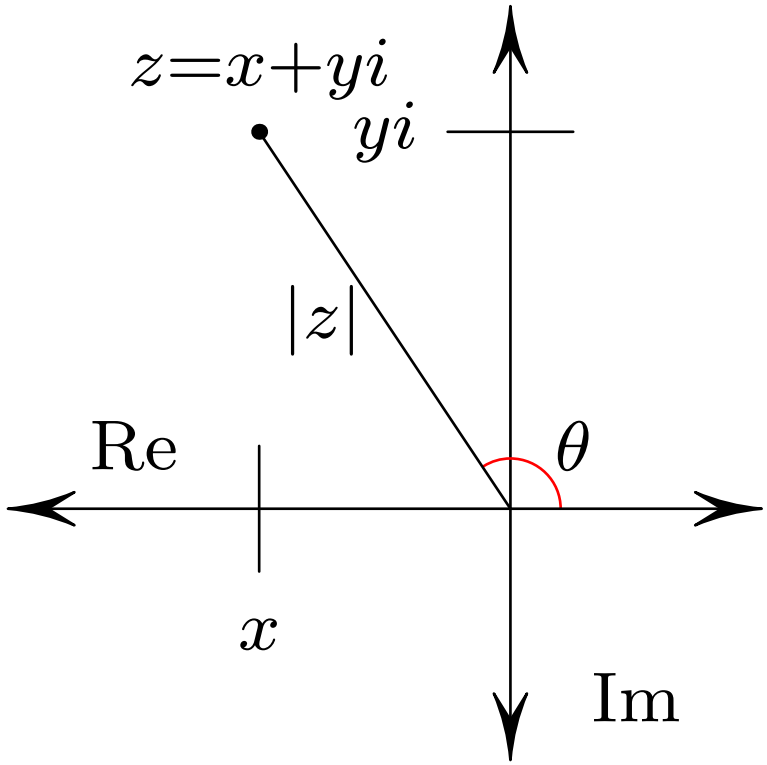


The angles, however, distinguish them!

Angle (argument)

Lets specify exactly what we mean:
Note that any complex point z also creates an angle θ which is measured anticlockwise ↺ from the real positive axis.

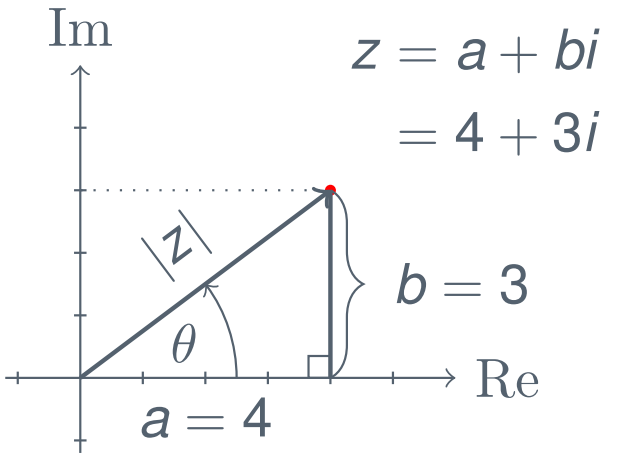
Definition: The angle θ for z is called the argument of z, denoted Arg (z).



How to find Arg(z)?

For a complex number $z = a + bi$, we can think of a as its distance from the x -Axis and b as its distance from the y -Axis. This creates a **rightangle** triangle (see img on RHS).

Note that in our case $a \geq 0$ and $b \geq 0$. This is important!



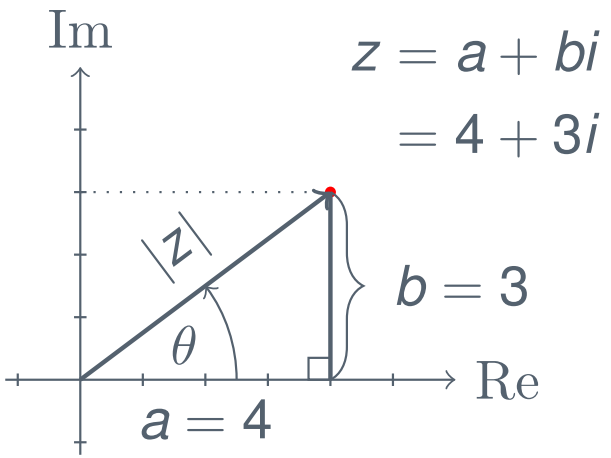
How to find Arg(z)?

By their respective definitions, we have $\cos(\theta) = \frac{a}{|z|}$ and $\sin(\theta) = \frac{b}{|z|}$. Hence

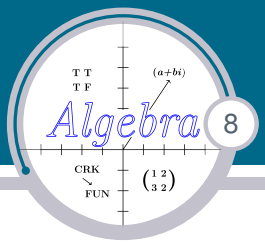
$$\begin{aligned} z &= a + bi = |z| \left(\frac{a}{|z|} + \frac{b}{|z|} i \right) \\ &= |z| \left(\cos(\theta) + i \sin(\theta) \right) \end{aligned}$$

This is effectively the *poolar coordinates* of z!

But how does this help us find Arg(z) (the angle θ)?



How to find Arg(z)?

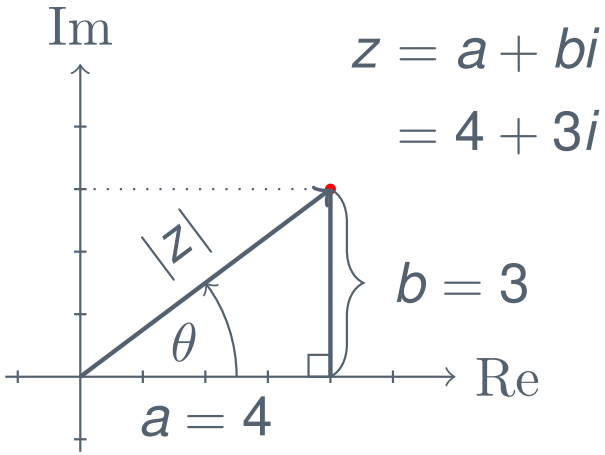


Simply recalling the definition

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

gives us the answer, since we now have

$$\begin{aligned} \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{\frac{b}{|z|}}{\frac{a}{|z|}} = \frac{b}{a}. \end{aligned}$$



As such, the inverse of $\tan$, being $\tan^{-1}$ (also called $\arctan$) now yields

$$\theta = \tan^{-1}\left(\frac{b}{a}\right).$$

CAREFUL!

NOTE!

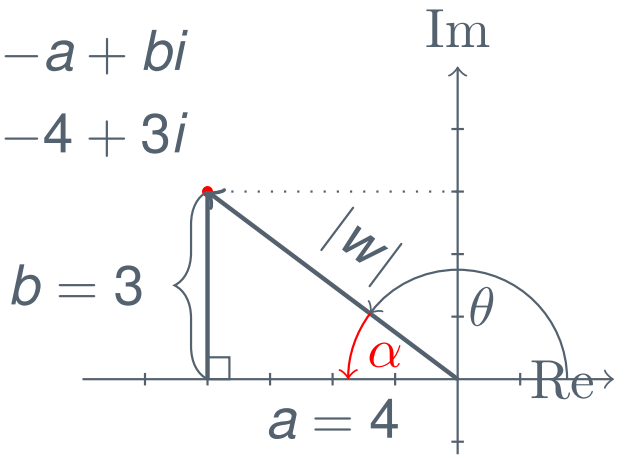
The above only works in Q1: the *first quadrant* (when a and b are positive, see next slide).

Looking at the image, we can see that if we calculate

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

for $a < 0$ (e.g., -4 in our case), we do NOT get θ, but α.

$$w = -a + bi$$
$$= -4 + 3i$$



Q2: Second quadrant

However, we **do** have

$$\theta = \pi - \alpha.$$

Recall

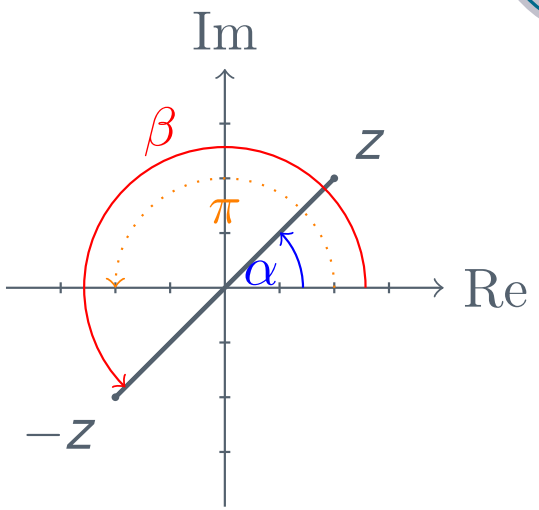
Recall that $2\pi = 360\ °$ is the full circle. Hence $\pi = 180\ °$.

An other view on why $\tan^{-1}\left(\left|\frac{b}{a}\right|\right)$ alone can not be enough.

Let $z = a + bi$. Then $-z = -a - bi$. If we simply define

$$\text{Arg}(z) = \tan^{-1}\left(\frac{b}{a}\right),$$

then we would have



$$\alpha = \text{Arg}(z) = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-b}{-a}\right) = \text{Arg}(-z) = \beta.$$

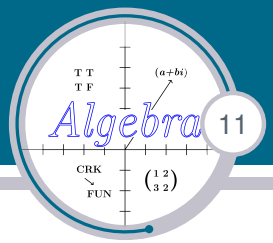
Looking at the picture, however, we can clearly see that the *arguments* of z and $-z$ are **not** the same.

Q3: Third quadrant

We do, however, have

$$\text{Arg}(-z) = \pi + \text{Arg}(z)$$

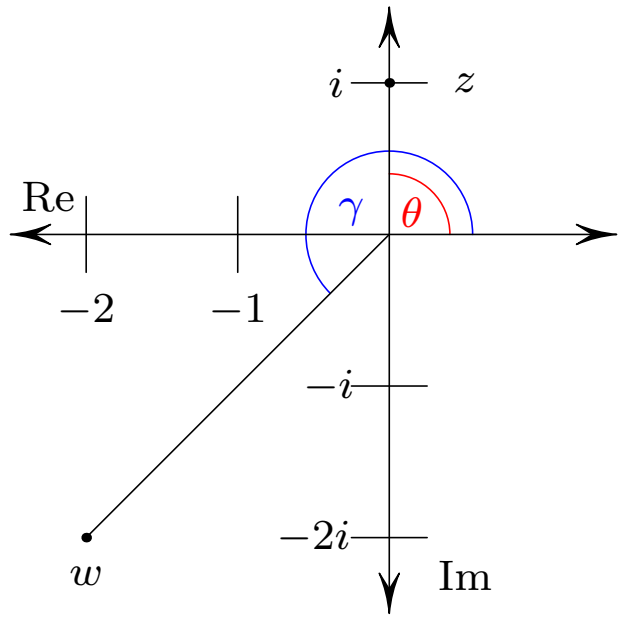
Argument example



Let $z = i$ and $w = -2 - 2i$, what are their arguments?

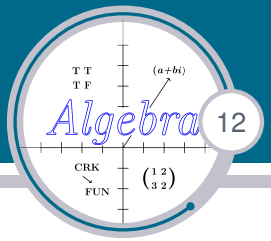
We can see that Arg $(z) = \theta = \dfrac{\pi}{2}$ or 90°, (I will only use radians (not degrees) in lectures.)

Also, we have
Arg$(w) = \gamma = \pi + \dfrac{\pi}{4} = \dfrac{5\pi}{4}$, as



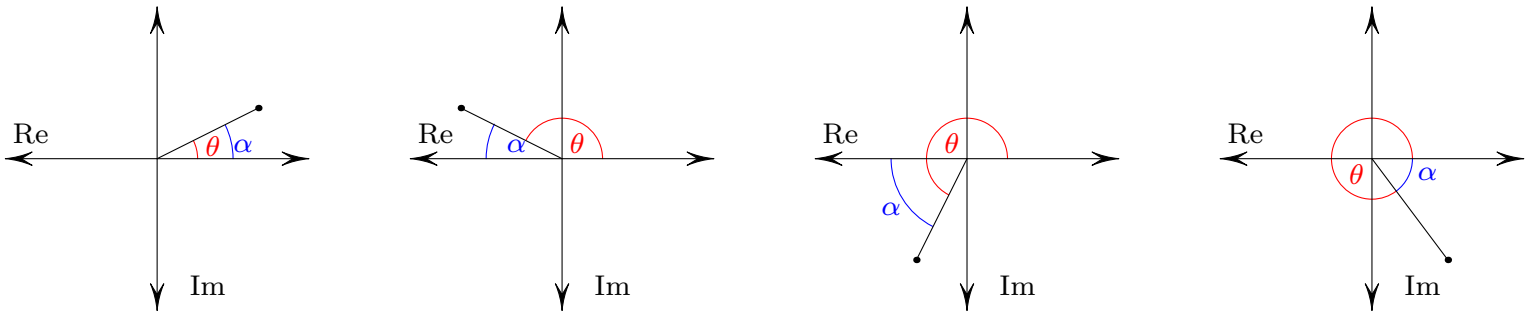
$$\frac{\pi}{4} = \tan^{-1}\left(\left|\frac{-2}{-2}\right|\right) \left(\text{or } \arctan\left(\left|\frac{-2}{-2}\right|\right)\right).$$

Determine Arg(z) – Algorithm



Steps to determine Arg(z) for $z = x + yi$:

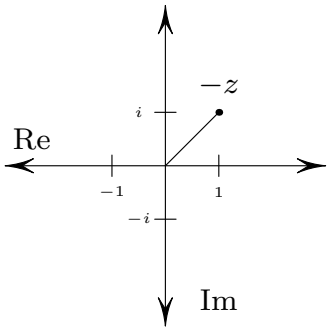
1. Locate/plot z on an Argand diagram.
2. Calculate $\alpha = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$.
3. Interpret the angle according to which quadrant z lies in.

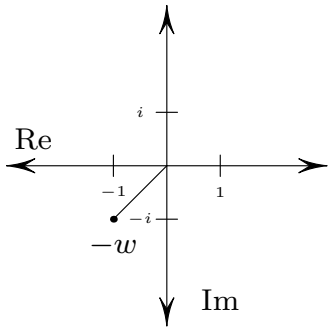


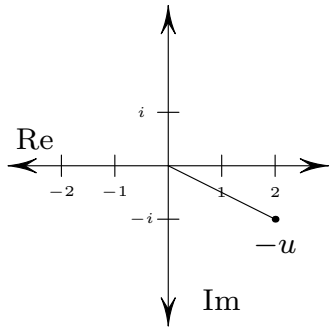
Q1: Arg(z) = α. Q2:Arg(z) = $\pi-\alpha$. Q3: Arg(z) = $\pi+\alpha$. Q4:Arg(z) = $2\pi-\alpha$.

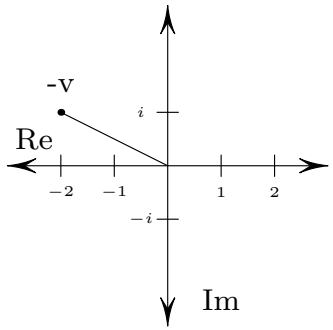
Exercise: plot arguments

Let $z = -1 - i$, $w = 1 + i$, $u = -2 + i$ and $v = 2 - i$.
Plot $-z$, $-w$, $-u$ and $-v$ on separate Argand diagrams.

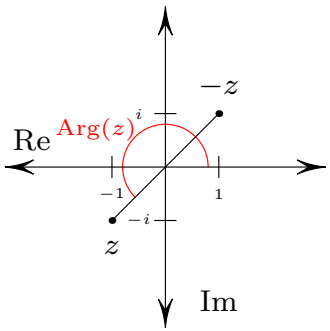


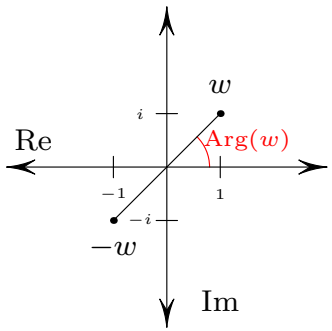


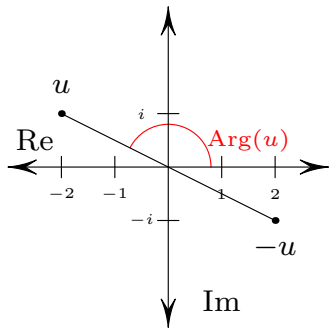


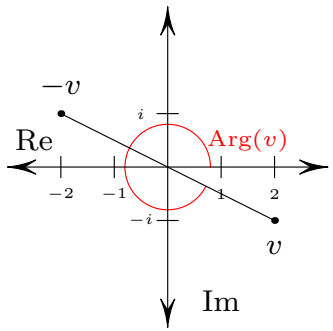


Indicate Arg(z), Arg(w), Arg(u) and Arg(v) on those Argand diagrams.









Plot the sum of two complex numbers

Let $z = -1 - i$, $w = 1 + i$, $u = -2 + i$ and $v = 2 - i$. Plot $u - z$ and $w + v$.

$$u-z = (-2+i)-(-1-i)=(-2-(-1))+(1-(-1))i = -1+2i.$$
$$w+v = (1 + i) + (2 - i) = 3.$$

