Semester II 2026

MA135/MA160 - Algebra II



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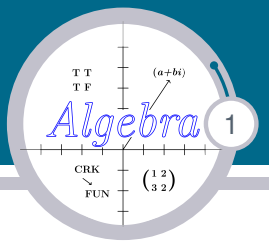
Lecture 10: Complex numbers

Recap (arithmetic)



$z = a + bi$ and $w = c + di$

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i.$$
$$z - w = (a + bi) - (c + di) = (a - c) + (b - d)i.$$
$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$
$$\frac{z}{w} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + (bc - ad)i}{c^2 - d^2}.$$

$\overline{w} = a - bi$ is the conjugate of $w = a + bi$

Recall that $\overline{w} = a - bi$ is the conjugate of $w = a + bi$. Note that the conjugate $\overline{w}$ is the reflection of w through the real axis of an Argand diagram.

Exercise

Let $z = 5 - 2i$ and $w = -3 + i$. Calculate

1. $z + w$ and $z - w$
2. zw
3. $\frac{z}{w}$.

1. $$\begin{aligned} z + w &= (5 - 2i) + (-3 + i) = (5 - 3) + (-2i + i) = 2 - i \\ z - w &= (5 - 2i) - (-3 + i) = (5 + 3) + (-2i - i) = 8 - 3i. \end{aligned}$$

2. $$\begin{aligned} zw &= (5 - 2i)(-3 + i) = 5(-3 + i) - 2i(-3 + i) \\ &= -15 + 5i + 6i - 2i^2 = -15 + 11i + 2 = -13 + 11i. \end{aligned}$$

3. $$\begin{aligned} \frac{z}{w} &= \frac{(5 - 2i)}{(-3 + i)} = \frac{(5 - 2i)}{(-3 + i)}\frac{(-3 - i)}{(-3 - i)} = \frac{-15 - 5i + 6i + 2i^2}{(-3)^2 + 1^2} \\ &= \frac{-15 + i - 2}{10} = \frac{-17 + i}{10} = -\frac{17}{10} + \frac{1}{10}i. \end{aligned}$$

Example

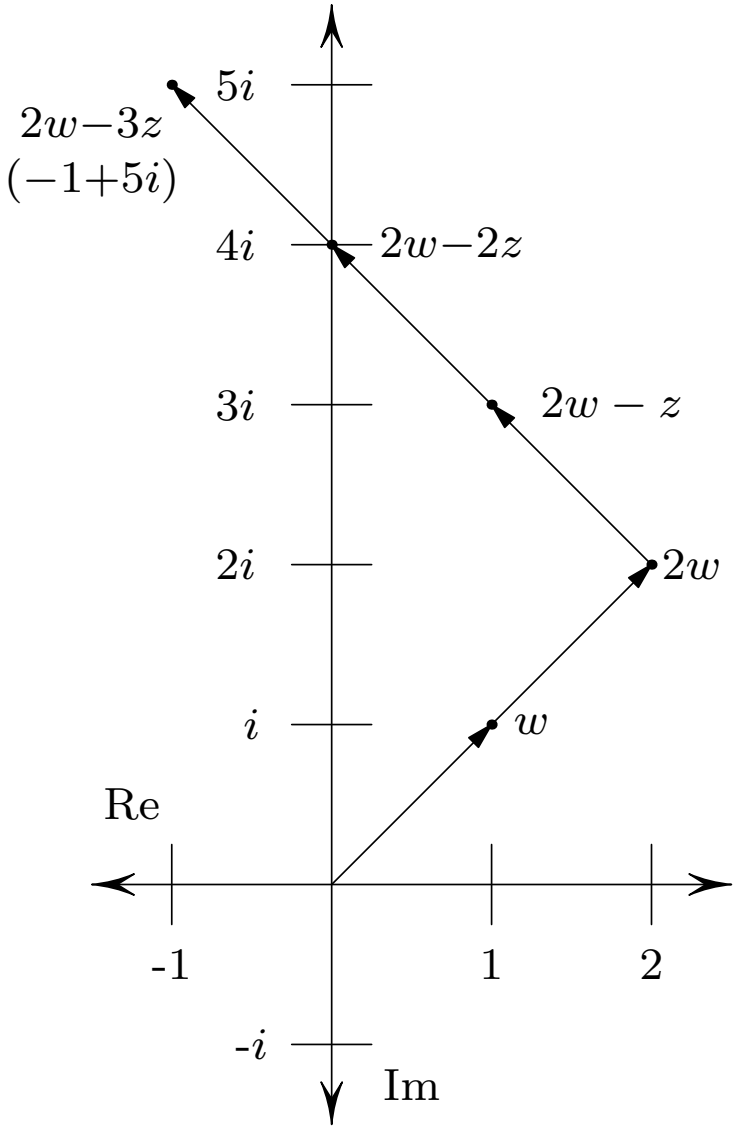
Let $w = 1 + i$ and $z = 1 - i$, and calculate $2w - 3z$.

Solution:

$$\begin{aligned} 2w - 3z &= 2(1 + i) - 3(1 - i) \\ &= (2 + 2i) - (3 - 3i) \\ &= (2 - 3) + (2i + 3i) \\ &= -1 + 5i. \end{aligned}$$

(Note that $-z = -(1 - i) = -1 + i$)

$$2w = 2 + 2i \xrightarrow{-z} 1 + 3i \xrightarrow{-z} 0 + 4i \xrightarrow{-z} -1 + 5i$$

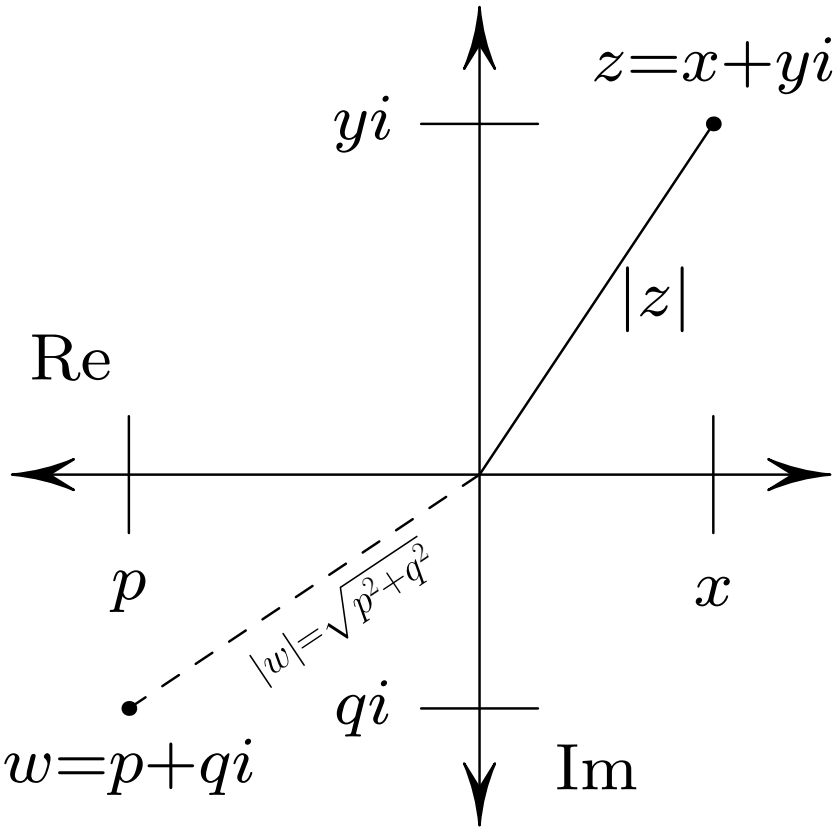


Modulus

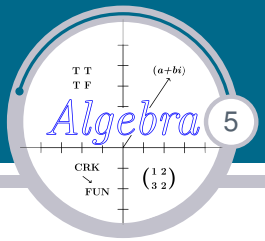
Definition: For $z = x + yi$ we define the modulus, $|z|$, to be $|z| = \sqrt{x^2 + y^2}$.

If we plot the point $z = x + yi$ on an Argand diagram then $|z|$ is the distance between the origin, $o := (0, 0)$, and the point z, i.e., the length of the straight line between o and z.

Note, $|w| \geq 0$, even though both $p, q < 0$, since it is a measurement of length and by convention $\sqrt{}$ gives the positive square root.



Modulus examples

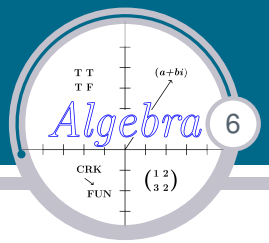


Find $|z|$ for $z = 3 - 4i$ and $|w|$ for $w = -\frac{2}{3} + \frac{4}{3}i$.

$$|z| = |3 - 4i| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

$$\begin{aligned} |w| = \left|\left|-\frac{2}{3} + \frac{4}{3}i\right|\right| &= \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} \\ &= \sqrt{\frac{4}{9} + \frac{16}{9}} = \sqrt{\frac{20}{9}} = \frac{\sqrt{20}}{3}. \end{aligned}$$

Modulus example



Find $|z|$ for

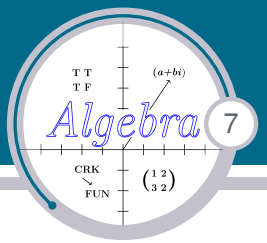
$$z = \frac{2+i}{3-2i}.$$

Solution: First need to state z in the form $x + yi$.

$$\begin{aligned}\frac{2+i}{3-2i} &= \frac{(2+i)}{(3-2i)}\frac{(3+2i)}{(3+2i)} = \frac{2(3+2i)+i(3+2i)}{3^2+2^2} \\ &= \frac{6+4i+3i+2i^2}{13} = \frac{6+7i-2}{13} \\ &= \frac{4+7i}{13} = \frac{4}{13} + \frac{7}{13}i.\end{aligned}$$

$$\begin{aligned}|z| &= \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{7}{13}\right)^2} = \sqrt{\left(\frac{16}{169}\right) + \left(\frac{49}{169}\right)} \\ &= \sqrt{\left(\frac{65}{169}\right)} = \frac{\sqrt{65}}{13}.\end{aligned}$$

Modulus exercises



Find $|z|$ for $z = 2 + 5i$ and $|w|$ for $w = \dfrac{1+i}{1-i}$

$$|z| = |2 + 5i| = \sqrt{2^2 + (5)^2} = \sqrt{4 + 25} = \sqrt{29}.$$

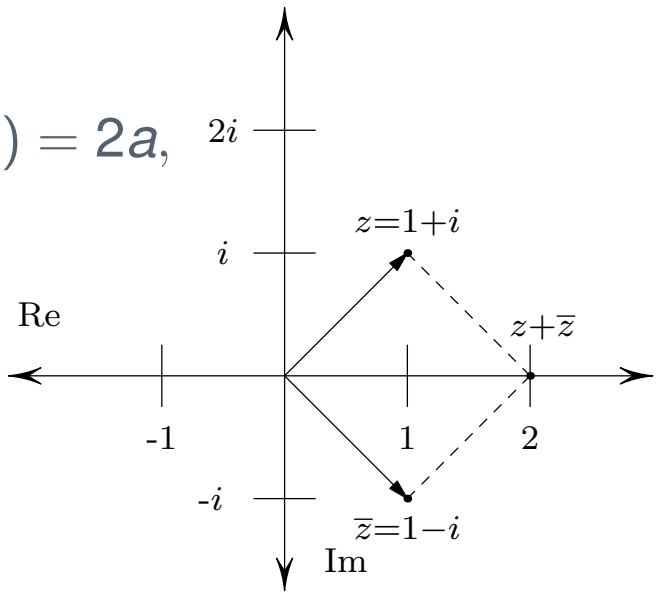
$$\begin{aligned} w &= \frac{1+i}{1-i} = \frac{1+i}{1-i}\frac{1+i}{1+i} \\ &= \frac{1(1+i) + i(1+i)}{1^2 + (-1)^2} = \frac{1+i+i+i^2}{2} \\ &= \frac{1+2i-1}{2} = \frac{2i}{2} = i \end{aligned}$$

Relation between z and $\overline{z}$

Recall that a complex number $z = a + bi$ has a complex conjugate $\overline{z} = a - bi$. As

$$z+\overline{z} = (a+bi)+(a-bi) = (a+a)+(bi-bi) = 2a,$$

$z + \overline{z}$ is always a real number. This is also clear geometrically, as addition corresponds to taking the verteces of a parallelogram

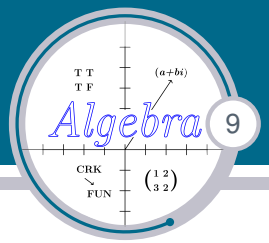


We still need to understand multiplication geometrically, but algebraically, we can see that

$$\overline{z}z = \overline{(a+bi)}(a+bi) = (a-bi)(a+bi) = a^2 - b^2 i^2 = a^2 + b^2$$

and as such, $\overline{z}z$ is likewise always a real number! (also positive).

Relation between $\overline{z}$ and $|z|$



Looking at

$$\overline{z}z = \overline{(a+bi)}(a+bi) = (a-bi)(a+bi) = a^2 - b^2 i^2 = a^2 + b^2$$

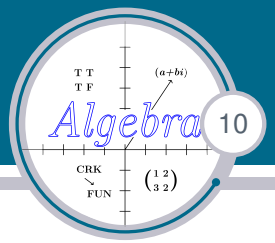
and

$$|z| = \sqrt{a^2 + b^2}$$

we can immediately see that

$$|z| = \sqrt{\overline{z}z} \qquad \text{or equivalelently} \qquad |z|^2 = \overline{z}z$$

The multiplicative nature of the modulus



Using the equation we just derived $|z| = \sqrt{\overline{z}z}$, we can immediatlely deduce the following:

The modulus is multiplicative

For any complex numbers z, w, we have

$$|zw| = |z| \cdot |w|.$$

Proof

$$|zw| = \sqrt{\overline{zw}zw} = \sqrt{\overline{z}\overline{z}\overline{w}w} = \sqrt{\overline{z}z}\sqrt{\overline{w}w} = |z| \cdot |w|$$

Of course, division is just an other form of multiplication. Hence,

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

Think about how much simpler we can now calculate $|z|$ for $z = \frac{2+i}{3-2i}$

$$
\begin{aligned}
|z| &= \left|\frac{2+i}{3-2i}\right| \\
&= \frac{|2+i|}{|3-2i|} \\
&= \frac{\sqrt{4+1}}{\sqrt{9+4}} \\
&= \frac{\sqrt{5}\sqrt{13}}{\sqrt{13}\sqrt{13}} \\
&= \frac{\sqrt{65}}{13}.
\end{aligned}
$$

Here, the last two lines are just needed to ensure we reduce the fraction to a "normal" form, meaning, we do not want a root in the denominator.