Stacks, Topoi and Monoid Schemes

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Abstract

This habilitation thesis explores several areas of algebra, in particular algebraic structures arising in the context of algebraic geometry. The main focus is given to the study of certain combinatorial geometric objects, called monoid schemes, and related objects. A monoid scheme is an algebraic object that can be described as a (ring) scheme without addition. Monoid schemes are related to many areas of algebra, most notably $\mathcal{F}_1$-geometry, toric varieties, tropical geometry and logarithmic geometry.

Indeed, the theory of monoid schemes generalises the theory of toric varieties, with the monoid scheme playing the role of a fan in toric varieties. As such, a monoid scheme can be seen as the combinatorial data, from which it is possible to create geometric realisations, much as in the theory of toric varieties.

The two main approaches taken in this thesis to study these objects are based on 2-category theory and topos theory. The former is used in the analysis of homotopy of their geometric realisations, mainly in calculating their fundamental groupoid. The latter is used to obtain a reconstruction type theory, with topos points and localisations playing a crucial role.

In order to do so, both 2-category theory and topos theory will be explored in their own right, and other related questions and approaches will be studied as well. For example, the relation between colimits and 2-colimits will be explored in more detail. This will be used to give an algorithmic method of calculating the fundamental groupoid. This method will then be applied in the case of monoid schemes, and their realisations.

Other algebraic structures that will be explored are quadratic algebras, and more general 2 dimensional torsors over Hopf algebras. Their isomorphism classes form a group, which will be studied in this habilitation thesis using methods of 2-category theory. In particular, using methods of 2-categorical homological algebra.

The last algebraic structure that will be explored in this thesis is the pre-addition on elliptic curves. This structure is very close to abelian groups, though it differs in some key aspects, as will be shown. Towards the end, the notion of elliptic rings is introduced, showing that it is possible to do both geometry and arithmetics on these objects.

This habilitation is based on several research papers I have written. They are attached at the end, in their original published form.

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Introduction

General overview

This thesis starts by exploring an interesting little invariant of commutative rings, using methods of 2-categorical homological algebra. Outside of its intrinsic interest, it serves as a simple demonstration of the use of 2-category theory, which is going to be a running theme throughout this dissertation. As 2-categories, and (co)-stacks in particular, were a central theme of my PhD thesis, it is not surprising that this subject features heavily in some of my first papers after my PhD.

More specifically, the thesis starts with the study of the group of isomorphism classes of torsors of finite flat group schemes of rank 2 over a commutative ring R. This, in particular, generalises the group of separable quadratic algebras, which was introduced by Bass in [4, Section 4.3] and was extensively studied by several authors (see for example [26], [65], and references therein), using methods of 2-dimensional categorical algebra. My approach, however, yields new results even in this case. The methods introduced herein are based on the fact that stackification is an exact 2-functor.

To be more precise, let R be a commutative ring and denote by $\mathsf{Qu}(R)$ the group of isomorphism classes of separable quadratic R-algebras [26, Ch. 12], and by $\mathsf{Dis}(R)$ the group of isomorphism classes of non-singular symmetric bilinear R-modules, which are finitely generated and projective of rank 1 [26, Ch. 4C]. According to [26, Statement 12.6, p.178], one has an exact sequence of abelian groups

$$\mathsf{Qu}(R) \to \mathsf{Dis}(R) \to \mathsf{Dis}(R/4R).$$

The goal is to extend this important exact sequence on the left hand side. In particular, it will follows from the results presented here, that the following is true:

Theorem 1. *If 2 is not a zero-divisor in R, one has an exact sequence of abelian groups*

$$0 \to \mathbb{Z}_2(R) \to \mu_2(R) \to \mathfrak{I}(\mu_2(R/4R) \to \mu_2(R/2R)) \to \mathsf{Qu}(R) \to \mathsf{Dis}(R) \to \mathsf{Dis}(R/4R).$$

Here,

$$\mu_2(R) = \{r \in R \mid r^2 = 1\} \text{ and } \mathbb{Z}_2(R) = \{r \in R \mid r^2 = r\}.$$

The operation in $\mathbb{Z}_2$ is given by $r +_1 s = r + s - 2rs$ (see [26, p. 202]). The first non-trivial map is given by $r \mapsto 1 - 2r$.

In order to obtain the above, we are going to define a symmetric categorical group $\mathfrak{Qu}(R)$ such that the groups $\mathrm{Qu}(R)$ and $\mathbb{Z}_2(R)$ are its π_0 and π_1 respectively (see Section 1.4.4 for symmetric categorical groups). Recall that $\pi_0(\mathbb{C})$ of a small category is defined as the set of isomorphisms classes of the objects of $\mathbb{C}$, where as $\pi_1(\mathbb{C}, C)$ is the set of automorphisms of $\mathbb{C}$. Note that in a categorical group, the automorphisms of any two objects are isomorphic. As such, we can simply talk about $\pi_1(\mathbb{C})$ for $\mathbb{C}$ a categorical group.

Similarly, behind of the groups $\mathrm{Dis}(R)$ and $\mu_2(R)$, there is a symmetric categorical group $\mathfrak{Dis}(R)$, while the groups $\mathrm{Dis}(R/4R)$ and $\mathfrak{I}(\mu_2(R/4R) \to \mu_2(R/2R))$ are obtained from the symmetric categorical group $\mathfrak{Dis}^{-2,1}(R)$. Hence, it suffices to show that $\mathfrak{Qu}(R)$, $\mathfrak{Dis}(R)$ and $\mathfrak{Dis}^{-2,1}(R)$ fit in an exact sequence of symmetric categorical groups, and then apply the classical six-term exact sequence [21, p.84] involving π_i, $i = 0, 1$. The symmetric categorical groups $\mathfrak{Qu}(R)$ and $\mathfrak{Dis}(R)$ are classical and they appear (at least implicitly) in many papers, for example in [26]. However, $\mathfrak{Dis}^{-2,1}(R)$ is new. The next observation is that, by varying R, these symmetric categorical groups form stacks in the Zariski topology. A stack is just the 2-categorical analogue of a sheaf. Stacks and their duals, costacks, will feature in this thesis at other places as well.

As it turns out, the stacks we obtain in the manner discussed above can also be seen as the stackifications of the analogue prestacks of symmetric categorical groups, based on free modules instated of projective ones. It is a fairly easy exercise to show that the 'free' versions of the above symmetric categorical groups fit in an exact sequence. Since stackification sends exact sequences to exact ones, our desired result will follow. This is one of the many demonstrations of the use of 2-category theory.

As already mentioned, we will actually work in a more general setting than that of separable quadratic algebras. In fact, we will fix a commutative and cocommutative Hopf algebra J, which is free of rank 2 as an R-module. It is a classical fact that any such Hopf algebra is uniquely determined by a pair (p, q) of elements of R such that $pq + 2 = 0$ (see [38, Theorem 1.2]). Recall that a Hopf algebra H is a bialgebra (compatible associative algebra with unit and a coassociative coalgebra with counit), together with an antipide $(H \to H)$.

The main object of interest in this setting will be the group $\mathrm{Qu}^{pq}(R)$ of J-Galois algebras, with the main result of this chapter being Theorem 2.4.8. Since $\mathrm{Qu}(R) = \mathrm{Qu}^{-2,1}(R)$, Theorem 1 is a specialization of Theorem 2.4.8.

Having obtained these results, the focus will shift towards studying 2-categories in more detail, specifically, 2-colimits. These are important to this dissertation due to the results obtained in my PhD thesis, where I was able to prove the following:

Theorem 2. *The fundamental groupoid Π_1, regarded as a 2-functor*

$$\Pi_1 : \mathfrak{Off}(X) \to \mathrm{Groupoids},$$

is the 2-terminal costack.

To explain some of the above terminology, a costack is the dual of a stack. The latter is itself the 2-categorical version of a sheaf. When deciphered, the costack condition can be seen

as the 2-categorical version of the Seifert-van Kampen theorem. More on stacks, costacks and 2-categories in general can be found in Chapter 1.

As such, the above theorem states that the Seifert-van Kampen theorem is the defining property of the fundamental groupoid. As a reminder, the fundamental groupoid is defined to be a category whose objects are the points of a given topological space, and whose morphisms are homotopy classes of paths between the given objects (being points). It can readily be shown that this is equivalent to taking the fundamental group at every connected component simultaneously. As such, it carries the combined information of π_0 and of π_1 of every connected component.

Though we have only mentioned this result in the case of topological spaces, the same axiomatisation is true for the étale fundamental groupoid as well.

It is relevant to this dissertation since outside of conceptual reasons, Theorem 2 has strong calculatory applications. This is demonstrated in Chapter 5, which deals with the results of a joint paper by Prof. Brenner from the University of Osnabrück and myself. In order to do so, however, we will need to understand 2-colimits better, see Formula (1) below.

It will be the aim of the next major chapter of this work to simplify the calculation of 2-colimits. This will be achieved though 2 main results: The first one will enable us to reduce calculating the 2-colimit to calculating the colimit. The second one will be a truncation result, which states that only groupoids participating in the first 3 codimensions are relevant to the 2-colimit - much like the classical result on colimits, which states that only the first 2 codimensions factor into the result. All of my work in this section will only deal with the case when the 2-colimit is taken over a Boolean poset. Note, however, that this setting is sufficient to calculate the fundamental groupoid of a finite cover, which was precisely the goal for me.

To be more specific, let X be a topological space and $\mathcal{U} = \{U_1, \cdots, U_n\} \in \mathrm{Cov}(X)$ an open covering of X. We have shown in [50] that under some small assumptions on X, there are equivalences of categories

$$\Pi_1(X) \simeq 2\text{-}\operatorname*{colim}_{\mathfrak{B}(\mathbf{n})} \Pi_1^* \simeq \operatorname{colim}_{\mathfrak{B}(\mathbf{n})} \Pi_1^*. \tag{1}$$

Here, $\mathfrak{B}(\mathbf{n})$ denotes the poset of proper subsets of $\{1, \cdots, n\}$ and $\Pi_1^* : \mathfrak{B}(\mathbf{n}) \to \mathfrak{Grpd}$ is given by

$$\Pi_1^*(J) = \Pi_1\left(\bigcap_{j \notin J} U_j\right)$$

Of course, that $\Pi_1(X)$ is isomorphic to $\operatorname{colim} \Pi_1^*$ is an easy consequence of the famous reformulation of the classical Seifert-van Kampen theorem by Brown and Salleh [11],[12].

However, showing that $\Pi_1(X)$ is equivalent to the 2-colimit, rather than the colimit, has some advantages. In its core, these are derived from the fact that the 2-colimit is defined up to an equivalence of categories, rather than isomorphisms. As such, it is also invariant under equivalences. More precisely, working with 2-colimits enables us to replace the participating groupoids $\Pi_1(U_i)$, of the 2-colimit with equivalent ones, not just isomorphic ones. In its essence, this is the reason that allows us to axiomatise the fundamental groupoid in [50] (and also the étale fundamental groupoid in [52]). It is, indeed, also what allows us to simplify

calculations. We will see a glimpse of this in Example 3.2.9. The full scale application of this will be seen in chapter 5, which will also serve as a bridge towards the second major theme of this dissertation: monoid schemes and the closely related binoid schemes.

The main point is that, using Formula (1), we can usually replace groupoids having uncountably many objects (which appears in Brown-Salleh formula) with groupoids having only finitely many objects. The downside, at least on its face, is that, calculating 2-colimits is harder than calculating colimits. This can, however, be alleviated and this is precisely the aim of chapter 3.

Let $\mathfrak{F} : \mathfrak{B}(\mathbf{n}) \to \mathfrak{Grpd}$ be a strict 2-functor in the 2-category of small groupoids. It will be show in Theorem 3.2.6, that the colimit and 2-colimit of $\mathfrak{F}$ are equivalent if a certain injectivity condition is satisfied. Though this condition will seem fairly restrictive, it should be pointed out that any 2-functor can be deformed to satisfy this condition. This is due to the fact that injectivity will only be required on objects and it is an obvious and well-known fact that any functor $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ between groupoids (or categories) can easily be replaced with an equivalent functor $\mathcal{F}' : \mathcal{G} \to \mathcal{H}'$ that is injective on objects, and for which $\mathcal{H}$ and $\mathcal{H}'$ are equivalent groupoids. As such, this is only a theoretical restriction that will not be a great hindrance in practice.

We will proceed with an other simplification, showing that it is possible to truncate the 2-colimit at "depth 3". To be more precise, let $\mathfrak{B}(\mathbf{n})$ be the poset of proper subset of the set $\{1, \ldots, n\}$. We can consider a new subposet $\mathfrak{B}(\mathbf{n}, \mathbf{n-3})$ of $\{1, \ldots, n\}$, which consists of proper subsets of $\{1, \cdots, n\}$, with cardinality $\geq n-3$. Proposition 3.2.14 gives us an equivalence of categories

$$2\text{-colim}\,\mathfrak{F} \simeq 2\text{-colim}\,\mathfrak{F}',$$

where $\mathfrak{F}' : \mathfrak{B}(\mathbf{n}, \mathbf{n-3}) \to \mathfrak{Grpd}$ is the restriction of $\mathfrak{F} : \mathfrak{B}(\mathbf{n}) \to \mathfrak{Grpd}$. Combining this with the previous result will enable us to simplify the injectivity condition under which $2\text{-colim}\,\mathfrak{F} \simeq \text{colim}\,\mathfrak{F}$. In particular, the new condition is of polynomial complexity, whereas the original condition without this truncation, Increases exponentially in complexity as n increases.

All this will allows us to simplify the calculation of the fundamental group(oid) in many cases. This is especially true if we are given a so called *discrete covering* for X. By discrete, we mean a covering $\{U_i\}$ of X, where for every i, j, k, the fundamental groupoids $\Pi_1(U_i)$, $\Pi_1(U_{ij})$ and $\Pi_1(U_{ijk})$ are discrete. We call a groupoid discrete if the automorphism group of every point is trivial. Note, however, that π_0 need not be trivial. Of course, by definition, every manifold admits such a covering. The major obstruction is explicitly knowing it. If such a covering is given however, the calculation of the fundamental group(oid) reduces to simple combinatorics.

As mentioned above, while we will demonstrate this in a small example in chapter 3, the main application will be seen when discussing the results of a joint paper with Prof. Brenner. There, we introduce a class of topological spaces which not only admit a discrete covering, but where we can explicitly write it out.

To state our result, we first need to introduce some constructions that will be central to this dissertation:

The notion of a monoid scheme was introduced by Deitmar [19] in 2005, following the pioneering work of Kato [35], who proved the link between monoid schemes and toric varieties in 1994. This link was further explored in [18]. This is arguably one of its most important aspect. There are, however, many others. In part due to the generality of monoids, the applications and importance of monoid schemes (and of related concepts) is indeed still increasing. They are used in algebraic K-theory [18], are a fundamental part in various models of algebraic geometry over a "field with one element" [15], [16], [19], and have strong uses in areas such as tropical- [47] and logarithmic geometry [1], [34] and absolute mathematics [39].

In its essence, a monoid scheme is a sheaf of commutative monoids that mimics a (ring) scheme. Indeed, there is canonical functor from (ring) schemes to monoid schemes, given by forgetting the multiplication. It has a left adjoint, which is called the *realisation* and is locally given by $M \mapsto \mathbb{Z}[M]$. Of course, a monoid scheme can be realised over an arbitrary ring R, not just $\mathbb{Z}$. In this way, if the participating monoids were so called toric monoids, we obtain a toric variety. Indeed, all toric varieties can be obtained as such, see [18], [35].

A binoid is a monoid with an absorbing element. We denote it with 0 due to the parallels and links with algebraic geometry we are drawing. Binoid homomorphisms are required to respect this element. We are able to construct binoid schemes much in the same way we construct monoid schemes: By endowing the set of prime ideals (which, unlike in the world of monoids may not be empty) with the Zariski topology, defining affine binoid schemes through localisations, and then gluing them. We can also realise binoid schemes in exactly the same way as well: For a commutative ring K and a binoid M, we can consider the K-algebra $\mathsf{K}[M]$. The only thing to point out here is that, we identify the zero of the binoid with the zero of the ring, much as we do with the unit. Likewise, for a binoid scheme X, we can consider $\mathsf{K}[X]$, which we call the realisation of X. Further details to these constructions can be found in [18].

There are many reasons to be interested in binoids rather than monoids. Chief amongst them is that they are more general and include many additional important classes, such as the Stanley-Reisner Rings, while still being within the scope of the same tools. Another reason in our case is that it will help us in our proof, see Section 5.2.5.

Above, I mentioned a realisation of monoid-, respectively binoid schemes. A closely related, but distinct, construction is to consider $\mathsf{KSpec}(M) := \mathsf{Hom}(M, \mathsf{K})$. Here, K is seen as a multiplicative binoid. This is a subset of K^n, with n being the number of generators of M. We have $\mathsf{KSpec}(M) \simeq \mathsf{KSpec}(\mathsf{K}[M])$, where the latter denotes the set of K-algebra homomorphisms from $\mathsf{K}[M]$ to K. To see the difference, consider the following simple example: Let $M := \langle t \rangle \simeq \mathbb{N}$. Then $\mathsf{KSpec}(M) \simeq \mathsf{K}$, whereas $\mathsf{K}[M] \simeq \mathsf{K}[t]$. It is basically the zeroset of our binoid. Naturally, we can also consider $\mathsf{K}X$, for a binoid scheme X. It should be thought of as the set of K-valued points of the scheme $\mathsf{K}[X]$.

If K is equipped with a topology, $\mathsf{K}X$ will inherit it. We will focus on the cases when $\mathsf{K} = \mathbb{R}, \mathbb{C}$, the real or complex numbers. Our aim is to calculate $\pi_1(\mathsf{K}X)$, where the topology of $\mathsf{K}X$ is induced by the topology of K^n. We will actually calculate the more general $\Pi_1(\mathsf{K}X)$, where Π_1 denotes the fundamental groupoid.

To give our reason as to why we consider the more general case, we first point out that the

main advantages of the fundamental groupoid is that it is base-point agnostic. This has far reaching implications. One such implication, and crucial to our case, is that the Seifert-van Kampen theorem no longer requires a common point. This allows us to use the affine covering of a binoid scheme X to calculate the fundamental group(oid) of KX.

As such, our focus will become the affine case KSpec(M). One of our main theorems is the reduction of the computation of the homotopy of general binoids to grouplike binoids in Theorem 5.2.7. A grouplike binoid is merely a group with an absorbing element formally attached.

This is proven in several steps: We first prove it in the special case when M is integral and separated. Then, we will remove these conditions one after the other using induction arguments, primarily on the number of prime ideals.

We will simplify things further for K = $\mathbb{R}$. One of our main results in this topic will be Corollary 5.4.1, which states that $\Pi_1(\mathbb{R}\mathrm{Spec}(M))$ is a discrete groupoid for every binoid M. We will also fully describe the functors between the fundamental groupoids, induced by the gluing homomorphisms, see Subsection 5.4.1. To sum it up, for every quasi-separable binoid scheme X, $\mathbb{R}X$ admits a natural discrete covering, which we are able to know in full. This enables us to give a method for calculating the fundamental group(oid) of real binoid schemes.

This calculation is especially clear in the case of Stanley-Reisner Rings. These are rings associated to simplicial complexes Δ. Our method simplifies in this case, allowing us to describe the fundamental groupoid in terms of the simplicial complex directly. Of course, the fundamental group of $\mathbb{R}(\Delta)$ can also be calculated in other ways. The aim of our calculations is to give a demonstration of our proposed methods.

Ancillary, this also enables us to simplify the calculation of the homology of $\mathbb{R}X$, in Subsection 5.5.2.2.

Here, the works by V. Uma [72] need be mentioned as well, where the author focuses on the fundamental group of real toric varieties, coming from smooth fans. However, even in that case, our methods are completely different. Moreover, the general idea of calculating the fundamental groupoid from a discrete covering has independent interest.

Having studied the homotopy (and to some degree homology) of KX, we proceed to study monoid schemes more in-depth. As mentioned above, there is a second realisation of monoid schemes, K$[X]$. This is an object I have studied in my PhD thesis. In particular, I studied the vector bundles over X and K$[X]$, and how they relate to each other [49]. As such, studying the quasi-coherent sheaves over X seemed like an appropriate continuation. This is indeed the point of origin of the next topic in this dissertation.

The quasi-coherent sheaves over monoid schemes have already been considered in [15], where some basic facts about such sheaves were established.

We start by studying some global properties of quasi-coherent sheaves over monoid schemes. To explain my motivation and results, let us recall a classical result of algebraic geometry: It was established by Gabriel in [20], that the category of quasi-coherent sheaves over a Noetherian scheme (in the classical sense) is a locally Noetherian abelian category. Further, the set of isomorphism classes of indecomposable injective objects of said category is in a one to one correspondence with the points of the underlying topological space of a given scheme. Reconstruction theorems are very important in algebraic geometry and have been studied by

numerous authors. We mention Rosenberg [61] and Brandenburg [9], though there are many others.

In the vein of Gabriel, in order to prove a reconstruction type theorem, we consider the category $\mathfrak{Qc}(X)$ of quasi-coherent sheaves over a monoid scheme X. If $X = \mathrm{Spec}(M)$ is an affine monoid scheme, then $\mathfrak{Qc}(\mathrm{Spec}(M))$ is equivalent to the category $\mathscr{S}_M$ of M-sets. This is naturally a topos. Indeed, it is one of the main examples of a topos. It should be mentioned that, unless otherwise stated, a topos for us is assumed to be a Grothendieck topos. This fits well with the general philosophy of monoid schemes, which is to "forget addition" in constructions of classical algebraic geometry.

We prove in Theorem 8.1.5, that this is also true globally, meaning that the category $\mathfrak{Qc}(X)$ is also a topos, under some finiteness assumptions. Moreover, it is a so called locally s-Noetherian topos, if X is an s-Noetherian monoid scheme. I refer to the main body of this dissertation for the definition of an s-Noetherian monoid scheme, respectively topos.

While the approach by Gabriel, to use indecomposable injective objects does not work in the case of Topoi, one can use the more manifest notion of a topos point. Recall that there exists a general notion of a point in topos theory, called a topos point. The category of all topos points of a topos $\mathscr{T}$ will be denoted by $\mathbf{Pts}(\mathscr{T})$. They are a core construction of any topos, as the name would indicate. The category $\mathbf{Pts}(\mathscr{S}_G)$ of a group G, as an example, is equivalent to the category $\mathfrak{G}$, having a single object x, with $\mathrm{Aut}(x) = \mathrm{End}(x) = G$, see Example 7.3.5. This has been known for a long time.

We will study the topos points of the topos $\mathfrak{Qc}(X)$ of quasi-coherent sheaves over an s-Noetherian monoid scheme X. As the recent work of Connes and Consani [16] demonstrates, this problem is quite interesting, even in the affine case.

Let x be a point of a monoid scheme X. The stalk at x can be considered as a functor

$$\mathrm{Stalk}(x) : \mathfrak{Qc}(X) \to \mathscr{S}.$$

Here, $\mathscr{S}$ refers to the topos of sets. It is straightforward to see that this is the inverse image part of a point of the topos $\mathfrak{Qc}(X)$. It is well known that the inverse image uniquely determines a point. Based on this, the above can be restated as saying that stalks determine points. The question we aim to answer is the inverse: Which points of $\mathfrak{Qc}(X)$ are induced by stalks of X and specifically, under what conditions on X is every topos point of this type. In general, there are 'hidden points', even in the affine case, as can already be concluded from a work by A. Connes and C. Consani (see Section 7.6). It will be one of the main results of this dissertation to show that under some finiteness assumptions on X, every topos point of $\mathfrak{Qc}(X)$ correspond to a stalk of X. In the affine case, this finiteness condition says that the Monoid $M/M^\times$ is finitely generated, where $M^\times$ denotes the subgroup of invertible elements. This is one of our main result, which is proven in Sections 7.3 and 8.3, in the affine and quasi-projective case, respectively.

In addition to the conceptual nature of this result, it also significantly simplifies the calculation of $\mathbf{Pts}(\mathscr{S}_M)$, for finitely generated M. In this case, understanding the topos points of M-sets reduces to understanding the prime ideals of M. This, on the other hand, is very simple if the monoid M is given in terms of generators and relations, using the results I obtained in [48].

As the first approximation of the reconstruction theorem, we will derive from these results that, under some finiteness assumptions, two monoid schemes X and Y are isomorphic if and only if their associated topoi of quasi-coherent sheaves $\mathfrak{Qc}(X)$ and $\mathfrak{Qc}(Y)$ are equivalent, see Theorem 8.3.11.

We will also explore the non-finitely generated case briefly as well. We will classify the topos points of free commutative monoids, using equivalence classes of certain maps.

We continue with the reconstruction of monoid schemes, proceeding to reconstruct open subsets and finally the structure sheaf.

To state these results, let us fix some terminology and notations. See any book in topos theory, or the main body of this dissertation for the terms used here. As mentioned above, $\mathscr{S}$ denotes the category of sets in this dissertation, when regarded as the terminal topos. This notation is a specialisation of ${}_M\mathscr{S}$ and $\mathscr{S}_M$, used throughout this work, which denote topoi of left and right M-sets respectively. For a category A, $\mathcal{Z}(\mathsf{A})$ will denote its centre. This is, by definition, the commutative monoid of all natural transformation of the identity functor Id_A to itself. We will denote the collection of all isomorphism classes of points of a topos $\mathscr{T}$ by $\mathsf{F}(\mathscr{T})$, while $\mathfrak{Loc}(\mathscr{T})$ will denote the poset of all localisations of $\mathscr{T}$. For a localisation $\mathscr{D}$ of $\mathscr{T}$ and a topos point $p = (p_*, p^*) : \mathsf{Sets} \to \mathscr{T}$ of $\mathscr{T}$, we write $p \pitchfork \mathscr{D}$ if $p_*(S) \in \mathscr{D}$ for every set $S \in \mathsf{Sets}$.

Let X be a monoid scheme. the poset of all open subsets of X will be denoted by $\mathfrak{Off}(X)$. The structure sheaf of X is denoted by $\mathcal{O}_X$. A monoid scheme is called of *finite type* if it is separated and can be covered by a finite number of affine sub-schemes $\mathsf{Spec}(M_i)$ such that each M_i is a finitely generated monoid.

The following is the reconstruction theorem proven in this thesis:

Theorem 3. *Let X be a Noetherian monoid scheme. The following results hold:*

i) The category $\mathfrak{Qc}(X)$ is a topos.

ii) The map $\Phi_X : \mathfrak{Off}(X) \to \mathfrak{Loc}(\mathfrak{Qc}(X))$, given by

$$\Phi_X(U) = \mathfrak{Qc}(U),$$

is an isomorphism of posets.

iii) If U is an open subset of X and $\mathscr{D} = \Phi_X(U)$ is the corresponding localisation of $\mathfrak{Qc}(X)$, then

$$\mathcal{O}_X(U) = \mathcal{Z}(\mathscr{D}).$$

iv) For any point $x \in X$, the functor

$$\mathfrak{Qc}(X) \to \mathsf{Sets}, \quad \mathcal{A} \mapsto \mathcal{A}_x,$$

which assigns to a quasi-coherent sheaf $\mathcal{A}$ the stalk $\mathcal{A}_x$ at x, is the inverse part of a point of the topos $\mathfrak{Qc}(X)$. Furthermore, the corresponding map $\Pi : X \to \mathsf{F}(\mathfrak{Qc}(\mathsf{X}))$ on the underlying topological space of X is a bijection if X is quasi-projective.

v) If $x \in X$ and U is an open subset of X, then $x \in U$ if and only if $\Pi(x) \pitchfork \Phi_X(U)$.

We have already briefly discussed parts i) and iv). Let us mention the other parts as well: Part ii) will be proven as Corollary 8.4.11 and in Subsection 7.3.9, in the affine and general case, respectively. Said subsection is basically just the proof of the remainder of this theorem. The third part is proven in Section 8.5.2 as Lemma 8.5.3. The last part is also proven in Subsection 7.3.9.

Following this, the next major goal will be to study 'non-commutative monoid schemes'. Much like in classical algebraic geometry, a sensible approach for this seems to be via the study of quasi-coherent sheaves over these elusive objects. Recall that in the affine case, quasi-coherent sheaves were just M-sets. This is a topos irrespective of the commutativity of M. When working in the non-commutative case, we will restrict ourselves to the affine setting, though there will be a brief reminder of the non-affine commutative setting.

As previously mentioned in passing, monoid schemes can be seen as a natural generalisation of toric varieties. As such, a better understanding of the non-commutative case could also be seen as progress towards understanding 'non-commutative toric varieties'.

We can talk about points, localisations and the centre for any topos $\mathscr{T}$. We try to define a topology on the isomorphism classes of topos points $\mathsf{F}(\mathscr{T})$ in the following way: Declare a subset U of $\mathsf{F}(\mathscr{T})$ to be open, if there exist a localisation $\mathscr{T}$ of $\mathscr{T}$, such that $\mathsf{p}_*(S) \in \mathscr{T}$ for any point $\mathsf{p} : \mathsf{Sets} \to \mathscr{T}$ of U and any set S. Of course, it is not expected for this to always work: $\mathsf{F}(\mathscr{T})$ or $\mathfrak{Loc}(\mathscr{T})$ could be a class and not a set, or on the other side, a topos might not have any points at all. Even under the assumption of enough, but not too many, points, there are other difficulties as well. For example, when $\mathscr{T} = \mathsf{Sh}(X)$ is the topos of sheaves over a sober topological space X, then $\mathsf{F}(\mathscr{T})$ is in a one-to-one correspondence with the points of X. In general, however, there are more localisations of $\mathsf{Sh}(X)$ than X has open subsets.

The problem of equipping $\mathsf{F}(\mathscr{T})$ with a topology is, of course, not new and already appeared in [27, Section 2.6], where a topological space structure on the set $\mathsf{F}(\mathscr{T})$ was defined based on the subobjects of the terminal object of $\mathscr{T}$. In the case when $\mathscr{T}$ is the topos $\mathsf{Sh}(X)$ of sheaves over a sober topological space X, the space constructed in [27, Section 2.6] is homeomorphic to X, see [27, Remark 2.6.2]. However, in many other cases, the above topology can be quite trivial. Particularly, when $\mathscr{T} = \mathscr{S}_M$ is the topos of right M-sets, the obtained topological space will only every have two open sets, though $\mathsf{F}(\mathscr{S}_\mathsf{M})$ can be arbitrarily big. Hence, it is natural to search for other topologies on $\mathsf{F}(\mathscr{T})$.

In recent years, considerable attention was given to understanding the topos points of $\mathscr{S}_M$, and the possible topological structures associated to said points. For example, Connes-Consani [17] and Le Bruyn [13] studied two interesting topologies in the case when $M = \mathbb{N}_+^\times$ is the multiplicative monoid of strictly positive natural numbers. See also a related paper by Hemelaer [29].

It is due to these reasons we consider the localisations and points of the topos $\mathscr{S}_M$ of M-sets, for a non-commutative monoid M. We will first consider the case when M is a left zero monoid. That is, $M = \{1\} \cup S$, where S is any set and the multiplication is given by

$st = s$ for all $s, t \in S$. Here, we assume M is non-commutative, that is $|S| \geq 2$.

Already the topoi $\mathscr{S}_M$ and ${}_M\mathscr{S}$ of right and left M-sets are quite different: This can already be observed from the cardinality of the subobject classifier of $\mathscr{S}_M$, which is equal to $1 + 2^{|S|}$, see the equality (7.1). The subobject classifier of ${}_M\mathscr{S}$, on the other hand, only has three elements, see Proposition 7.3.21. Nonetheless, we will see that $\mathsf{F}({}_M\mathscr{S}) = \mathsf{F}(\mathscr{S}_M)$ and $\mathsf{F}(\mathscr{S}_M)$ is independent from S. In more concrete terms, we prove that both topoi $\mathscr{S}_M$ and ${}_M\mathscr{S}$ have exactly two points (up to isomorphism) (see Corollary 7.3.17 and Lemma 7.3.18), three localisations (see Propositions 7.3.20 and 7.3.21) and in both cases $\mathcal{O}$ is the constant sheaf corresponding to the trivial monoid. This shows that, in general, even if $\mathsf{F}(\mathscr{T})$ is a well-defined set, it carries limited information on $\mathscr{T}$.

There are some interesting questions that arise from these results: Two such are the following: Are localisations (respectively isomorphism classes of points) of the topoi $\mathscr{S}_M$ and ${}_M\mathscr{S}$ bijective sets?

The main results of this section give an affirmative answer to the above two questions, for finite M. This is achieved by proving that for finite M, the isomorphism classes of points of $\mathscr{S}_M$ is in a one-to-one correspondence with Green's $\mathscr{J}$-equivalence classes of idempotents: (that is, $e \mathscr{J} f$ if and only if $MeM = MfM$). We also prove that for a finite monoid M, there is a bijection between the set of localisations $\mathfrak{Loc}(\mathscr{S}_M)$ and the set of all two-sided idempotent ideals of M. It follows that, as topological spaces, $\mathcal{F}({}_M\mathscr{S})$ and $\mathcal{F}(\mathscr{S}_M)$ are both homeomorphic to the ordered topology of the poset of principal idempotent ideals.

As pointed out by the referee of my corresponding paper [58], some of the results obtained in this section were already known and written either directly or indirectly. This was not known to me previously and so, the proofs often differ as well. More details on the exact relation to the previous work can be found before each of the mentioned postulates: 1) Proposition 7.4.10; 2) Proposition 7.4.11 i); 3) Theorem 7.4.15; 4) Proposition 7.4.18; 5) Corollary 7.4.25.

We move on to a topic a little removed from the rest. Let $\mathcal{E}$ be an elliptic curve defined over a field K. It is well known that the set $\mathcal{E}(K)$ of K-points of $\mathcal{E}$ has a group structure, denoted by $+$. The operation $+$ is based on a binary operation $*$, called the chord-tangent law [31]. We then fix a point $0 \in \mathcal{E}(K)$, which will become the unit of $+$, and iterate $*$. Even without iterating $*$, the chord-tangent law satisfies some identities, spelled out in [64].

This gives rise to an algebraic structure which shall be referred to as an elliptic group structure (see Definition 9.1.1) in this thesis. Any elliptic group, together with a chosen point 0, gives rise to an abelian group by setting $a+b := 0*(a*b)$. It is well known that the choice of the distinguished point is not of paramount importance, since the corresponding groups are always isomorphic. However, after passing from an elliptic group to the corresponding abelian group, we loose some information since non-isomorphic elliptic groups can have isomorphic associated abelian groups, see Example 9.1.4. Such situations can even be realised geometrically, as the same example indicates. This motivates us to investigate the algebraic properties of elliptic groups.

We will show that elliptic groups are closely related to abelian groups. We will describe free elliptic groups, automorphisms of various elliptic groups and we give a full list of finitely generated indecomposable elliptic groups. We will further show that any finitely generated elliptic group is isomorphic to a product of indecomposable ones. We will also give an explicit

construction of the coproduct of elliptic groups in many cases. Unlike for abelian groups, this will differs from the product.

I should also point out that, open finishing my investigation, I discovered that elliptic groups were studied by other authors under the name "totally symmetric and entropic quasigroups", see [63] and the literature cited therein. Results from Sections 9.1, 9.2 and 9.5 are known [63]. Likewise, Corollary 9.3.3 is also proven in [63]. I included detailed account of these facts for the convenience of the reader, and to have it all expressed in terms of elliptic groups.

The thesis is organised as follows:

Chapter 1

We start with a revision of 2-category theory in Section 1.1, covering topics such as groupoids, which will feature heavily in the latter sections. In particular, in the work around quadratic algebras and the fundamental groupoid of binoid schemes. We then move on to some general talk about the 2-categorical version of limits and colimits in Section 1.2. The aim is to recall the general construction of 2-limits and 2-colimits, as they feature in various parts of the dissertation. Predominantly, they will be of core interest in our discussion about the comparison of the colimit and 2-colimit.

Outside of that, they will also enable us to talk about stacks and costacks. Recall that a stack is the 2-categorical analogue of a sheaf and a costack is, as the name would indicate, its categorical dual. Section 1.3 deals precisely with establishing some basic results about these important objects.

In particular, we will recall the notion of Grothendieck topologies and pretopologies, including associated notions, such as Sieves, in Paragraph 1.3.1. These will play a role in both our discussion on quadratic algebras and topoi.

Moving on in Section 1.4, we discuss the notion of categorical groups and associated constructions. These are central to the first major part of this dissertation: The study of rank 2 torsors of finite flat groups schemes.

In particular, we will recall what 2-exactness means in terms of symmetric categorical groups and the obtains exact sequence of abelian groups, see Formula (1.2).

The results of Section 1.1 are meant to fix notation and to recall some basic concepts. Most of it is well known. It should also be mentioned that the first 2 subsections can already be found in my PhD thesis in a nearly identical form. An exception is Paragraph 1.3.3. Outside of not appearing in my PhD thesis, it might even be new in general.

Chapter 2

With this, we will start with the core part of this habilitation thesis, and talk about new results. The first topic centres around quadratic algebras and its generalisations towards Hopf algebras, in Chapter 2.

The results of this chapter are, in essence, my paper [53]. After a brief introduction, we start our discussion by recalling some well known results about Hopf algebra, including

coalgebras, comodules, cocommutativity, Bialgebras and Galois algebras.

We then specialise to the case of dimension 2. These are all determined by pairs of elements (p, q) such that $pq = -2$, something we recall. Likewise, we discuss under what assumptions on p and q, we obtain bialgebra structures and Hopf algebra structures. We also highlight the standout case $p = -2$ $q = 1$, as this corresponds to the case of quadratic algebras. Though we are exploring a more general setting, the case of quadratic algebras is in many ways the main object of our interest.

As mentioned previously, we wish to do so using stacks and 2-categorical homological algebras. This this end, we are going to introduce categorical groups, whose π_0 and π_1 will agree with the groups on interest in the general theory of quadratic algebras, such as the ones appearing in the classical exact sequence appearing in Hahn's book [26, Statement 12.6, p.178]. The first such categorical group will be $\mathfrak{Qu}_f^{pq}(R)$ in Section 2.3, whose π_0 is the group of isomorphism classes of free quadratic algebras, in the case $p = -2, q = 1$. Its π_1 is $\mathbb{Z}_{pq}(R) := \{r \in R \mid r^2 = qr\}$. Next, we will introduce the simple categorical groups $\mathfrak{G}(R)$ and $\mathfrak{G}^p(R)$, associated to the group homomorphisms $R \xrightarrow{\mathsf{sq}} R$ and $R/pR \xrightarrow{\mathsf{sq}} R/pR$, introduced in Subsections 2.3.1 and 2.3.2 respectively.

Likewise, in the above subsections, we are talking about homomorphisms from and to $\mathfrak{Qu}_f^{pq}(R)$. This allows us to construct a 2-exact sequence, whose associated long exact sequence generalises the known exact sequence on the level of abelian groups. This is discussed in Subsections 2.3.3 and 2.3.4. Before proceeding to the main part of our first topic, being the projective case, we make a little detour in Subsection 2.3.5, to generalise a sequence

$$0 \to \mathsf{V}(R) \xrightarrow{\rho} \mathsf{Qu}_f(R) \xrightarrow{\omega} \mathsf{S}(R) \to 1$$

found in [26, Exercises 6-10, p.41] to arbitrary $J^{p,q}$-Galois algebras $\mathfrak{Qu}_r^{pq}(R)$, for certain types of Galois algebras $J^{p,q}$. As the obtained result is a short exact sequence with $\mathfrak{Qu}_r^{pq}(R)$ being in the middle, it can be helpful for calculations in many situations.

Finally, we will move towards the projective case, introducing the categorical group $\mathfrak{Qu}^{pq}(R)$, but regarding it as a stack. Here, we are exploiting the generally well known idea that projective means "locally free", which in turn implies that it is the stackification of the free case. This is also the case in our setting, as shown in Corollary 2.4.3.

As the penultimate part in this chapter, in Subsections 2.4.3 and 2.4.4, we recall the stack $\mathfrak{Pic}(R)$, being the stackification of $R \to \mathfrak{G}(R)$. Using this, we are able to define the discriminant $\mathfrak{Dis}(R)$ as the 2-kernel of $\mathsf{sq} : \mathfrak{Pic}(R) \to \mathfrak{Pic}(R)$. In order to get an exact sequence, we need a third piece. This is not the 2-kernel of $\mathsf{sq} : \mathfrak{Pic}(R/pR) \to \mathfrak{Pic}(R/pR)$, as might be expected from the free case.

Rather, we need to modify it a little, defining $\mathfrak{Dis}^{pq}(R)$ in Subsection 2.4.5 as the 2-kernel of $\mathsf{sq} : \mathfrak{Pic}(R/pR) \to \mathfrak{Pic}(R/p^2R)$.

All of these enable us to construct our 2-exact sequence in Subsection 2.4.6, whose associated long exact sequence in abelian groups gives us our desired result in Theorem 2.4.8

Chapter 3

Moving on to the next topic, the main aim of Chapter 3 is to compare the colimits and the 2-colimits of strict 2-functors (of groupoids) with each other. As a reminder, a strict 2-functor is one that respects compositions. It can also be thought of as an ordinary functor "lifted up" to a 2-functor. This construction is very much like considering a map between sets as a functor between discrete groupoids (meaning, the only morphisms are identities).

Our motivation for this is to calculate the fundamental groupoid of binoid schemes. We will talk more about monoids/binoids in later subsections. But ultimately, we will wish to to move freely between colimits and 2-colimits, and try to determine when this is possible. This way, one can use the strengths of both construction, - the relative simplicity of colimits, and the freedom of 2-colimits to replace groupoids with equivalent ones.

Of main interest to us is the case when our source 2-category actually comes from a Boolean poset, though we will also mention the case when the source category is coming form a filtered poset.

We start by recalling the construction of limits/2-limits and 2-limit/2-colimits in Section 3.1, but specialised to the simplified setting when the domain is a poset. In this capacity, we will also introduce the Grothendieck construction in Paragraph 3.1.4.1.

The main result of this section is Theorem 3.2.6, which assesses that for punctured Boolean posts (meaning the maximal element is missing), formally being $\mathfrak{B}(\mathbf{n}) \coloneqq \{X \subsetneq \{1,\dots,n\}\}$, the colimit and 2-colimit agree if a certain injectivity condition is satisfied. This is proven through induction, with the induction step being done separately for the colimit and the 2-colimit.

The base step, being essentially the pushout/2-pushout, is done in Section 3.2.1. In this case, the injectivity condition, mentioned above, becomes rather clear: One of the two functors needs to be injective on objects. The injectivity step is done by partitioning the punctured Boolean poset $\mathfrak{B}(\mathbf{n})$ as

$$\mathfrak{B}(\mathbf{n}) = \mathfrak{B}(\mathbf{n-1}) \coprod \mathfrak{B}'(\mathbf{n-1}) \coprod \{\mathbf{n-1}\}.$$

Here, $\mathfrak{B}'(\mathbf{n-1})$ is the set of proper subsets of $\{1,\dots,n\}$, which specifically contain $n-1$ as an element. The others are self explanatory. This itself can be arranged as s poset and the idea then is to use our base step for this pushout/2-pushout.

Actually, we will not work with the pushout/2-pushout in the induction steps in Subsections 3.2.3 and 3.2.4. Rather, we will use the familiar trick of considering the limits/2-limits and than, via the Yoneda Lemma, obtain our desired result for the colimit. As usual, this is done because the limit/2-limit is, as expected, easier to work with.

Having obtained these results, we will prove an analogue of a classical result of category theory, for 2-categories: In Subsection 3.2.5, we will prove that the 2-limit, and consequently the 2-colimit, is determined by the subposet consisting of elements with codimension 3. To make it more explicit, we can truncate our source poset $\mathfrak{B}(\mathbf{n})$ to

$$\mathfrak{B}(n,k) \coloneqq \{X \subsetneq \{1,\dots,n\} \mid \ |X| \geq k\},$$

without changing the 2-limit, and subsequently the 2-colimit. This is, of course, reminiscent to the classical result that says that posets can be truncated to codimension 2, without changing

the limit/colimit.

This has implications for our main result and we simplify it in Proposition 3.2.16.

Lastly, we also consider the filtered case and show that if I is a filtered poset, the colimit and 2-colimit of a functor $\mathfrak{F} : I \to \mathfrak{Cat}$ is an equivalence of categories. This is Proposition 3.2.17 in Section 3.2.6. This result was already published in my PhD thesis. I included it for completeness sake.

As mentioned before, our desire to better understanding the colimit will be clarified in a subsequent chapter. In order to state it, we will need to introduce the second part our our dissertation: the world of monoids and its relation to algebraic geometry, and later topos theory.

Chapter 4

We start Chapter 4 by introducing the notion of a monoid scheme and recalling some core results in Section 4.1. Many of the results of this section can already be found in my PhD thesis, though the formulations are often different. There are, however, a few new results as well.

We will start by considering the affine case in Subsection 4.1.1. Recalling that Spec(M) is a natural semi-lattice, including its topology, we will recall some results about posets, lattices, poset/order topologies and idempotents. Especially Paragraph 4.1.1.3 deals with the structure of prime ideals. We will also recall how localisations work in Paragraph 4.1.1.4 and the structure sheaf in Paragraph 4.1.2.1. Some of our simplifications include reduceing sheaves to covariant functors over M^{sl}, in the finitely generated case. We use the notation M^{sl} for the associated semi-lattice of M. This is the semilattice obtained from M by imposing the relation $m^2 \sim m$ for every $m \in M$.

We move on to Binoids in Section 4.2. These very closely related to monoids, being merely monoids with an absorbing element. This absorbing element is, as may be expected, required to be respected by all of the core constructions: Homomorphisms, ideals and localisations, just to name a few. It is even required to be respected the both realisations. We mentioned one before, being locally given by the monoid/binoid algebras. We can also consider a different type of realisation, one that is locally given by

$$\mathsf{Hom}_{\mathsf{Mn}}(M, R) = \mathsf{Hom}_{\mathsf{R\text{-}Alg}}(R[M], R).$$

Before proceeding with that, we will discuss some of the similarities between binoids and monoids, and mention that we can translate most results without any effort.

Chapter 5

We start our study of binoid schemes in earnest in Chapter 5. The results of this chapter are all part of a joint paper with Prof. Brenner from the University of Osnabrück.

We start with a few technical lemmas and constructions, which will be used in latter parts of the chapter. This includes the introduction of so called pivotal elements in Subsection 5.2.2.

As mentioned above, our aim is to calculate the fundamental groupoid of binoid varieties. We first do this in the affine setting. Then, using our results on 2-colimits, we calculate the fundamental groupoid in the general setting. Even in the affine case, however, we need to build up to the general case. We will actually first start with separated and integral binoids and then, remove both conditions. Pivotal elements play an important role for removing the separatedness condition.

We will proceed to decompose the spectrum of a binoid by its idempotents in Subsection 5.2.4. This decomposition will carry over to the fundamental groupoid. The idea will be to calculate the fundamental groupoid for each of these cosummands separately, and then, obtain the fundamental groupoid as the disjoint union.

Subsection 5.2.5 is where we will use binoids not merely because they are more general. Rather, we will need the added generality to obtain a certain map, which has no counterpart in monoids. We also state one of the central results of this chapter, Theorem 5.2.7.

As mentioned above, we start by proving it in stepwise generalisations. We start with the case of separated and integral binoids in Subsection 5.2.6, and remove these conditions in Paragraphs 5.2.7.1 and 5.2.7.2.

The results mentioned thus far were for realisations over both real and complex numbers. Over the reals, however, we can proceed significantly further. In Section 5.3, we will show that the fundamental groupoid does not change if we tensor our binoid with C_2°. This is the binoid equivalent of the group with 2 elements (though in our case, it has 3 elements, as 0 must be formally attached). Our results of the affine case culminate in Corollary 5.4.1.

In order to understand schemes, morphism naturally play a vital role. As such, we proceed to study them in Subsection 5.4.1, before proceeding to schemes in Section 5.5. The first thing we study in this capacity is a small offshoot of the main part. In Subsection 5.5.2, we consider the homology of $\mathbb{R}X$, reducing its calculation to the affine case.

We then proceed to the main part in Subsection 5.5.3. We are able to calculate the fundamental groupoid of a finitely generated binoid scheme over the real numbers in a nearly algorithmic way. The Key here is our result in the affine case and our work in the previous chapter with 2-colimits.

Finally, we give a few applications in Section 5.6, including using Stanley-Reisner rings to calculate the fundamental groupoid of simplicial sets, using the methods developed in this chapter. We end with a few examples.

Chapter 6

We proceed with an introductory chapter focused around topoi. We state several general results and constructions, which will be used for latter parts of the dissertation.

We start by considering localisations of categories in Subsection 6.1.1. These will play a central role throughout the rest of this thesis. All of the results in this subsection are naturally well known. We then proceed to talk a little about elementary topoi in Subsection 6.1.2, though for us, a topos will generally mean Grothendieck topos. The main reason we introduce elementary topoi is because the subobject classifiers will play an important role for us. As we will also consider the Lawvere-Tierney topologies in our discussion on sieves and topologies, this subsection serves as a general reminder.

We will, of course, also talk about appropriate functors between topoi, be it elementary or Grothendieck's version. These can be found in Subsections 6.1.2 and 6.1.5, respectively.

We proceed with Subsection 6.2. The results of this subsection are presumably also well known, though a source is not known to me. We introduce the notion of so called s-Noetherian topoi, mimicking the well known notion of Noetherian Abelian categories. However, it should be remarked and emphasised that there are two different natural ways to generalise Noetherian Abelian Categories. We have chosen to consider the stabilisation condition of ascending sequences given via subobjects. An alternative way is to consider quotients. These are not equivalent in the world of topoi and the latter is presumably richer. This also explains why we used the qualifier s in front of Noetherian.

In this subsection, we prove some general results about these types of Topoi, about their generators and localisations. Likewise we consider their gluings and show that gluing along localisations preserves s-Noetherianness. This is done in Subsections 6.2.1, 6.2.2. We also consider the centre of these topoi, including its behaviour under gluing and localisations in Subsection 6.2.3.

We need these topoi because they correspond to the category of actions on s-Noetherian monoids (and their gluings). This is initially discussed as an example in Subsection **??**. It is, however, our main motivation for considering s-Noetherian topoi.

Chapter 7

We will systematically discuss the link between monoids and topoi for the first time in this chapter. We start by recalling the well-known fact that the category of M-sets is a natural topos and explore some of the basic topos theoretic constructions of it in Section 7.1. These are generally specialisations of our discussions in Chapter 6 and include such things as topologies, subobject classifiers and localisations. For the latter, we also contrast it with localisations of monoids, which we discussed in Paragraph 4.1.1.4.

We proceed to introduce s-Noetherian monoids in Section **??**. This class of monoids is going to play an important role for us, as we will restrict ourselves to it in some of our main theorems. The acts over such a monoid will form an s-Noetherian topos and among other things, we will discuss its subobject classifier in Subsection 7.2.1.

We proceed to study the topos points of M-sets in Section 7.3. These are some of the most central objects in our study of monoids using topos theory. We first connect these objects with filtered M-sets in Subsection 7.3.1 and using this observation, we give a classification of the topos points of M-sets in certain cases:

At first, we consider idempotents of monoids and show that these define topos points in Subsection 7.3.2. We also characterise when exactly two idempotents induce equivalent topos points in Lemma 7.3.14. Of special interest is this construction to us in the finite case, as in this setting, all topos points of M-sets are induced by idempotents. This is done in Subsection 7.4. It should be pointed out that commutativity is not assumed in these parts.

We also consider an other class of monoids before proceeding to commutative monoids. The upper mentioned class is the case of left-zero monoids. These are fairly simple monoids, which are obtained by sets, where the multiplication is defined as $st = s$ for all elements (save the unit, which we formally adjoin afterwards). This class is studied in Subsection 7.3.3. The

fact that these are binoids will not be emphasised, as we do not consider binoid actions, though most anything obtained here should be freely translatable.

For left-zero monoids, we will also characterise the localisations and the subobject classifiers of the topoi $\mathscr{S}_M$ and ${}_M\mathscr{S}$. Recall that these refer to the topoi of left and right actions, respectively. An other class of monoids that we will explore more in depth are finite monoids. For these, will will give a full description of the localisations of the topos $\mathscr{S}_M$.

The commutative setting will be dealt with at a later point. Idempotents once again make an appearance, especially idempotent ideals in Subsection 7.4.7, where we show that they correspond exactly to the localisations of $\mathscr{S}_M$, for finite M. This is done in Lemma 7.4.17 and Proposition 7.4.18.

We start our study of the commutative case in Subsection 7.5. Here, we first establish a link between the prime ideals of monoids and the points of the topoi of acts over them. Recalling that prime ideals of monoids carry natural monoidal structures (actually more), we start by observing that filtered M-sets (which correspond to topos points) also have a natural monoidal structure in terms of the tensor product. We than show that this structure is preserved when constructing topos points out of prime ideals.

We proceed to consider the case of *almost finitely generated* commutative monoids in Subsection 7.5.2. These are monoids, where the quotient M/M^* is finitely generated. For us, they will be the central objects from now on, as they are the ones we will prove the reconstruction theorem for. We start by establishing certain important facts for the topos points of these types of monoids. In particular, we prove Theorem 7.5.10, which establishes an isomorphism between topos points and prime ideals. More precisely, we prove that the topos points (which are functors $f^* : \mathscr{S}_M \to \mathscr{S}$) are exactly isomorphic to functors of the form $(-)_\mathfrak{p} : \mathscr{S}_M \to \mathscr{S}$, being given by $A \to U(A_\mathfrak{p})$. Here, $U(A_\mathfrak{p})$ denotes regarding the localisation $A_\mathfrak{p}$ as a set (acting with the forgetful functor U).

Recalling that prime ideals of monoids have a natural ordering as well, we proceed to study topos points in this regard. We introduce a construction for general topoi and prove that in the case when our topos comes from $\mathscr{S}_M$, for a commutative monoid M, this construction defines an ordering. If we regard prime ideals as topos points, this ordering agrees with the natural ordering (under inclusion) of prime ideals.

As such, using our results on the prime ideals of finitely generated commutative monoids, we show that this construction on topos points yields the Zariski topology in the case of almost finitely generated commutative monoids.

Though we will restrict ourselves again to the almost finitely generated case when proceeding with schemes, we will make a small excursion towards infinitely generated commutative monoids in Subsection 7.6. Among other things, we will show that some finiteness assumption on M is essential in order to obtain a result like Proposition 7.5.14. In other words, for topos points to be in a one-to-one correspondence with prime ideals.

We show that, already in the free case (in particular, for $M = \mathbb{N}^{(m)}$ the multiplicative monoid of the natural numbers), there are more topos points than prim ideals. This is a direct consequence of Lemma 7.6.9. Moreover, in Theorem 7.6.5 and the subsequent subsection (7.6.4), we give a full description of how topos points of non-finite generated monoids look like.

Chapter 8

We proceed to the penultimate chapter. Here, we wish to prove Theorem 3, which is our version of the reconstruction theorem. To do this, we will start by recalling the definition and basic results on quasi-coherent sheaves and line bundles in Section 8.1. The main result of this section is Theorem 8.1.5, which is part i) of our Theorem 3.

We will move on to study graded monoids in Section 8.2. We will need these for part iv) of the above theorem. As the bijection on the level of points is only shown for quasi-projective monoid schemes, we will need to study such monoid schemes. We will construct them using the theory of graded monoids, hence our study of them. We will introduce and prove basic results on the graded versions of our objects, such as of sets, monoids, maps, products, ideals and even localisations. We will also introduce the Tot functor (meaning total), which will allow us to go back to the world of monoids, when needed.

The quasi-coherent sheaves over quasi-projective monoid schemes $\mathfrak{Qc}(\mathsf{Proj}(M_*))$ form a topos as well, and we shall study this structure in Section 8.3, especially Subsection 8.3.5. The main results here are Theorems 8.3.9 8.3.11. The latter can be seen as a weak version of the reconstruction theorem. The former is part iv) of Theorem 3

We will make a short return to the affine case in Section 8.4. We will start by constructing a Galois connection between the subset of $\mathsf{Spec}(M)$, which is a natural topos (semi-lattice) and the poset of Grothendieck topologies on M, when regarded as a one-object category. This is done in Proposition 8.4.2 and is specialised in Corollary 8.4.7 to the case of open subsets of $\mathsf{Spec}(M)$ and stable topologies. Our work in this subsection culminates with Corollary 8.4.11, where we prove part ii) of our reconstruction theorem, in the affine case.

The remaining parts, being iv), v) and the general case of ii) are proven in Section 8.5.2, especially in Subsection 7.3.9.

Chapter 9

This brings us to our last chapter, which is also rather removed from the rest of the topics discussed in this habilitation thesis. It deals with the natural structure of elliptic curves, called the chord-tangent law. As a reminder, it is through fixing a point and iterating this chord-tangent law that one defines the group structure of elliptic curves. In doing so, however, one looses some information, hence our study of this object. It is also an interesting algebraic structure in its own right.

We will start in Section 9.1 by solving an exercise from the textbook [64], which is the basic motivation to introduce elliptic groups. We will establish a relationship between abelian groups and elliptic groups in Section 9.2. Section 9.3 will focus on examining the morphisms and isomorphisms of elliptic groups. We will proceed in Section 9.4 by studying the case of flex points. That is, when a point o is given such that $o * o = o$.

In the following section, being 9.5, we will equip the set of all homomorphisms $\mathsf{Hom}_{\mathsf{El}}(S, T)$ with the structure of an elliptic curve. In Section 9.6, we will give a brief result about congruences. The proceeding Section 9.7 will give an explicit construction of free elliptic groups.

Section 9.8 will deal with the study of finitely generated elliptic groups and the direct

product decomposition of such objects. In Section 9.9, we will study coproducts and give several explicit constructions. We will then move on to study the automorphism group in Section 9.10 and in Section 9.10, we will consider the tensor product. This will allows us to define elliptic rings, which we will study in more detail in Section 9.12, including the notion of primes and the fundamental theorem of arithmetic. We will finish at that point, having come to the cusp of developing a new flavour of algebraic geometry.

Chapter 1

Preliminary material on 2-category theory and stacks

For the purposes of this habilitation dissertation, we are going to assume the general results of (ordinary) category theory to be well-known. While 2-category theory is also familiar to a fairly broad audience, we will try to give most of the definitions, and at least state the results used. Sections 1.1, 1.2 and 1.3, with the exception of Subsection 1.3.3, can already be found in my PhD thesis [51], verbatum. It has been included for the convenience of the reader.

1.1 2-Categories

1.1.1 Definition and Basic Results on 2-Categories

This subsection (1.1.1), as well as the next (1.3.2), follow the second chapter (Lecture) of [45] very closely, even on notations.

Definition 1.1.1. *A (strict) 2-category* $\mathfrak{C}$ *consists of the following:*

1. *A class of objects, denoted by* $\mathrm{obj}(\mathfrak{C})$*;*
2. *A family of categories* $\mathrm{Hom}_{\mathfrak{C}}(X,Y)$*, indexed by* $\mathrm{obj}(\mathfrak{C}) \times \mathrm{obj}(\mathfrak{C})$
   - *i) objects of* $\mathrm{Hom}_{\mathfrak{C}}(X,Y)$ *are called morphisms of* $\mathfrak{C}$*, or sometimes 1-cells,*
   - *ii) morphisms of* $\mathrm{Hom}_{\mathfrak{C}}(X,Y)$ *are called 2-morphisms of* $\mathfrak{C}$*, or 2-cells;*
3. *A family of functors*

$$\mu_{x,y,z} : \mathrm{Hom}_{\mathfrak{C}}(X,Y) \times \mathrm{Hom}_{\mathfrak{C}}(Y,Z) \to \mathrm{Hom}_{\mathfrak{C}}(X,Z)$$

indexed by $\mathrm{obj}(\mathfrak{C}) \times \mathrm{obj}(\mathfrak{C}) \times \mathrm{obj}(\mathfrak{C})$*, called composition functors, such that the following*

diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{C}}(X,Y)\times\mathrm{Hom}_{\mathfrak{C}}(Y,Z)\times\mathrm{Hom}_{\mathfrak{C}}(Z,T) & \xrightarrow{\mathrm{Id}\times\mu_{Y,Z,T}} & \mathrm{Hom}_{\mathfrak{C}}(X,Y)\times\mathrm{Hom}_{\mathfrak{C}}(Y,T) \\ \downarrow{\scriptstyle \mu_{X,Y,Z}\times\mathrm{Id}} & & \downarrow{\scriptstyle \mu_{X,Y,T}} \\ \mathrm{Hom}_{\mathfrak{C}}(X,Z)\times\mathrm{Hom}_{\mathfrak{C}}(Z,T) & \xrightarrow[\mu_{X,Z,T}]{} & \mathrm{Hom}_{\mathfrak{C}}(X,T) \end{array}$$

is commutative for all 4-tuples (X, Y, Z, T) *of objects of* $\mathfrak{C}$*;*

4. *For each object* X *of* $\mathfrak{C}$*, there is an object* i_X *of* $\mathrm{Hom}_{\mathfrak{C}}(X,X)$ *such that the following diagrams*

$$\begin{array}{ccc} \mathbf{1}\times\mathrm{Hom}_{\mathfrak{C}}(X,Y) & \xrightarrow{i_X\times\mathrm{Id}} & \mathrm{Hom}_{\mathfrak{C}}(X,X)\times\mathrm{Hom}_{\mathfrak{C}}(X,Y) \\ & {\scriptstyle =}\searrow \quad \swarrow{\scriptstyle \mu_{X,X,Y}} & \\ & \mathrm{Hom}_{\mathfrak{C}}(X,Y) & \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{C}}(Y,X)\times\mathbf{1} & \xrightarrow{\mathrm{Id}\times i_X} & \mathrm{Hom}_{\mathfrak{C}}(Y,X)\times\mathrm{Hom}_{\mathfrak{C}}(X,X) \\ & {\scriptstyle =}\searrow \quad \swarrow{\scriptstyle \mu_{Y,X,X}} & \\ & \mathrm{Hom}_{\mathfrak{C}}(Y,X) & \end{array}$$

commute. Here, $\mathbf{1}$ *denotes the category with one object and only the identity morphism and* $i_X : 1 \to \mathrm{Hom}_{\mathfrak{C}}(X,X)$ *is the unique functor which sends the object* $\mathbf{1}$ *to* Id_X*.*

Whenever any one of our arrows is invertible, we add the prefix iso. For example, an invertible morphism (i.e. object of $\mathrm{Hom}_{\mathfrak{C}}(X,Y)$) is called an *isomorphism*, an invertible 2-morphism a *2-isomorphism*, and so on.

Definition 1.1.2. *A morphism* $F : A \to B$ *in a 2-category is called an* equivalence*, if there exists a morphism* $G : B \to A$ *and two 2-isomorphisms* $f : F \circ G \to \mathrm{Id}_B$ *and* $g : G \circ F \to \mathrm{Id}_A$*.*

Note that, there exists a more general version of 2-categoris. In this, composition need not be associative, but only associative up to a 2-isomorphism, which satisfies a coherence condition. Since we will only deal with strict 2-categories, we will not define the more general version. Indeed, throughout this dissertation, we will usually try to define the least general case that we need.

Definition 1.1.3. *Let* $\mathfrak{C}$ *and* $\mathfrak{D}$ *be two 2-categories. We say* $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ *is a (covariant)* 2-functor *if*

- $\mathfrak{F}$ *takes objects to objects, morphisms to morphisms and 2-morphisms to 2-morphisms.*

- *For all chains $A \xrightarrow{i} B \xrightarrow{j} C$ in $\mathfrak{C}$, we have a 2-isomorphism $\tau_{i,j} : j_* i_* \Rightarrow (ji)_*$, where i_* denotes $\mathfrak{F}(i)$ such that for all triple compositions $A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} D$, the diagram*

$$\begin{array}{ccc} (kji)_* & \xRightarrow{\tau_{k,ji}} & k_*(ji)_* \\ \Downarrow{\scriptstyle \tau_{i,kj}} & & \Downarrow{\scriptstyle k_*\tau_{i,j}} \\ (k_* i_*) & \xRightarrow{\tau_{j,k}i^*} & k^* j^* i^* \end{array}$$

  commutes. We call this the coherent compatibility condition *.*

- *On 2-morphisms we require that $\mathfrak{F}$ respects the composition.*

Definition 1.1.4. *A 2-functor $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ is said to be* strict, *if for all $i, j \in I$, the isomorphism $\tau_{i,j}$ is the identity.*

If a strict 2-functor is invertible, that is, if we have $\mathfrak{G} : \mathfrak{D} \to \mathfrak{C}$ with $\mathfrak{F} \circ \mathfrak{G} = \mathrm{Id}$ and $\mathfrak{G} \circ \mathfrak{F} = \mathrm{Id}$, then $\mathfrak{F}$ is called a 2-*isomorphism*. In this case, we say that $\mathfrak{C}$ and $\mathfrak{D}$ are 2-*isomorphic*.

Remark 1.1.5. *It should be pointed out, that we use the term 2-isomorphism for two different concepts. However, since there can be no ambiguity from the context, we will keep this terminology for simplicity.*

Definition 1.1.6. *Let $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ and $\mathfrak{G} : \mathfrak{C} \to \mathfrak{D}$ be two 2-functors. A* natural transformation *$\phi : \mathfrak{F} \Rightarrow \mathfrak{G}$ of 2-functors is the following data:*

- *A morphism $\phi_U : \mathfrak{F}(U) \to \mathfrak{G}(U)$ for each $U \in \mathrm{obj}(\mathfrak{C})$,*
- *A 2-isomorphism $\alpha_i : i_* \phi_V \Rightarrow \phi_U i_*$ for each $i : V \to U$, i.e.*

$$\begin{array}{ccc} \mathfrak{F}(V) & \xrightarrow{\phi_V} & \mathfrak{G}(V) \\ {\scriptstyle i_*}\downarrow & \swarrow{\scriptstyle \alpha_i} & \downarrow{\scriptstyle i_*} \\ \mathfrak{F}(U) & \xrightarrow[\phi_U]{} & \mathfrak{G}(U), \end{array}$$

such that for each composition $W \xrightarrow{j} V \xrightarrow{i} U$, the diagram

$$\begin{array}{ccccc} & & (ij)_*\phi_W & \xRightarrow{\alpha_{ij}} & \phi_U(ij)_* \\ & \nearrow{\scriptstyle \tau_{j,i}\phi_W} & & & \nwarrow{\scriptstyle \phi_U\tau_{j,i}} \\ i_*j_*\phi_W & & \xRightarrow{i_*\alpha_j} & i_*\phi_V j_* & \xRightarrow{\alpha_i j_*} & \phi_U i_* j_* \end{array}$$

commutes.

An invertible natural transformation is called a *natural isomorphism*.

Definition 1.1.7. *Let $\phi, \psi : \mathfrak{F} \Rightarrow \mathfrak{G}$ be two natural transformations of two 2-functors. A* natural 2-transformation *$\mu : \phi \to \psi$, also known as a fibred transformation or modification, consists of 2-morphisms*

$$\mu_U : \phi_U \Rightarrow \psi_U$$

for each $U \in \mathrm{obj}(\mathfrak{C})$, such that for every $i : V \to U$, the diagram

$$\begin{array}{ccc} i_*\phi_V & \xRightarrow{\alpha_i} & \phi_U i_* \\ {\scriptstyle i_*\mu_V}\Downarrow & & \Downarrow{\scriptstyle \mu_U i_*} \\ i_*\psi_V & \xRightarrow[\beta_i]{} & \psi_U i_* \end{array}$$

commutes.

Following the notations, if μ is invertible, we will call it a *natural 2-isomorphism*.

Definition 1.1.8. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two 2-categories and $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ a 2-functor between them. We say that $\mathfrak{F}$ is a 2-*equivalence*, if there exists a 2-functor $\mathfrak{G} : \mathfrak{D} \to \mathfrak{C}$, and two natural isomorphisms $\phi : \mathfrak{F} \circ \mathfrak{G} \Rightarrow \mathrm{Id}$, $\varphi : \mathfrak{G} \circ \mathfrak{F} \Rightarrow \mathrm{Id}$. If that is the case, $\mathfrak{C}$ and $\mathfrak{D}$ are said to be 2-*equivalent.

Clearly, if ϕ and φ can be taken to be the identity, we get our definition of a 2-isomorphism, below Definition 1.1.3.

We have the following well-known (see for example in [40, Prop. 1.5.13]) result.

Proposition 1.1.9. *The 2-functor $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ is a 2-equivalence of 2-categories if and only if the following two conditions hold:*

- *For every $A, A' \in \mathrm{obj}(\mathcal{A})$ we have an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(\mathfrak{F}(A), \mathfrak{F}(A'));$$

- *For every object $B \in \mathcal{B}$, there exists an object $B' \in \mathcal{B}$ such that B' is equivalent to B and B' is in the image of $\mathfrak{F}$.*

When $\mathfrak{F}$ satisfies the first condition, we say that $\mathfrak{F}$ is *full and faithful*. When it satisfies the second, we call $\mathfrak{F}$ *essentially surjective*.

1.1.2 Groupoids

Definition 1.1.10. *Let $\mathfrak{G}$ be a category. We say that $\mathfrak{G}$ is a* groupoid*, if every morphism in $\mathfrak{G}$ is an isomorphism.*

Note, however, that not every object is isomorphic. If $x, y \in \mathfrak{G}$ are isomorphic, we also say that they are connected. Being isomorphic (or connected) is an equivalence relation. The set of equivalence classes of a small groupoid is a denoted by $\pi_0(\mathfrak{G})$. In a similar vein, we use the notation $\pi_1(\mathfrak{G}, x)$ for $\mathrm{Aut}(x)$, $x \in \mathfrak{G}$.

The following important example explains the above notations:

1.1.2.1 The fundamental groupoid

Let X be a topological space. Define $\Pi_1(X)$ to be the groupoid whose objects are the points of X and for every $x, y \in \Pi_1(X)$, we set $\mathrm{Hom}(x, y)$ to be the homotopy classes of paths from x to y. It is clear that this defines a groupoid since if there exists a path $\gamma : x \to y$, there exists a path $\gamma^{-1} : y \to x$. It is immediate that $\pi_0(\Pi_1(X)) = \pi_0(X)$ is the set of connected components of X and for any object/point x, the automorphisms of x are exactly the homotopy classes of loops at x. In other words, we have $\pi_1(\Pi_1(X), x) = \pi_1(X, x)$. It is now clear why this groupoid is called the *fundamental groupoid*.

Indeed, the fundamental groupoid can be seen as 'the collection of fundamental groups at every connected component'. But, $\Pi_1(X)$ has an other advantage over $\pi_1(X, x)$ than just generality: The fundamental groupoid is not dependant on a basepoint $x \in X$. As such, it has far nicer categorical properties, see Theorem 1.3.15 below.

1.1.2.2 Connected Groupoids

Let G be a group. We can define a groupoid $\mathfrak{G}$, which only has one object x, and whose endomorphisms (speak automorphisms) are exactly G. Composition is given by the group law. As such, groupoids generalise groups in a very natural way. Indeed, one can think of a group as a groupoid with only a single connected component due to the following simple, and well-known lemma.

Lemma 1.1.11. *Let $\mathfrak{G}$ be a groupoid such that for every pair of objects $x, y \in \mathfrak{G}$, there exists a morphism $\phi : x \to y$. Then $\mathfrak{G}$ is equivalent to a groupoid coming from a group.*

A groupoid satisfying the condition of Lemma 1.1.11 is called a *connected groupoid*.

Like with any category, if we take a groupoid $\mathcal{G}$ and take its skeleton (that is we choose representatives of isomorphism classes of objects), we will get an equivalent (but not isomorphic) groupoid $\mathcal{G}'$. In this way, we can think of a groupoid as a *disjoint collection of groups*. More precisely, every groupoid $\mathcal{G}$ is equivalent to the 2-coproduct (see section 1.2.4 below) $\coprod_i \underline{G_i}$ of groupoids coming from groups.

Reversely, we can also add isomorphic objects to our groupoid without fundamentally changing it. Using this, we can make a functor injective on objects, which will be useful to calculate 2-colimits, see Theorem 3.2.6.

A connected groupoid such that $\mathrm{Aut}(x)$ is trivial is called a *trivial* or *simply connected* groupoid. It is equivalent to the groupoid denoted by $\mathbf{1}$, which has only one object and only the identity morphism.

1.1.3 Discrete Groupoids

We say that a groupoid is *discrete*, if for every object x of our groupoid, $\mathrm{Aut}(x)$ only has the identity morphism. These groupoids can be thought of as coming from sets and will play a vital part in calculating the fundamental groupoid $\Pi_1(\mathbb{R}X)$. Using these, we defined the following types of coverings. We will only give them for topological spaces, as it is all we will need in this thesis. However, it is clear, that they can be generalised to sites.

Definition 1.1.12. *Let X be a topological space. An open covering $\mathcal{U} := \{U_i\}$ of X is said to be* discrete *(or* discrete of depth 3 *to be more precise) if for every i, j, k, the following fundamental groupoids are discrete groupoids:*

$$\Pi_1(U_i), \Pi_1(U_{ij}) \text{ and } \Pi_1(U_{ijk}).$$

Here, we use the shorthand notation $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

1.2 2-Limits and 2-Colimits

Just like the limit and colimit of a functor are of fundamental importance in category theory, so too are the 2-limit and 2-colimit of a 2-functor in 2-category theory. If the 2-functor is strict, however, we can also talk about its limit and colimit.

Analogous to how a set S can be seen as a category $\mathcal{S}$, whose objects are the elements of S and the only morphisms are identities, a category can be 'lifted' to a 2-category. Here, the objects and morphisms remain the same and the 2-morphisms are just taken to be the identities. Hence, we can talk about 2-functors between categories and 2-categories.

The aim of this section is to give the definitions and constructions of the various limits of 2-categories. Note that, just like in the last subsection, we will not define the most general cases, but only up to the generality we will actually need. Most notably, this means that the source of our functors will be categories, rather then 2-categories.

We will give the definitions, as well as constructions, of the limit/2-limit and the colimit of 2-functors. However, we will not give the general construction for the 2-colimit. The reason is that it is not an easy construction, and as we will show in Subsection 3, we will not actually need it for our purposes.

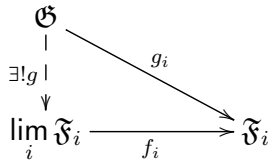
We will now fix the notations, that will be used throughout this section. Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a covariant 2-functor from the category I to the 2-category $\mathfrak{Cat}$ of categories. For an element $i \in I$, let $\mathfrak{F}_i$ be the value of $\mathfrak{F}$ at i. For a morphism $\psi : i \to j$, let $\psi_* : \mathfrak{F}_i \to \mathfrak{F}_j$ be the induced functor. For any chain $i \xrightarrow{\psi} j \xrightarrow{\nu} k$, one has the natural transformation $\mu_{\psi,\nu} : \nu_*\psi_* \Rightarrow (\nu\psi)_*$, satisfying the coherence condition.

1.2.1 Limits of Categories

Definition 1.2.1. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a strict 2-functor. The limit of $\mathfrak{F}$ is a category $\varprojlim_i \mathfrak{F}_i$, together with a collection of morphisms $f_i : \varprojlim_i \mathfrak{F}_i \to \mathfrak{F}_i$ such that $\psi_* \circ f_i = f_j$. This data has to satisfy the following property:*

- *For any category $\mathfrak{G}$ with $g_i : \mathfrak{G} \to \mathfrak{F}_i$ satisfying $\psi_* \circ g_i = g_j$, there is a unique morphism*

$g : \mathfrak{G} \to \lim_i \mathfrak{F}_i$, *such that for all* i, *the diagram*



commutes.

Proposition 1.2.2. *Limits exist and are unique up to isomorphisms. Objects of the category* $\lim_i \mathfrak{F}_i$ *are families* (x_i), *where* x_i *is an object of the category* $\mathfrak{F}_i$, *such that for any map* $\psi : i \to j$, *one has* $\psi_*(x_i) = x_j$. *A morphism* $(x_i) \to (y_i)$ *is a family* (f_i), *with* $f_i : x_i \to y_i$ *a morphism of* $\mathfrak{F}_i$, *such that for any* $\psi : i \to j$, *one has* $\psi_*(f_i) = f_j$.

This is well known, see for example [21, p.5]. To define limits for other 2-categories, such as the 2-category of groupoids, one only needs to consider the uniqueness condition in said 2-category. Alternatively, one can define limits as follows:

Definition 1.2.3. *Let* $\mathfrak{F} : I \to \mathcal{A}$ *be a strict covariant 2-functor in a general 2-category* $\mathcal{A}$. *The limit of* $\mathfrak{F}$ *is the following:*

- *An object* $\lim_i \mathfrak{F}_i$ *in* $\mathcal{A}$, *together with morphisms* $f_i : \lim_i \mathfrak{F}_i \to \mathfrak{F}_i$, *satisfying* $f_i \circ \psi_* = f_j$;
- *For all objects* $A \in \mathcal{A}$, *the functor*

$$\mathrm{Hom}_{\mathcal{A}}(A, \lim_i \mathfrak{F}_i) \to \lim_i \mathrm{Hom}_{\mathcal{A}}(A, \mathfrak{F}_i),$$

  given by $\chi \mapsto (\chi \circ f_i)$. *Where* $\chi \in \mathrm{Hom}_{\mathcal{A}}(A, \lim_i \mathfrak{F}_i)$, *is an isomorphism of categories.*

It is straightforward to see that when $\mathcal{A} = \mathfrak{C}$, these two definitions coincide.

An other way of saying that is as follows: For a strict 2-category $\mathcal{A}$ and $\mathcal{F} : I^{op} \to \mathcal{A}$ a strict 2-functor, we can consider the strict 2-functor $\mathcal{A} \to \mathcal{C}$ given by

$$A \mapsto \lim_i \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}(i)).$$

If this strict 2-functor is representable, the representable object is called the *limit* of $\mathcal{F}$. It is denoted by $\lim_i \mathcal{F}$. Thus,

$$\mathrm{Hom}_{\mathcal{A}}(A, \lim_i \mathcal{F}(i)) \simeq \lim_i \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}_i).$$

It is a straightforward exercise to check that the category $\lim_i \mathcal{F}$ we constructed is indeed the limit with this definition as well.

1.2.2 2-Limits of Categories

Definition 1.2.4. *The 2-limit of $\mathfrak{F} : I \to \mathfrak{Cat}$ is a category $2\text{-}\varprojlim_i \mathfrak{F}_i$, together with a family of functors $(f_i : 2\text{-}\varprojlim_i \mathfrak{F}_i \to \mathfrak{F}_i)_{i \in \mathsf{obj}(I)}$ and natural transformations*

$$(\zeta_\psi : \psi_* \circ f_i \Rightarrow f_j)_{\psi \in \mathsf{Hom}_I(i,j)},$$

satisfying the coherence condition for any composable maps $i \xrightarrow{\psi} j \xrightarrow{\nu} k$ in I. In other words, $\zeta_\nu \zeta_\psi \mu_{\psi,\nu} = \zeta_{\nu\psi}$. These data must satisfy the following properties:

- *For any category $\mathfrak{G}$ with $g_i : \mathfrak{G} \to \mathfrak{F}_i$ and compatible $\eta_\psi : \psi_* \circ g_i \Rightarrow g_j$, we have a functor $a : \mathfrak{G} \to 2\text{-}\varprojlim_i \mathfrak{F}_i$ and a natural isomorphism $a_i : g_i \Rightarrow f_i \circ a$ such that the diagram*

$$\begin{array}{ccc} \psi_* \circ g_i & \xRightarrow{\eta_\psi} & g_j \\ {\scriptstyle \psi \circ a_i}\Downarrow & & \Downarrow{\scriptstyle a_j} \\ \psi_* \circ f_i \circ a & \xRightarrow[\zeta_\psi \circ a]{} & f_j \circ a \end{array}$$

  commutes.

- *If there is given a category $\mathfrak{G}$ and two functors $a, b : \mathfrak{G} \to 2\text{-}\varprojlim_i \mathfrak{F}_i$, then for any collection of natural transformations*

$$\alpha_i : f_i \circ a \Rightarrow f_i \circ b,$$

  for which the diagram

$$\begin{array}{ccc} \psi_* \circ f_i \circ a & \xRightarrow{\psi_*(\alpha_i)} & \psi_* \circ f_i \circ b \\ {\scriptstyle \zeta_\psi \circ a}\Downarrow & & \Downarrow{\scriptstyle \zeta_\psi \circ b} \\ f_j \circ a & \xRightarrow[\alpha_j]{} & f_j \circ b \end{array}$$

  commutes, there exists a unique natural transformation $\alpha : a \Rightarrow b$ such that $\alpha_i = f_i \alpha$.

Proposition 1.2.5. *2-Limits exist in $\mathfrak{Cat}$ and*

- *Objects of the category $2\text{-}\varprojlim_i \mathfrak{F}_i$ are collections (x_i, ξ_ψ), where x_i is an object of $\mathfrak{F}_i$, while $\xi_\psi : \psi_*(x_i) \to x_j$ for $\psi : i \to j$ is an isomorphism of the category $\mathfrak{F}_j$, satisfying the* 1-cocycle condition. *That is, for any $i \xrightarrow{\psi} j \xrightarrow{\nu} k$, the diagram:*

$$\begin{array}{ccc} \nu_*(\psi_*(x_i)) & \xrightarrow{\nu_*(\xi_\psi)} & \nu_*(x_j) \\ \downarrow{\scriptstyle \mu_{\psi,\nu}} & & \downarrow{\scriptstyle \xi_\nu} \\ (\nu\psi)_*(x_i) & \xrightarrow[\xi_{\psi\nu}]{} & x_k \end{array}$$

  commutes.

- *Morphisms from (x_i, ξ_ψ) to (y_i, η_ψ) are collections of morphisms $(f_i : x_i \to y_i)$ such that for any $\psi : i \to j$, the following is a commutative diagram:*

$$\begin{array}{ccc} \psi_*(x_i) & \xrightarrow{\xi_\psi} & x_j \\ {\scriptstyle \psi_*(f_i)}\downarrow & & \downarrow{\scriptstyle f_j} \\ \psi_*(y_i) & \xrightarrow[\eta_\psi]{} & y_j. \end{array}$$

This result is well known and is a particular case of a far more general construction given in [70]. Comparing this construction to the one given for limits shows us immediately that the following holds:

Lemma 1.2.6. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a strict 2-functor. The functor*

$$\gamma : \lim \mathfrak{F} \to 2\text{-}\lim \mathfrak{F},$$

given by

$$\gamma(x_i) = (x_i, \mathsf{Id}),$$

is full and faithful.

Remark 1.2.7. *Let $f : A \to B$ and $g : B \to C$ be two morphisms in the 2-limit of $\mathfrak{F} : I \to \mathfrak{Cat}$. The composition is defined componentwise. To see that, consider the following commutative diagram:*

$$\begin{array}{ccc} \psi_*(A_i) & \xrightarrow{\alpha_\psi} & A_j \\ {\scriptstyle \psi_*(f_i)}\downarrow & & \downarrow{\scriptstyle f_j} \\ \psi_*(B_i) & \xrightarrow{\beta_\psi} & B_j \\ {\scriptstyle \psi_*(g_i)}\downarrow & & \downarrow{\scriptstyle g_j} \\ \psi_*(C_i) & \xrightarrow{\gamma_\psi} & C_j. \end{array}$$

The functoriality of ψ_*, being the induced functor of $\psi : i \to j$, yields $\psi_*(g_i)\psi_*(f_i) = \psi_*(g_i f_i)$. Hence, $(g_i f_i)$ defines a morphism in the 2-limit of $\mathfrak{F}$, which we denote by gf.

Remark 1.2.8. *Let (x_i, ξ_ψ) and (y_i, ν_ψ) be objects in the 2-limit of $\mathfrak{F} : I \to \mathfrak{Cat}$. From Proposition 1.2.5, we immediately see that we have the following exact sequence of sets:*

$$\mathsf{Hom}_{2\text{-}\lim_i \mathfrak{F}_i}(x, y) \longrightarrow \prod_i \mathsf{Hom}_{\mathfrak{F}_i}(x_i, y_i) \rightrightarrows \prod_{\psi : i \to j} \mathsf{Hom}_{\mathfrak{F}_j}(\psi_*(x_i), y_j).$$

Based on the above, we can define the 2-limit of general 2-cateogories as follows:

Definition 1.2.9. *Let $\mathfrak{F} : I \to \mathcal{A}$ be a covariant 2-functor in a general 2-category $\mathcal{A}$. The 2-limit of $\mathfrak{F}$ is the following data:*

- *An object $2\text{-}\lim_i \mathfrak{F}_i$ with morphisms $f_i : 2\text{-}\lim_i \mathfrak{F}_i \to \mathfrak{F}_i$ and 2-morphisms $\zeta_\psi : f_i \circ \psi_* \Rightarrow f_j$, satisfying the coherence condition, meaning $\zeta_\nu \zeta_\psi \mu_{\psi,\nu} = \zeta_{\nu\psi}$;*

- *Additionally we require that for all objects $A \in \mathcal{A}$, the functor*

$$\mathrm{Hom}_{\mathcal{A}}(A, 2\text{-}\varprojlim_i \mathfrak{F}_i) \to 2\text{-}\varprojlim_i \mathrm{Hom}_{\mathcal{A}}(A, \mathfrak{F}_i)$$

given by $\kappa : \chi \to (\chi \circ f_i, \chi \circ \zeta_\phi)$, where $\chi \in \mathrm{Hom}_{\mathcal{A}}(A, 2\text{-}\varprojlim_i \mathfrak{F}_i)$, is an equivalence of categories.

It can be shown that this definition agrees with the above definition in the case when $\mathcal{A}$ is the 2-category of categories. We have the following fact which is straightforward to check:

Proposition 1.2.10. *The assignment*

$$\mathfrak{L} : \{\textit{2-Functors over } I \textit{ with values in categories}\} \to \mathfrak{Cat}$$

defined in the following way:

i) $\{\mathfrak{F} : I \to \mathfrak{Cat}\} \mapsto 2\text{-}\varprojlim \mathfrak{F}$ *on objects;*

ii) $\{\varphi : \mathfrak{F} \to \mathfrak{G}\} \mapsto \{\varphi' : 2\text{-}\varprojlim \mathfrak{F} \to 2\text{-}\varprojlim \mathfrak{G}\}$ *given by*

$$x = (x_i, \xi) \mapsto (\varphi(x_i), \varphi(\xi)), \quad \{f : x \to y\} = (f_i) \mapsto (\varphi(f_i))$$

on morphisms, and

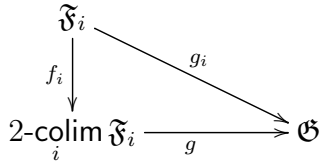
iii) $\{\lambda : \varphi \Rightarrow \phi\} \mapsto \{\lambda' : \varphi' \Rightarrow \phi'\}$ *given by* $\lambda'(\varphi'(x)) = \lambda(\varphi(x_i), \varphi(\xi))$ *on 2-morphisms,*

defines a 2-functor.

1.2.3 Colimits of Categories

Definition 1.2.11. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a strict 2-functor. The colimit of $\mathfrak{F}$ is a category $2\text{-}\varinjlim_i \mathfrak{F}_i$, together with morphisms $f_i : \mathfrak{F}_i \to 2\text{-}\varinjlim_i \mathfrak{F}_i$ such that $f_i = f_j \circ \psi_*$. This data must satisfy the following property:*

- *For any other category $\mathfrak{G}$ with $g_i : \mathfrak{F}_i \to \mathfrak{G}$ satisfying $g_i = g_j \circ \psi_*$, we have a unique morphism $g : \varinjlim_i \mathfrak{F}_i \to \mathfrak{G}$ such that for all i, the diagram*



commutes.

Alternatively we can define the colimit as follows:

Definition 1.2.12. *Let $\mathfrak{F} : I \to \mathcal{A}$ be a strict 2-functor and $\mathcal{A}$ a general 2-category. The colimit of $\mathfrak{F}$ is:*

- *An object* $\operatorname{colim}_i \mathfrak{F}_i$ *of* $\mathcal{A}$*, together with a collection of functors*

$$f_i : \mathfrak{F}_i \to \operatorname{colim}_i \mathfrak{F}_i$$

such that $f_i = f_j \circ \psi_*$.

- *Additionally, one requires that for any object* G *of* $\mathcal{A}$*, the canonical functor*

$$c : \mathrm{Hom}_{\mathcal{A}}(\operatorname{colim}_i \mathfrak{F}_i, G) \to \lim_i \mathrm{Hom}_{\mathcal{A}}(\mathfrak{F}_i, G),$$

given by $c(\chi) = (\chi \circ f_i)$*, is an isomorphism of categories. Here,* $\chi : \operatorname{colim}_i \mathfrak{F}_i \to G$ *is is a morphism in* $\mathcal{A}$*,*

It is well known that colimits exist in the 2-categories of groupoids and categories [21, p. 36].

1.2.3.1 Construction in small categories

The construction of the colimit of a diagram of small categories, is quite a bit harder then the limit. First, we will need to define what a graph is, then take the colimit of graphs, make a free category out of that obtained graph and finally quotient the obtained category in such a way, that compositions are respected. The following construction is, of course, well known. See for example [21, p.4].

Definition 1.2.13. *Let* A^0 *and* A^1 *be sets and*

$$A^1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} A^0 \xrightarrow{\ i\ } A^1$$

a diagram. We say that it is a graph *if* $d^0 \circ i = d^1 \circ i = \mathrm{Id}_{A^1}$*, and denote it with* A^* *for short.*

Elements of A_0 are called objects, while elements of A_1 are called arrows. For an arrow $f \in A_1$, $d^0(f)$ and $d^1(f)$ are called the domain and range of f. For an object $a \in A_0$, the arrow $i(a)$ is called the identity arrow of a.

Definition 1.2.14. *Let* A^* *and* B^* *be graphs. A morphism of graphs* $f^* : A^* \to B^*$ *is defined to be a pair of maps* $f^0 : A^0 \to B^0$ *and* $f^1 : A^1 \to B^1$ *such that*

- $f^0 \circ d^i = d^i \circ f^1$ *for* $i = 0, 1$ *and*
- $i \circ f^1 = f^0 \circ i$.

Hence we can talk about the category of graphs, which we denote by Graphs. There is an obvious forgetful functor $U : \mathfrak{Cat} \to$ Graphs, which assigns to a small category C the sets C^0, C^1 of objects and arrows. The forgetful functor has a left adjoint functor $F :$ Graphs $\to \mathfrak{Cat}$ given by assigning to a graph A^* the free category generated by A^*.

Objects of $F(A^*)$ are the elements of A^0. For two objects $a, b \in A^0$, a morphism from a to b is defined to be a sequence $(f^1, \ldots, f^k)$, where $f^1, \ldots, f^k \in A^1, d^0(f^k) = a, d_1(f^1) = b$ and $d^1(f^i) = d^0(f^{i-1}), 1 \leq i \leq k$. Moreover, if $k > 1$, then $f^i \in A^1 \setminus i(A^0), i = 1, \ldots, k$. Composition of $(f^1, \cdots, f^k)$ and $(g^1, \cdots, g^n)$ is defined in the obvious way if $k, n \geq 2$. If $k = 1$ and $f^1 = i(a)$ or $n = 1$ and $g^1 = i(b)$, then the composition is $(g^1, \cdots, g^n)$ or $(f^1, \cdots, f^k)$ respectively.

Let $\mathfrak{G} : I \to$ Graphs be a functor. For any $i \in I$, one has a graph $\mathfrak{G}_i$ with set of objects $\mathfrak{G}_i^0$ and set of arrows $\mathfrak{G}_i^1$. In this way, one obtains two functors

$$\mathfrak{G}^0, \mathfrak{G}^1 : I \to \text{Sets}$$

and natural transformations

$$\mathfrak{G}^1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} \mathfrak{G}^0 \xrightarrow{i} \mathfrak{G}^1 .$$

Now passing to the colimits, we obtain a diagram of sets

$$\operatorname{colim} \mathfrak{G}^1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} \operatorname{colim} \mathfrak{G}^0 \xrightarrow{i} \operatorname{colim} \mathfrak{G}^1 ,$$

which is a graph, and it is the colimit in the category Graphs. Assume $\mathfrak{F} : I \to \mathfrak{Cat}$ is a functor. To take the colimit of $\mathfrak{F}$, we first take the colimit of $\mathfrak{G} :\equiv U \circ \mathfrak{G} : I \to$ Graphs and obtain

$$\mathfrak{G}^1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} \mathfrak{G}^0 \xrightarrow{i} \mathfrak{G}^1 .$$

Denote the graph by B. We can form a free category $F(B)$. For any i, we have a morphism of graphs $\alpha_i : U(\mathfrak{F}_i) \to F(B)$. Note that every category can be seen as a graph by forgetting the composition requirement. The colimit of $\mathfrak{F} : I \to \mathfrak{Cat}$ is the quotient of $F(B)$ by the minimal congruence on $F(B)$ under which $\alpha_i(g) \circ \alpha_i(f) \sim \alpha_i(g \circ f)$, for all $i \in I$ and all composable morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathfrak{F}_i$.

1.2.3.2 The Filtered Case

Recall that a non-empty category I is called *filtered*, if for every two objects $i, j \in I$, there exists an object $k \in I$ and arrows $i \to k \leftarrow j$. Furthermore, for every pair of parallel arrows $f, g : i \to j$, there needs to exist an object k and an arrow $h : j \to k$ such that $hg = hf$.

Let I be a filtered category and $\mathfrak{F} : I \to \mathfrak{Cat}$ a strict 2-functor. For any $i \in Obj(I)$, any object $A \in Obj(\mathfrak{F}_i)$ defines an object $[A]$ in $\operatorname{colim} \mathfrak{F}_i$. For $B \in Obj(\mathfrak{F}_i)$, one has $[A] = [B]$ if and only if there are morphisms $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$ such that $\alpha_*(A) = \beta_*(B)$ in $\mathfrak{F}_k$. Any morphism $f : A \to A'$ of the category $\mathfrak{F}_i$ defines the morphism

$$[f] : [A] \to [A'].$$

For a morphism $g : B \to B'$ of the category $\mathfrak{F}_j$, one has $[f] = [g]$, if and only if there are

morphism $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$ such that

$$\alpha_*(A) = \beta_*(B), \alpha_*(A') = \beta_*(B'), \alpha_*(f) = \beta_*(g).$$

If $f : A \to A'$ is a morphism in $\mathfrak{F}_i$ and $g : B \to B'$ is a morphism in F_j such that $[A'] = [B']$, then the composite $[g] \circ [f] : [A] \to [B']$ is defined as follows: Choose morphisms $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$ such that $\alpha_*(A') = \beta_*(B)$. Then

$$[g] \circ [f] := [\beta_* g \circ \alpha_* f].$$

1.2.4 2-Colimits of Categories

Definition 1.2.15. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor. The 2-colimit of $\mathfrak{F}$ is the following:*

- *A category $2\text{-}\operatorname{colim}_i \mathfrak{F}_i$, together with a family of functors*

  $$\alpha_i : \mathfrak{F}_i \to 2\text{-}\operatorname{colim}_i \mathfrak{F}_i$$

  and natural transformations $\lambda_\psi : \alpha_j \psi_* \Rightarrow \alpha_i$, satisfying the following condition: For any $i \xrightarrow{\psi} j \xrightarrow{\nu} k$, the following diagram

  $$\begin{array}{ccc} \alpha_k(\nu\psi)_* & \xRightarrow{\lambda_{\nu\psi}} & \alpha_i \\ {\scriptstyle \alpha_k \circ \mu_{\psi,\nu}} \Uparrow & & \Uparrow {\scriptstyle \lambda_\psi} \\ \alpha_k \nu_* \psi_* & \xRightarrow[\lambda_{\nu_*} \circ \psi_*]{} & \alpha_j \psi_* \end{array}$$

  commutes.

- *For any category C, the canonical functor*

  $$\kappa : \operatorname{Hom}_{\mathfrak{Cat}}(2\text{-}\operatorname{colim}_i \mathfrak{F}_i, C) \to 2\text{-}\lim_i \operatorname{Hom}_{\mathfrak{Cat}}(\mathfrak{F}_i, C)$$

  is an equivalence of categories. Here κ is given by $\kappa(\chi) = (\chi \circ \alpha_i, \chi_i \circ \lambda_\psi)$.

It is well known that the 2-colimit exists and is unique up to an equivalence of categories, see, [14, pp. 192-193]. The analogous statement for groupoids holds as well [25, Exposé VI, Section 6].

1.2.4.1 The Filtered Case

Let I be a filtered category and $\mathfrak{F} : I \to \mathfrak{Cat}$ a 2-functor in the 2-category of small categories. The set of objects of the category $2\text{-}\operatorname{colim} \mathfrak{F}$ is the disjoint union of the objects of $\mathfrak{F}_i$. Thus, any object $A \in \mathfrak{F}_i$ is also an object of $2\text{-}\operatorname{colim} \mathfrak{F}$. Take $A \in \mathfrak{F}_i$ and $B \in \mathfrak{F}_j$. Any diagram of the form

$$i \xrightarrow{\alpha} k \xleftarrow{\beta} j$$

and any morphism $\alpha_*(A) \xrightarrow{h} \beta_*(B)$ in $\mathfrak{F}_k$ determines a morphism $A \to B$ in the category 2-colim $\mathfrak{F}$. This morphism will be denoted by $[\alpha, \beta, h]$. If we take an other morphism $[\alpha', \beta', h']$ with $i \xrightarrow{\alpha'} k' \xleftarrow{\beta'} j$ and $h' \in \mathsf{Hom}_{\mathfrak{F}_k}(\alpha'_*(A), \beta'_*(B))$, then

$$[\alpha, \beta, h] = [\alpha', \beta', h']$$

if and only if there exists a diagram $k \xrightarrow{\gamma} l \xleftarrow{\gamma'} k'$ such that $\gamma\alpha = \gamma'\alpha', \gamma\beta = \gamma'\beta'$ and that the following is a commutative diagram:

$$\begin{array}{ccccccc}
(\gamma\alpha)_*(A) & \longrightarrow & \gamma_*\alpha_*(A) & \xrightarrow{\gamma_* h} & \gamma_*\beta_*(B) & \longrightarrow & (\gamma\beta)_*(B) \\
\mathsf{Id}\downarrow & & & & & & \downarrow\mathsf{Id} \\
(\gamma'\alpha')_*(A) & \longrightarrow & \gamma'_*\alpha'_*(A) & \xrightarrow[\gamma'_* h']{} & \gamma'_*\beta'_*(B) & \longrightarrow & (\gamma'\beta')_*(B).
\end{array}$$

It follows from the definition, that if $[\alpha, \beta, h] : A \to B$ is a morphism in 2-colim $\mathfrak{F}$, then for every morphism $\nu : k \to l$, one has

$$[\alpha, \beta, h] = [\nu\alpha, \nu\beta, \nu_*(h)].$$

To define the composition in 2-colim $\mathfrak{F}$, let $[\alpha, \beta, h] : A \to B$ and $[\gamma, \delta, g] : B \to C$ be morphism in 2-colim $\mathfrak{F}$, where $h : \alpha_*(A) \to \beta_*(A)$ and $g : \gamma_*(B) \to \delta_*(C)$ are morphism in $\mathfrak{F}_k$ and $\mathfrak{F}_m$ respectively. Here, $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$ and $j \xrightarrow{\gamma} m \xleftarrow{\delta} l$ are diagrams in I. Since I is filtered, there are morphisms $k \xrightarrow{\mu} n \xleftarrow{\zeta} m$. Using the second condition of filteredness, there exists a morphism $\eta : n \to t$ in I such that $\eta\mu\beta = \eta\zeta\gamma$. Then one puts

$$[\gamma, \delta, g] \circ [\alpha, \beta, h] = [\eta\mu\alpha, \eta\mu\beta = \eta\zeta\gamma, \eta_*\zeta_* g \circ \eta_*\mu_* h].$$

Proposition 1.2.16. *Let I be a filtered category and $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor. Take $\mathcal{A} \in \mathfrak{F}_i$ and $\alpha : i \to j$, a morphism in I. Then, A and $\alpha_*(A)$ are isomorphic in the 2-colimit of $\mathfrak{F}$.*

Proof. The diagram $i \xrightarrow{\alpha} j \xleftarrow{\mathsf{Id}} j$ in I, induces the map $[\alpha, \mathsf{Id}_j, \mathsf{Id}_{\alpha_*(A)}] : A \to \alpha_*(A)$. Its inverse is given by $[\mathsf{Id}_j, \alpha, \mathsf{Id}_{\alpha_*(A)}] : \alpha_*(A) \to A$, which is coming from the diagram $j \xrightarrow{\mathsf{Id}} k \xleftarrow{\alpha} i$. $\mathcal{Q.E.D}$

By the very definition of morphisms in the 2-colimit, we have the following result:

Proposition 1.2.17. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a filtered 2-colimit. Take $A \in \mathfrak{F}_i$ and $B \in \mathfrak{F}_j$ and denote by $I_{i,j} := \{i \xrightarrow{\alpha} k \xrightarrow{\beta} j\}$. We have*

$$\mathsf{Hom}_{2\text{-}\operatorname{colim}_i \mathfrak{F}_i}(A, B) \simeq \operatorname*{colim}_{k \in I_{i,j}} \mathsf{Hom}_{\mathfrak{F}_k}(\alpha_*(A), \beta_*(B)).$$

We have the following straightforward to check fact:

Proposition 1.2.18. *The assignment*

$$\mathfrak{L} : \{\textit{2-Functors over } I \textit{ with values in categories}\} \to \mathfrak{Cat}$$

defined in the following way:

i) $\{\mathfrak{F} : I \to \mathfrak{Cat}\} \mapsto 2\text{-colim}\,\mathfrak{F}$ *on objects;*

ii) $\{\varphi : \mathfrak{F} \to \mathfrak{G}\} \mapsto \{\varphi' : 2\text{-colim}\,\mathfrak{F} \to 2\text{-colim}\,\mathfrak{G}\}$ *given by*

$$x = x_i \mapsto \varphi(x_i), \quad [\alpha, \beta, f : x_i \to y_j] \mapsto [\varphi(\alpha), \varphi(\beta), \varphi(f)]$$

on morphisms, and

iii) $\{\lambda : \varphi \Rightarrow \phi\} \mapsto \{\lambda' : \varphi' \Rightarrow \phi'\}$ *given by* $\lambda'(\varphi'(x)) = \lambda'(\varphi'(x_i)) = \lambda(\varphi(x_i))$ *on 2-morphisms,*

defines a 2-functor.

1.3 Stacks and Costacks

In this section we will talk about the 2-categorical analogues of sheaves and its dual notion cosheaves, which are called stacks and costacks respectively. We will define these using the 2-limit and 2-colimit constructed in the previous section.

As we will show, a costack is a very important construction. Indeed, saying that a 2-functor is a costack, is equivalent to saying that it satisfies a slightly reformulated version of the Seifert-van Kampen theorem.

Stacks and costacks will play a major role for us, which might not be a big surprise, as they are the 2-categorical versions of sheaves and cosheaves. We start by briefly recollecting some basic facts on sites and (co)sheaves.

1.3.1 Sites, Sheaves and Cosheaves

Let $\mathbf{X}$ be a small category. A set valued *presheaf* on $\mathbf{X}$ is a functor $P : \mathbf{X}^{op} \to \mathrm{Sets}$. Any object $X \in \mathbf{X}$ gives us the presheaf $\mathrm{Hom}_{\mathbf{X}}(-, X)$. Presheaves of this form a called *representable*. Presheaves and natural transformations between them form a category which we denote by $\mathrm{Psh}(X)$.

Sieaves For an object U in $\mathbf{X}$, a *sieve* S on U is a subfunctor of the representable functor $\mathrm{Hom}_{\mathbf{X}}(-, U)$. Thus, for any object $V \in \mathbf{X}$, there is given a collection $S(V)$ of morphisms $f : V \to U$ such that for any morphism $g : W \to V$ of $\mathbf{X}$, the composition $fg \in S(W)$.

Let S be a sieve on U and $h : W \to U$ a morphism in $\mathbf{X}$. We can form the pull-back diagram in the category of presheaves (which always exists):

$$\begin{array}{ccc} h^*S & \hookrightarrow & \mathrm{Hom}_{\mathbf{X}}(-, W) \\ \downarrow & & \downarrow{\scriptstyle h^*} \\ S & \hookrightarrow & \mathrm{Hom}_{\mathbf{X}}(-, U). \end{array}$$

Thus, h^*S is a sieve on W, as the pullback of an injective map is injective. It is given by

$$(h^*S)(Y) = \{\phi : Y \to W \mid (h\phi : Y \to U) \in S(Y)\}.$$

Grothendieck Topologies Recall that a *Grothendieck topology* J on $\mathbf{X}$, assigns to each object U of $\mathbf{X}$ a collection JU of sieaves on U, called *covering sieves* of U, such that the following axioms hold:

i) For any object U of $\mathbf{X}$, the representable functor $\mathrm{Hom}_{\mathbf{X}}(-, U)$ is a covering sieve, that is $\mathrm{Hom}_{\mathbf{X}}(-, U) \in JU$.

ii) If S is a covering sieve on U and $h : W \to U$ is a morphism, the pull-back sieve $h^*(S)$ is a covering sieve on W.

iii) Let S be a covering sieve on U and T any sieve on U. Suppose that for every morphism $h : W \to U$ in $S(W)$, the pull-back sieve $h^*(T)$ is a covering sieve on W. Then T is a covering sieve on U.

Grothendieck Pretopologies A bit more intuitive than a Grothendieck topology is the notion of a Grothendieck pretopology. Both of these constructions admit the notion of sheaves over them. It is well known that these two notions of sheaves are equivalent. Moreover, many of the important examples of Grothendieck topologies come from Grothendieck pretopologies.

We recall that a *Grothendieck pretopology* on a category $\mathbf{X}$ is given by fixing families $\{U_i \to U\}$ of morphisms with a common codomain, called *covering families*of U for every object U of $\mathbf{X}$. These covering families are denoted by $\mathrm{Cov}(U)$ and satisfy the following conditions:

i) The singleton family $\{V \to U\}$ is a covering family $\{V \to U\} \in \mathrm{Cov}(U)$ for any object U and any isomorphism $f : V \to U$.

ii) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ is a covering family on A and $h : V \to U$ is any morphism, there exists a pull-back diagram

$$\begin{array}{ccc} h^*U_i & \longrightarrow & V \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U \end{array}$$

in $\mathbf{X}$ and the family $\{h^*U_i \to V\}_{i \in I} \in \mathrm{Cov}(V)$ is a covering family of V.

iii) If $\{U_i \xrightarrow{\phi_i} U\}_{i \in I} \in \mathrm{Cov}(U)$ is a covering family of U and for every $i \in I$, there is given a covering family $\{V_{ij} \xrightarrow{\psi} U_i\}_{ij \in I_j} \in \mathrm{Cov}(U_i)$, then $\{V_{ij} \xrightarrow{\psi\phi} U\} \in \mathrm{Cov}(U)$.

From Grothendieck Pretopologies to Grothendieck Topologies Given a collection of arrows $\{U_i \to U\}$, one can define a sieve S of U by letting $S(W)$ be the set of all morphisms $W \to U$, that factor through some arrow $U_i \to U$. With this in mind, one sees that any Grothendieck pretopology defines a Grothendieck topology, by declaring a sieve $S \subseteq \mathrm{Hom}_X(-, U)$ to be a covering sieve, if it came from a covering family.

Any topological space (in the classical sense) gives rise to a Grothendieck pretopology and hence, also a Grothendieck topology. This is done through the category $\mathfrak{Open}(X)$, whose objects are open subsets of X and morphisms are exactly inclusions. Since the intersection of open subsets is again open, pull-backs exist in $\mathfrak{Open}(X)$. The Grothendieck pretopology is given by declaring families $\{U_i \to U\}$ to be coverings, if and only if $\bigcup U_i = U$.

Sites A *site* $(\mathbf{X}, J)$ (or simply $\mathbf{X}$ when J is clear from the context) consists of a small category $\mathbf{X}$, equipped with a Grothendieck topology J.

Sheaves A presheaf $F : \mathbf{X}^{op} \to$ Sets is said to be a *sheaf* for the topology J, if for any covering sieve S of an object U, every morphism $\alpha : S \to F$ of presheaves has a unique extension $\mathrm{Hom}_{\mathbf{X}}(-, U) \to F$. This can be expressed by the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\subseteq} & \mathrm{Hom}_{\mathbf{X}}(-, U) \\ & \searrow & \downarrow \exists! \\ & & F. \end{array}$$

The category of sheaves on $\mathbf{X}$ with respect to the topology J is denoted by $\mathrm{Sh}(\mathbf{X}, J)$, or simply $\mathrm{Sh}(\mathbf{X})$, when there is no ambiguity about J.

In the specific case, when J comes from a Grothendieck pretopology, we can give a more direct description. We first fix some notation.

Let $\mathcal{U} = \{U_i \xrightarrow{f_i} U\}_{i \in I}$ be a family of arrows with a common codomain. For $i, j \in I$, we can form the pull-back

$$\begin{array}{ccc} U_i \times_U U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_j & \longrightarrow & U. \end{array}$$

This can alternatively be interpreted as the diagram

$$\coprod_{i,j} U_i \times_U U_j \rightrightarrows \coprod_i U_i \to U,$$

where the left parallel arrows comes from the projections in the pull-back diagram. Clearly, the two composite morphisms from $\coprod_{i,j} U_i \times_U U_j$ to U are equal. Applying the presheaf F to the diagram, we obtain a morphism

$$F(U) \to H^0(\mathcal{U}, F),$$

where $\mathcal{U}$ denote the family $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ and $H^0(\mathcal{U}, F)$ denotes the equaliser of the parallel arrows

$$\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j).$$

A presheaf $F : \mathbf{X}^{op} \to$ Sets is a sheaf if for any object U and any covering family $\{U_i \xrightarrow{f_i} U\}_{i\in I}$, the canonical map $F(U) \to H^0(\mathcal{U}, F)$ is a bijection. In other words, if the sequence

$$1 \to F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact.

Indeed, one can define presheaves and sheaves in any category $\mathbf{C}$, by requiring the equaliser of the above diagram, taken in the category $\mathbf{C}$, to exist in $\mathbf{C}$ and be isomorphic to $F(U)$.

This is equivalent to saying that for any object c of $\mathbf{C}$, the contravariant functor $U \mapsto \mathrm{Hom}_{\mathbf{C}}(c, F(U))$ is a sheaf on $\mathbf{X}$, with values in Sets. Thus, a presheaf $F : \mathbf{X}^{op} \to \mathbf{C}$ is a sheaf if for any object $c \in \mathbf{C}$, any open cover $\{U_i \to U\}_{i\in I}$ and any family of morphisms $\{f_i : c \to F(U_i)\}_{i\in I}$ in $\mathbf{C}$ for which

$$\begin{array}{ccc} c & \xrightarrow{f_j} & F(U_j) \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ F(U_i) & \longrightarrow & F(U_i \times_U U_j) \end{array}$$

is a commutative diagram, there exists a unique morphism $f : c \to F(U)$, such that the diagram

$$\begin{array}{ccc} c & \xrightarrow{f} & F(U) \\ & {\scriptstyle f_i}\searrow & \downarrow \\ & & F(U_i) \end{array}$$

commutes for all i.

The set-valued sheaves have the following very important property: Since any sheaf is also a presheaf, we have the inclusion functor

$$\mathsf{Sh}(\mathbf{X}, J) \subseteq \mathsf{Psh}(\mathbf{X}),$$

which has a finite-limits-preserving left adjoint functor. In particular, for any presheaf P, there exists a sheaf $\hat{P}$ and a morphism of presheaves $\alpha : P \to \hat{P}$, satisfying the following: For any morphism $\beta : P \to F$ to a sheaf F, there exists a unique morphism of sheaves $\theta : \hat{P} \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \hat{P} \\ & {\scriptstyle \beta}\searrow & \downarrow{\scriptstyle \theta} \\ & & F. \end{array}$$

The sheaf $\hat{P}$ can be constructed as follows: We first define a presheaf P' by

$$P'(U) := \operatorname*{colim}_{\mathcal{U}} H^0(\mathcal{U}, P),$$

where we are taking the colimit by varying through all the possible covering families $\mathcal{U} := \{U_i \to U\}$ of U. It is a well-known fact that

$$\hat{P} = P''.$$

Cosheaves Dual to the notion of a sheaf is the notion of a cosheaf. By dual we mean that it is a sheaf in the opposite 2-category. Nonetheless, we specify the definition here.

Definition 1.3.1. *Let $\mathbf{X}$ be a site and $G : \mathbf{X} \to \mathbf{A}$ be a functor with values in a category $\mathbf{A}$. We say that G is a* cosheaf *if for any covering $\mathcal{U} = \{U_i \to U\}$ of any object $U \in \mathbf{X}$, the diagram*

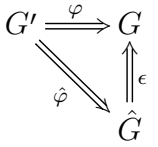
$$\coprod_{i,j} G(U_i \cap U_j) \underset{b_1}{\overset{b_0}{\rightrightarrows}} \coprod_i G(U_i) \xrightarrow{a} G(U) \longrightarrow \mathbf{1}$$

is exact, where $\mathbf{1}$ is the terminal object of $\mathbf{A}$.

An other way of saying that is that $G(U)$ is the coequaliser of b_0, b_1. That is to say, G is a cosheaf if and only if for any object S of $\mathbf{A}$ the presheaf F defined by $F(U) = \mathrm{Hom}_{\mathbf{A}}(G(U), S)$ is a sheaf of sets [14]. This in and of itself says that G, considered as a contravariant functor $\mathbf{X}^{op} \to \mathbf{A}^{op}$ is a sheaf with values in the opposite category of $\mathbf{A}$.

Definition 1.3.2. *Let $G : \mathbf{X} \to \mathbf{A}$ be a functor from a site $\mathbf{X}$ to a category $\mathbf{A}$. We say that $\hat{G} : X \to \mathbf{A}$ is the associated cosheaf of G, if:*

- *We are given a natural transformation $\epsilon : \hat{G} \Rightarrow G$;*
- *For any cosheaf $G' : \mathbf{X} \to \mathbf{A}$ and any natural transformation $\varphi : G' \Rightarrow G$, there is a unique natural transformation $\hat{\varphi} : G' \Rightarrow \hat{G}$ such that the diagram*



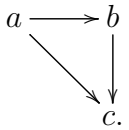
commutes.

The existence of $\hat{G}$ is delicate problem and in general it might depend on additional set theoretic axioms.

1.3.2 Stacks

As already mentioned, this subsection follows [45] closely, even on notation.

For $\mathbf{C}$ an object and $c \in \mathbf{C}$ an object, we can define the *slice category* $\mathbf{C}/c$, whose objects are arrows in $\mathbf{C}$ of the form $a \to c$ and morphisms are commutative diagrams



Let $(\mathbf{X}, J)$ be a site. We will assume that the Grothendieck topology is generated by a Grothendieck pretopology. In this case, for any object U of $\mathbf{X}$, the slice category $\mathbf{X}/U$ is also a site, where coverings of an object $V \to U$ are formed by the families $\{V_i \to V\}$ in $\mathbf{X}/U$, such that $\{V_i \to U\} \in \mathrm{Cov}(U)$. For simplicity, we will denote this site by $\mathbf{X}_{|U}$. This plays the analogue of restricting a topological space to an open subset.

Let $\mathfrak{F} : \mathbf{X}^{op} \to \mathfrak{Cat}$ a 2-functor, where $\mathfrak{Cat}$ is the 2-category of categories. This is called a *fibred category* over $\mathbf{X}$. Morphisms of fibred categories are simply natural transformations of corresponding 2-functors.

If $\mathfrak{F}$ is a strict 2-functor, we would call it a strict fibred category. It should be noted, to avoid any confusion, that this is often called a prestack (which we will call something else).

Lemma 1.3.3. *Let $\mathfrak{F}$ be a fibred category over $\mathbf{X}$, $U \in \mathbf{X}$ an object of $\mathbf{X}$ and A, B be objects of $\mathfrak{F}(U)$. The assignment*

$$(V \xrightarrow{i} U) \mapsto \mathrm{Hom}_{\mathfrak{F}(V)}(i^*(A), i^*(B)),$$

for any object $i : V \to U$ of the slice category $\mathbf{X}/U$, defines a presheaf on $\mathbf{X}|_U$. We denote it by $\mathrm{Hom}_{\mathfrak{F}}(A, B)$.

Before we define the notion of a stack, we need the notion of a prestack.

Definition 1.3.4 (Prestack)**.** *If the presheaf* $\mathrm{Hom}_{\mathfrak{F}}(A, B)$ *is in addition a sheaf, we say that $\mathfrak{F}$ is a* prestack.

It is clear that every prestack is a fibred category. Hence, we have the inclusion 2-functor

$$\mathcal{A} : \{\textit{Prestacks over } \mathbf{X}\} \to \{\textit{Fibred Categories over } \mathbf{X}\}$$

given in the obvious way. This 2-functor has a left adjoint, called *prestackification*, which is given by sheafifying the functors $\mathrm{Hom}_{\mathfrak{F}}(A, B)$ for all A and B. More precisely, to a contravariant 2-functor $\mathfrak{F} : \mathbf{X} \to \mathfrak{Cat}$, we assign $\overline{\mathfrak{F}} : \mathbf{X} \to \mathfrak{Cat}$. The objects of the category $\overline{\mathfrak{F}}(U)$ are the same as that of $\mathfrak{F}$, and for every pair of objects (A, B), we define $\mathrm{Hom}_{\overline{\mathfrak{F}}(U)}(A, B)$ to be the set of all sections over U of the sheafification of $\mathrm{Hom}_{\mathfrak{F}}(A, B)$.

Let $\mathbf{X}$ be a site and $\mathfrak{F} : \mathbf{X}^{op} \to \mathfrak{Cat}$ a 2-functor. Let U be an object in $\mathbf{X}$ and $\mathcal{U} = \{U_i \to U\}$ a covering of U. We can consider the following diagram:

$$\bigsqcap_{i \in I} \mathfrak{F}(U_i) \rightrightarrows \bigsqcap_{i,j \in I} \mathfrak{F}(U_{ij}) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} \bigsqcap_{i,j,k \in I} \mathfrak{F}(U_{ijk}),$$

where $U_{ij} := U_i \times_U U_j$ and $U_{ijk} := U_i \times_U U_j \times_U U_k$. We denote its 2-limit by $2\text{-lim}(\mathcal{U}, \mathfrak{F})$ or $\mathsf{Des}(\mathcal{U}, \mathfrak{F})$ (read *descent data*) interchangeably. The category $\mathsf{Des}(\mathcal{U}, \mathfrak{F})$ has

- as objects: Systems $(a, \theta) = (\{a_i\}, \{\theta_{ij}\})$, where each a_i is an object of $\mathfrak{F}(U_i)$ and

  $$\theta_{ij} : a_{j|U_{ij}} \xrightarrow{\sim} a_{i|U_{ij}}$$

  is an isomorphism (here, $a_{j|U_{ij}} = i_{ij}^*(a_j)$) for the projection $i_{ij} : U_{ij} \to U_j$, likewise the other term). These isomorphisms are required to satisfy the cocylce condition

  $$\theta_{ii} = \mathsf{Id}, \quad \theta_{ij} \circ \theta_{jk} = \theta_{ik} \ (\text{in } \mathfrak{F}(U_{ijk})).$$

- as morphisms: $(a, \theta) \xrightarrow{f} (b, \rho)$ is a family of arrows $f_i : a_i \to b_i$ in $\mathfrak{F}(U_i)$, for which $\rho_{ij} f_j = f_i \theta_{ij}$. That is to say, the diagrams

  $$\begin{array}{ccc} a_{j|U_{ij}} & \xrightarrow{f_j} & b_{j|U_{ij}} \\ {\scriptstyle \theta_{ij}}\downarrow & & \downarrow{\scriptstyle \rho_{ij}} \\ a_{i|U_{ij}} & \xrightarrow[f_i]{} & b_{i|U_{ij}} \end{array}$$

  commute.

Remark 1.3.5. *The above conditions are written in the case when our 2-functor is strict. In the general case, we also need to factor in the natural isomorphisms*

$$(\mathfrak{F}(U_i) \to \mathfrak{F}(U_{ijk})) \Rightarrow (\mathfrak{F}(U_i) \to \mathfrak{F}(U_{ij}) \to \mathfrak{F}(U_{ijk}))\,.$$

There is an obvious functor

$$D : \mathfrak{F}(U) \to \mathsf{Des}(\mathcal{U}, \mathfrak{F}),$$

which sends an object a of $\mathfrak{F}(U)$ to $(a_{|U_i}, \theta)$, where $\theta_{ij} = \mathsf{Id}_{U_{ij}}$.

Definition 1.3.6 (Stack)**.** *A fibred category $\mathfrak{F}$ over $\mathbf{X}$ is called a* stack*, if for all objects U of $\mathbf{X}$ and for all coverings $\mathcal{U}$ of U, the functor $\mathfrak{F}(U) \to \mathsf{Des}(\mathcal{U}, \mathfrak{F})$ is an equivalence of categories.*

Let $\mathbf{X}$ be a site. From Remark 1.2.8 on page 28, we immediately see that every stack over $\mathbf{X}$ is also a prestack over $\mathbf{X}$, i.e. we have an inclusion 2-functor:

$$\mathcal{B} : \{\textit{Stacks over } \mathbf{X}\} \to \{\textit{Prestacks over } \mathbf{X}\}.$$

Proposition 1.3.7 (Stackification)**.** *The 2-functor $\mathcal{B}$ has a left adjoint, called stackification. If $\mathfrak{F}$ is a prestack, its associated stack is defined by*

$$\hat{\mathfrak{F}}(U) := 2\text{-}\operatorname*{colim}_{\mathcal{U}}(\mathsf{Des}(\mathcal{U}, \mathfrak{F})).$$

For the proof of the above proposition, see [45, Theorem 2.1]. Just like for sheaves, the associated stack of a 2-functor can also be defined by its uniqueness property.

Definition 1.3.8 (Uniqueness Property)**.** *Let $\mathfrak{F}$ be a fibred category over a site $\mathbf{X}$. Then $\hat{\mathfrak{F}}$ is the associated stack of $\mathfrak{F}$ if we are given a natural 2-transformation $\mathfrak{F} \to \hat{\mathfrak{F}}$ of 2-functors, such that for every stack $\mathfrak{G}$ over $\mathbf{X}$, the induced functor*

$$\mathsf{Hom}_{\mathsf{St}}(\hat{\mathfrak{F}}, \mathfrak{G}) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Fb}}(\mathfrak{F}, \mathfrak{G}),$$

is an equivalence of categories. Here, St *denotes the 2-category of stacks and* Fb *the 2-category of fibred categories.*

By simply composing $\mathcal{A}$ and $\mathcal{B}$ and their adjoints, we can associate a stack to every fibred category over X. But there exists another, direct, way of stackifying a 2-functor, as shown in [71, Theorem 3.8]:

Proposition 1.3.9 (Direct stackification)**.** *Let $\mathfrak{F} : \mathbf{X}^{op} \to \mathfrak{Cat}$ be a 2-functor and define $\mathfrak{F}'(U) := 2\text{-}\mathsf{colim}_{\mathcal{U}}(\mathsf{Des}(\mathcal{U}, \mathfrak{F}))$. Clearly we can iterate this construction. Then,*

$$\hat{\mathfrak{F}} := \mathfrak{F}'''(U) = 2\text{-}\operatorname*{colim}_{\mathcal{U}}(\mathsf{Des}(\mathcal{U}, \mathfrak{F}''))$$

is the associated stack of $\mathfrak{F}$.

The stackification has a very nice characterisation in terms of weak equivalences [45], which we recall here: A morphism of fibred functors $\phi : \mathfrak{F} \to \mathfrak{F}'$ is called a *strong* equivalence, if for every object U of $\mathbf{X}$, the functor $\phi(U) : \mathfrak{F}(U) \to \mathfrak{F}'(U)$ is an equivalence of categories. Recall that this means that ϕ is full and faithful and essentially surjective.

A morphism $\phi : \mathfrak{F} \to \mathfrak{F}'$ is called a *weak equivalence*, if it is full and faithful and for any object $x \in \mathfrak{F}'(U)$, there is a covering $\{U_i \to U\}_{i \in I}$ and objects $x_i \in \mathfrak{F}(U_i)$ such that the "restriction" of x on U_i (meaning, the image of x under the morphism $\mathfrak{F}'(U) \to \mathfrak{F}'(U_i)$) is isomorphic to the object $\phi(U_i)(x_i)$ in $\mathfrak{F}'(U_i)$.

Lemma 1.3.10 ([45])**.** *Let $\phi : \mathfrak{F} \to \mathfrak{F}'$ be a morphism of fibred categories such that $\mathfrak{F}'$ is a stack and ϕ a weak equivalence. Then $\mathfrak{F}'$ is equivalent to the statification of $\mathfrak{F}'$.*

1.3.3 Between sheaves and stacks

The following result might be known to experts. However, a source is not known to me.

Theorem 1.3.11. *Let F be a sheaf on $\mathbf{X}$, with values in a category $\mathbf{C}$. Define $\mathfrak{F}$ by setting $\mathfrak{F}(U)$ to be the comma category $\mathbf{C}/F(U)$, and for a morphism $V \xrightarrow{i} U$ in $\mathbf{X}$, set $i^* : \mathfrak{F}(U) \to \mathfrak{F}(V)$ by sending*

$$(x \xrightarrow{a} F(U)) \mapsto (x \xrightarrow{a} F(U) \xrightarrow{F(i)} F(V)).$$

That is, $i^*(a) := F(i) \circ a$. Then $\mathfrak{F}$ is a stack.

Proof. Observe that $\mathfrak{F}$ is a strict 2-functor, thus τ_{ij} = Id in this case. Here, τ_{ij} is as in Definition 1.1.3. Take an open cover $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ of an object U of $\mathbf{X}$. Objects of the category Des$(\mathcal{U}, \mathfrak{F})$ are tripples $(x_i, a_i, \theta_{i,j})$, where x_i is an object of $\mathbf{C}$, a_i is a morphism $a_i : x_i \to F(U_i)$ of $\mathbf{C}$, and $\theta_{ij} : x_j \to x_i$ is an isomorphism of $\mathbf{C}$, for which the following diagram

$$\begin{array}{ccccc} x_j & \xrightarrow{a_j} & F(U_j) & \longrightarrow & F(U_{ij}) \\ {\scriptstyle\theta_{i,j}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Id}} \\ x_i & \xrightarrow[a_i]{} & F(U_i) & \longrightarrow & F(U_{ij}) \end{array} \tag{1.1}$$

commutes. Finally, one requires that the isomorphism θ satisfies the cocycle condition. That is, $\theta_{ki} = \theta_{ij}\theta_{kj}$.

Let $(x_i, a_i, \theta_{i,j})$ and $(y_i, b_i, \rho_{i,j})$ be objects of Des$(\mathfrak{U}, \mathfrak{F})$. A morphism $(x_i, a_i, \theta_{i,j}) \to (y_i, b_i, \rho_{i,j})$ is given by a family of arrows $f_i : x_i \to y_i$ of the category $\mathbf{C}$, such that the following diagrams

$$\begin{array}{ccc} x_i & \xrightarrow{f_i} & y_i \\ & {\scriptstyle a_i}\searrow \quad \swarrow{\scriptstyle b_i} & \\ & F(U_i) & \end{array}$$

$$\begin{array}{ccccc} x_j & & \xrightarrow{f_j} & & y_j \\ & \searrow & & \swarrow & \\ {\scriptstyle\theta_{ij}}\downarrow & & F(U_{ij}) & & \downarrow{\scriptstyle\rho_{ij}} \\ & \nearrow & & \nwarrow & \\ x_i & & \xrightarrow[f_i]{} & & y_i \end{array}$$

commute. Here, any unlabelled arrow is an obvious composition. For example, $x_j \to F(U_{ij})$ is $x_j \xrightarrow{a_j} F(U_j) \to F(U_{ij})$.

The functor

$$D : \mathfrak{F}(U) \to \text{Des}(\mathfrak{U}, \mathfrak{F})$$

is given by

$$D(a : x \to F(U)) = (x_i, a_i, \theta_{i,j})$$

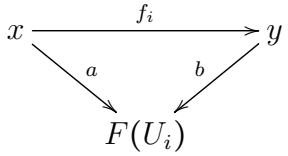
on objects. Here, $x_i = x$ for all i, the morphism $a_i : x \to F(U_i)$ is the composite $x \xrightarrow{a} F(U) \to F(U_i)$, while $\theta_{i,j}$ = Id. On morphisms, D is given as follows: Let $a : x \to F(U)$ and $b : y \to F(U)$ be two objects of $\mathfrak{F}(U)$ and let f be a morphism from $a : x \to F(U)$ to $b : y \to F(U)$. Thus, we have a commutative triangle

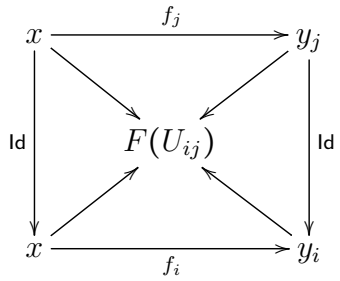
$$\begin{array}{ccc} x & \xrightarrow{f} & y\ . \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & F(U_i) & \end{array}$$

Then D sends f to the constant family $f_i = f$.

We need to show that D is an equivalence of categories. First, we will show that D is full and faithful. We will then prove that it is essentially surjective.

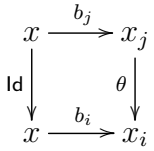
A morphism from $D(a : x \to F(U))$ to $D(b : y \to F(U))$ in the Des$(\mathfrak{U}, \mathfrak{F})$ is given by a family of arrows $f_i : x \to y$ of the category $\mathbf{C}$ such that the following diagrams



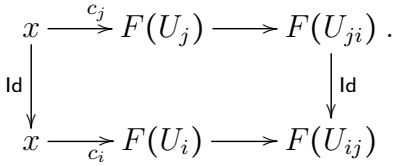


commute. It follows that $f_i = f_i$ for all i, j. Denote this common value by f. This defines a morphism in $F(U)$, proving that D is full and faithful.

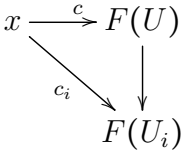
To prove essential surjectivity, take an object $(x_i, a_i, \theta_{i,j})$ of Des$(\mathfrak{U}, \mathfrak{F})$. Choose i_0 and set $x = x_{i_0}$, $b_i = \theta_{i,i_0} : x = x_{i_0} \to x_i$ and denote the composite $x \xrightarrow{b_i} x_i \xrightarrow{a_i} F(U_i)$ by c_i. Since θ satisfies the 1-cocycle condition (see above), the diagram



commutes. This, together with the commutative diagram (1.1), implies the commutativity of the diagram



By the definition of a sheaf, there exist a unique morphism $c : x \to F(U)$ such that the diagram



commutes for all i. The morphism $c : x \to P(U)$ is an object of $F(U)$, and we see that the collection of morphisms $\theta_{i_0 i}$ defines an isomorphism from $D(c : x \to F(U))$ to $(x_i, a_i, \theta_{i,j})$.
$\mathcal{Q}.\mathcal{E}.\mathcal{D}$

1.3.4 Costacks

1.3.4.1 Costacks

Let $\mathfrak{C}$ be a 2-category, $\mathbf{X}$ a site and $\mathfrak{F} : \mathbf{X} \to \mathfrak{C}$ a 2-functor. Dually to Subsection 1.3.2, in the case when $\mathfrak{C} = \mathfrak{Cat}$, we call this a *cofibred category* over $\mathbf{X}$.

take an object in $U \in \mathbf{X}$ with a covering $\mathcal{U} = \{U_i \to U\}$. Consider the following diagram in $\mathfrak{C}$:

$$\coprod_{i \in I} \mathfrak{F}(U_i) \leftleftarrows \coprod_{i,j \in I} \mathfrak{F}(U_{ij}) \lefttriplearrows \coprod_{i,j,k \in I} \mathfrak{F}(U_{ijk}),$$

where $U_{ij} := U_i \times_U U_j$ and $U_{ijk} := U_i \times_U U_j \times_U U_k$. We denote its 2-colimit by $2\text{-colim}(\mathcal{U}, \mathfrak{F})$ or $\mathsf{Codes}(\mathcal{U}, \mathfrak{F})$.

Definition 1.3.12 (Costack)**.** *A cofibred category $\mathfrak{F}$ over $\mathbf{X}$ is called a* costack*, if for every object U of $\mathbf{X}$ and every covering $\mathcal{U}$ of U, the functor $\mathfrak{F}(U) \leftarrow \mathsf{Codes}_{\mathcal{U}}(\mathcal{U}, \mathfrak{F})$ is an equivalence of categories.*

Due to the fact that $\mathsf{Hom}(-, A)$ is left exact, we immediately obtain the following lemma:

Lemma 1.3.13. *Let $\mathfrak{F}$ be a cofibred category over $\mathbf{X}$. Then $\mathfrak{F}$ is a costack if and only if for every object C, the assignment $U \mapsto \mathsf{Hom}_{\mathfrak{C}}(\mathfrak{F}(U), C)$ is a stack.*

It should be noted that since $\mathsf{Hom}(-, A)$ is not (in general) right exact, the duality between stacks and costacks breaks down here. Namely, it would not be sufficient to check that $U \mapsto \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{F}(U), -)$ is a costack for $\mathfrak{F}$ to be a stack.

Definition 1.3.14 (Uniqueness Property)**.** *Let $\mathfrak{F}$ be a cofibred category over a site $\mathbf{X}$. Then $\hat{\mathfrak{F}}$ is the associated costack of $\mathfrak{F}$ if we are given a natural transformation $\mathfrak{F} \Rightarrow \hat{\mathfrak{F}}$ such that for every costack $\mathfrak{G}$ over $\mathbf{X}$, the induced functor*

$$\mathsf{Hom}_{\mathsf{Cofb}}(\mathfrak{G}, \mathfrak{F}) \cong \mathsf{Hom}_{\mathsf{Cost}}(\mathfrak{G}, \hat{\mathfrak{F}}),$$

is an equivalence of categories. Here Cofb *denotes the 2-category of cofibred categories and* Cost *the 2-category of costacks.*

Note that if our 2-functor $\mathfrak{F}$ took values in groupoids, it would suffice to check it only for costacks with values in groupoids. Unfortunately, unlike for stacks, it is not known whether every cofibred category has an associated costack.

The fundamental groupoid as a 2-terminal costack As mentioned above, the fundamental groupoid has nicer categorical properties than the fundamental group. The following, which appeared already in my PhD thesis, and will be used in this habilitation thesis in Section 5, is one such examples. It says that the fundamental groupoids can be defined via a universal property (axiomatised):

Theorem 1.3.15 ([50], Theorem 2.6)**.** *Let X be a 'nice' topological space. The fundamental groupoid, when regarded as a 2-functor, is the 2-terminal costack over X, with values in groupoids.*

We use the word 'nice' to describe a topological space which admits a discrete covering. See Subsection 1.1.3 for the definition.

Recalling the definition of a costack, the above theorem directly implies that for any open covering $\mathcal{U} = \{U_i\}$ of any open subset $U \subseteq X$, we have an equivalence of groupoids

$$\Pi_1(U) \simeq \underset{U_i \in \mathcal{U}}{2\text{-colim}}\, \Pi_i(U_i).$$

Of course, this can now be interpreted as the 2-categorical version of he Seifert-van Kampen theorem. Though we will not use it in this dissertation, it should be mentioned that the analogous statement also holds for the algebraic (étale) fundamental groupoid.

1.4 Categorical Groups

1.4.1 Monoidal categories

We start by recalling the notion of (symmetric) monoidal categories. These are categories where we can talk about (commutative) monoids and monoid actions. We also need them in order to define categorical groups.

Definition 1.4.1. *A* monoidal category *is a category $\mathbf{C}$, equipped with a functor*

$$\odot : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

called the monoidal product*, and a distinguished object I, called the* unit object*, together with functorial isomorphisms*

$$\alpha_{A,B,C} : A \odot (B \odot C) \to (A \odot B) \odot C,$$

$$\lambda_A : I \odot A \to A, \quad \rho_A : A \odot I \to A,$$

where A, B, C are objects of $\mathbf{C}$. Here, $\alpha_{A,B,C}$ is called the associativity constraint *and λ_A, ρ_A the left and right* unit constraints*, respectively.*

One also requires that the following diagrams commute:

$$\begin{array}{ccccc}
A \odot (B \odot (C \odot D)) & \xrightarrow{\alpha_{A,B,C\odot D}} & (A \odot B) \odot (C \odot D) & \xrightarrow{\alpha_{A\odot B,C,D}} & ((A \odot B) \odot C) \odot D) \\
\downarrow{\scriptstyle \mathrm{Id}_A \odot \alpha_{B,C,D}} & & & & \uparrow{\scriptstyle \alpha_{A,B,C} \odot \mathrm{Id}_D} \\
A \odot ((B \odot C) \odot D) & & \xrightarrow[\alpha_{A,B\odot C,D}]{} & & (A \odot (B \odot C)) \odot D
\end{array}$$

$$\begin{array}{ccc} A \odot (I \odot B) & \xrightarrow{\alpha_{A,I,B}} & (A \odot I) \odot B\,. \\ & {\scriptstyle \mathsf{Id}_A \odot \lambda_B} \searrow \quad \swarrow {\scriptstyle \rho_A \odot \mathsf{Id}_B} & \\ & A \odot B & \end{array}$$

Examples of monoidal categories i) The category Sets is a monoidal category if we take $\odot$ to be the cartesian product $\times$ and I to be the singleton $\star$. In fact, $(\mathsf{Sets}, \times, \star)$ is a symmetric monoidal category (see the definition below).

ii) Let K be a commutative ring. One can take $\mathbf{C}$ to be the category of K-modules, $\otimes$ the tensor product over K and I be K, considered as a module over itself. This is again a symmetric monoidal category.

iii) Let $\mathbf{A}$ be a small category. Take $\mathbf{C}$ to be the category of endofunctors and I the identity functor. In this way, we obtain a monoidal category with respect to the composition, which is non-symmetric in general.

As we said above, one can define monoids and monoid actions in monoidal categories:

Definition 1.4.2. *Let $(\mathbf{C}, \odot, I)$ be a monoidal category. A* monoid *in $\mathbf{C}$ is an object M of $\mathbf{C}$, together with morphisms $\mu : M \odot M \to M$ and $\nu : I \to M$, for which the following diagrams commute*

$$\begin{array}{ccccc} M \odot (M \odot M) & \xrightarrow{\alpha} & (M \odot M) \odot M & \xrightarrow{\mu \odot \mathsf{Id}} & M \odot M \\ \downarrow{\scriptstyle \mathsf{Id} \odot \mu} & & & & \downarrow{\scriptstyle \mu} \\ M \odot M & & \xrightarrow{\qquad\qquad \mu \qquad\qquad} & & M \end{array}$$

$$\begin{array}{ccccc} M \odot I & \xrightarrow{\mathsf{Id}_M \odot \eta} & M \odot M & \xleftarrow{\eta \odot \mathsf{Id}_M} & I \odot M \\ & {\scriptstyle \rho_M} \searrow & \downarrow & \swarrow {\scriptstyle \lambda_M} & \\ & & M & & \end{array}$$

Moreover, let S be an object of S. We will say that a monoid object M *acts* on the left on S, if there is given a morphism $\kappa : M \odot S \to S$ for which the diagram

$$\begin{array}{ccccc} M \odot (M \odot S) & & \xrightarrow{\mathsf{Id} \odot \kappa} & & M \odot S \\ \downarrow{\scriptstyle \alpha} & & & \searrow{\scriptstyle \kappa} & \\ (M \odot M) \odot S & \xrightarrow{\mu \odot \mathsf{Id}} & M \odot S & \xrightarrow{\kappa} & S \end{array}$$

commutes.

The right action is defined in as expected.

Remark 1.4.3. *In the example* $(\mathsf{Sets}, \times, \star)$*, the above will give the classical notion of monoids and monoid actions (on sets).*

In the example $(K\text{-}\mathsf{Mod}, \otimes_K, K)$, we obtain the notion of a unital associative algebras and modules over it.

In example iii), monoids are known as monads or triples.

1.4.2 Monoidal functors

Definition 1.4.4. *Let $(\mathbf{C}, \odot, I)$ and $(\mathbf{C}', \odot, I')$ be monoidal categories. A* monoidal functor *from $(\mathbf{C}, \odot, I)$ to $(\mathbf{C}', \odot, I')$ is a functor $F : \mathbf{C} \to \mathbf{C}'$, together with a natural transformation*

$$\phi_{A,B} : F(A) \odot F(B) \to F(A \odot B)$$

and a morphism $\psi : I' \to F(I)$, such that the following diagrams commute:

$$\begin{array}{ccccccc}
F(A) \odot (F(B) \odot F(C)) & \xrightarrow{\alpha'} & (F(A) \odot F(B)) \odot F(C) & \xrightarrow{\phi \odot \mathsf{Id}} & F(A \odot B) \odot F(C) \\
\downarrow{\scriptstyle \mathsf{Id} \odot \phi} & & & & \downarrow{\scriptstyle \phi} \\
F(A) \odot F(B \odot C) & \xrightarrow{\phi} & F(A \odot (B \odot C)) & \xrightarrow{F(\alpha)} & F((A \odot B) \odot C)
\end{array}$$

$$\begin{array}{ccc}
F(A) \odot I' & \xrightarrow{\mathsf{Id} \odot \psi} & F(A) \odot F(I) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \phi} \\
F(A) & \xleftarrow[F(\rho)]{} & F(A \odot I)
\end{array}$$

$$\begin{array}{ccc}
I' \odot F(A) & \xrightarrow{\psi \odot \mathsf{Id}} & F(I) \odot F(A) \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \phi} \\
F(A) & \xleftarrow[F(\lambda)]{} & F(I \odot A)
\end{array}$$

If the natural transformation is a natural isomorphism, we call the monoidal functor *strong*. If it is even the identity, we call the monoidal functor *strict*. For more on this, the reader is reffered to [62, p. 161].

Throughout this dissertation, we will hence simply say monoidal functor instead of strong monoidal functor.

Let $(\mathbf{C}, \odot, I)$ and $(\mathbf{C}', \odot, I')$ be monoidal categories and let $F, T : \mathbf{C} \to \mathbf{C}'$ be monoidal functors. A natural transformation $\xi : F \to T$ is called *monoidal* if the following diagram

$$\begin{array}{ccc}
F(A) \odot F(B) & \xrightarrow{\phi} & F(A \odot B) \\
{\scriptstyle \xi(A) \odot \xi(B)}\downarrow & & \downarrow{\scriptstyle \xi(A \odot B)} \\
T(A) \odot T(B) & \xrightarrow{\phi} & T(A \odot B)
\end{array}$$

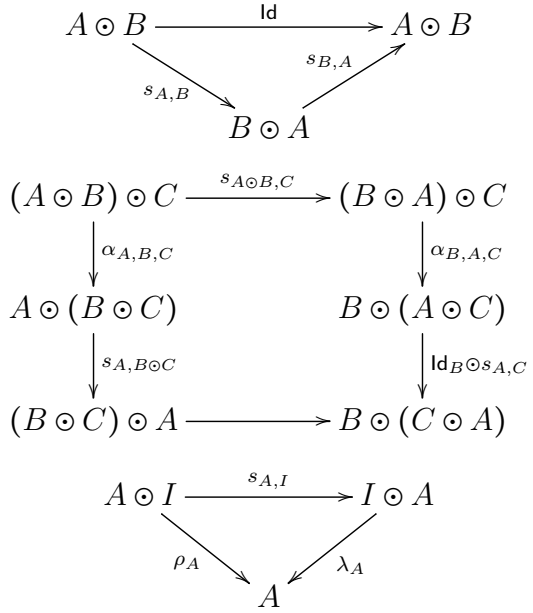
commutes.

1.4.3 Symmetric monoidal categories and symmetric monoidal functors

Definition 1.4.5. *A* symmetric monoidal category *is a monoidal category* $(\mathbf{C}, \odot, I)$*, together with natural (both in* A *and* B*) isomorphisms*

$$s_{A,B} : A \odot B \to B \odot A,$$

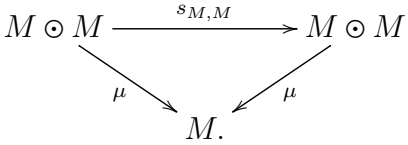
called symmetric constrains *which are functorial with respect to* A *and* B *and for which the following diagrams*



commute.

We also mention that if instead of the first diagram (triangle), we required the symmetric analogue of the second diagram to commute, we would get the notion of a *braided monoidal category*. It will, however, not be relevant for our work.

In symmetric monoidal categories, one can talk about *commutative monoids*. These are monoid objects $(M, \mu : M \odot M \to M, \mu : I \to M)$, for which the following diagram commutes:



Definition 1.4.6. *Assume* $(\mathbf{C}, \odot, I)$ *and* $(\mathbf{C}', \odot, I')$ *are symmetric monoidal categories. A monoidal functor* $(F : \mathbf{C} \to \mathbf{C}', \phi_{A,B} : F(A) \odot F(B) \to F(A \odot B), \varphi : I' \to F(I))$ *is* symmetric*,*

if the following diagram

$$\begin{array}{ccc} F(A)\odot F(B) & \xrightarrow{\phi} & F(A\odot B) \\ \downarrow{\scriptstyle s_{FA,FB}} & & \downarrow{\scriptstyle Fs_{A,B}} \\ F(B)\odot F(A) & \xrightarrow[\phi]{} & F(B\odot A) \end{array}$$

commutes.

1.4.4 Symmetric cat-groups

Definition 1.4.7. *A symmetric categorical group, is a groupoid $\mathfrak{G}$, equipped with a symmetric monoidal structure $+ : \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$, such that for each object X, the endofunctor $(-)+X : \mathfrak{G}\to\mathfrak{G}$ is an equivalence of categories [73].*

In particular, for each object X, one has an object Y and an isomorphism $X+Y\to 0$, where 0 is the neutral object with respect to the symmetric monoidal structure $+$. Unlike what one is used to for groups, the inverse is not unique. It is not even functorial. However, it is well-known that it is possible to choose the "inverses" in such as way, that it becomes functorial. To state it more formally, there exists a contravariant endofunctor $X\mapsto -X$ and functorial isomorphisms

$$m_X : X+(-X)\to 0, n_X : (-X)+X\to 0,$$

for which the following is a commutative diagram

$$\begin{array}{ccccc} X+(-X+X) & & \xrightarrow{\alpha} & & (X+(-X))+X \\ \downarrow{\scriptstyle \mathrm{Id}+n_X} & & & & \downarrow{\scriptstyle m_X+\mathrm{Id}} \\ X+0 & \xrightarrow[\rho]{} & X & \xleftarrow[\lambda]{} & 0+X \end{array}$$

Remark 1.4.8. *Throughout this dissertation, we will refer to a symmetric categorical group as simply a cat-group. As they will be the only categorical groups under consideration, it should, hopefully, not lead to any confusion or ambiguity.*

If $\mathfrak{G}$ and $\mathfrak{H}$ are cat-groups, a morphism $\alpha : \mathfrak{G}\to\mathfrak{H}$ is just a symmetric monoidal functor. Naturally, we can also talk about 2-morphisms between $\alpha,\beta : \mathfrak{G}\to\mathfrak{H}$, which are simply monoidal transformations.

π_0 and π_1 of cat-groups Let $\mathfrak{G}$ be a cat-group. The monoidal structure induces an abelian group structure on the set of connected components. This abelian group is denoted by $\pi_0(\mathfrak{G})$. We also have an abelian group $\pi_1(\mathfrak{G})$, which is the group of automorphisms of the neutral object of $\mathfrak{G}$. Since the functor $(-)+X : \mathfrak{G}\to\mathfrak{G}$ is an equivalence of categories for every object X, the induced homomorphism $\pi_1(\mathfrak{G})\to \mathrm{Aut}_{\mathfrak{G}}(X)$ is an isomorphism. Because of this, we will identify the automorphism group of any object with $\pi_1(\mathfrak{G})$. Recall also that if $\alpha : \mathfrak{G}\to\mathfrak{H}$

is a morphism of cat-groups, than α is an equivalence if and only if the induced morphisms

$$\pi_i(\mathfrak{G}) \to \pi_i(\mathfrak{H}), \quad i = 0, 1$$

are isomorphisms. This is a simple consequence of Proposition 1.1.9, as the surjectivity on π_0 implies essential surjectivity and the isomorphism of π_1, together with the injectivity of π_0 implies full an faithfulness. The injectivity (of π_0) condition is required for the full and faithfulness as well since otherwise it would be possible for non-connected objects in the source groupoid to become isomorphic in the target groupoid. In such a case, automorphisms would not be enough to distinguish them.

The 2-kernel For any morphism $\alpha : \mathfrak{G} \to \mathfrak{H}$ of cat-groups, one can define the cat-group $2\text{-}\ker(\alpha)$. Objects of $2\text{-}\ker(\alpha)$ are pairs (X, x), where X is an object of $\mathfrak{G}$, while $x : \alpha(X) \to 0$ is a morphism in $\mathfrak{H}$. A morphism $(X, x) \to (Y, y)$ in $2\text{-}\ker(\alpha)$ is a morphism $f : X \to Y$ in $\mathfrak{G}$, such that $x = y\alpha(f)$. It is immediate that the 2-kernel, much like the kernel, can be seen as a 2-limit. Specifically, it is the 2-pullback of the diagram

$$\begin{array}{ccc} & & \mathfrak{G} \\ & & \downarrow{\scriptstyle \alpha} \\ 0 & \xrightarrow[\alpha]{} & \mathfrak{H} \end{array}$$

It is also well-known ([21, p. 84], [73]), that there is an exact sequences of abelian groups

$$0 \to \pi_1(2\text{-}\ker(\alpha)) \to \pi_1(\mathfrak{G}) \to \pi_1(\mathfrak{H}) \to \pi_0(2\text{-}\ker(\alpha)) \to \pi_0(\mathfrak{G}) \to \pi_0(\mathfrak{H}). \tag{1.2}$$

Recall the definition of an exact sequence of cat-groups [73]: A diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \kappa \Uparrow & & \\ \mathfrak{G} & \xrightarrow[\alpha]{} & \mathfrak{H} & \xrightarrow[\beta]{} & \mathfrak{K} \end{array}$$

is called *exact* at $\mathfrak{H}$, provided the induced functor $\gamma : \mathfrak{G} \to 2\text{-}\ker(\beta)$ is essentially surjective. Here, κ is a natural isomorphism and

$$\gamma(X) = (\alpha(X), \kappa(X) : \beta\alpha(X) \to 0).$$

It follows that one has an exact sequence of abelian groups

$$\pi_0(\mathfrak{G}) \to \pi_0(\mathfrak{H}) \to \pi_0(\mathfrak{K}).$$

From group homomorphisms to cat-groups Let $\alpha : G_1 \to G_0$ be a group homomorphism of abelian groups. We can make a cat-group $\mathfrak{G}$, whose objects are the elements of G_0, and

for $x, y \in G_0$, we have

$$\mathrm{Hom}(x, y) = \{g \in G_1 \mid y + \alpha(g) = x\} = \alpha^{-1}(y - x).$$

The monoidal structure is induced by the additive group structures on G_1 and G_0. Here, the associativity and commutativity constraints are identities, meaning we have a strict monoid. We have $\pi_0(\mathfrak{G}) = \mathrm{Coker}(\alpha)$ and $\pi_1(\mathfrak{G}) = \ker(\alpha)$.

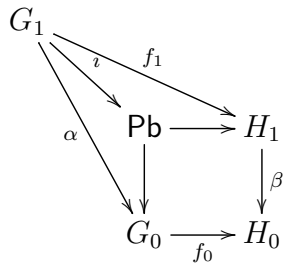
Let $\alpha : G_1 \to G_0$ and $\beta : H_1 \to H_0$ be morphisms of abelian groups and $\mathfrak{G}$ and $\mathfrak{H}$ their associated cat-groups. Observe that any commutative diagram of abelian groups and group homomorphisms

$$\begin{array}{ccc} G_1 & \xrightarrow{f_1} & H_1 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ G_0 & \xrightarrow[f_0]{} & H_0 \end{array}$$

gives rise to a morphism of the abelian cat-groups $\mathfrak{G} \to \mathfrak{H}$. We denote by Pb the pull-back of the diagram

$$\begin{array}{ccc} & & H_1 \\ & & \downarrow{\scriptstyle\beta} \\ G_0 & \longrightarrow & H_0 \end{array}$$

and denote by $\imath$ the canonical homomorphism $G_1 \to \mathrm{Pb}$, making the diagram



commute. We have the following fact:

Proposition 1.4.9. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be as above. We have an isomorphism of cat-groups*

$$2\text{-}\ker(\mathfrak{G} \to \mathfrak{H}) \cong \mathfrak{K},$$

where $\mathfrak{K}$ is the abelian cat-group associated to the morphism $\imath : G_1 \to \mathrm{Pb}$.

Proof. Objects of $2\text{-}\ker(\mathfrak{G} \to \mathfrak{H})$ are pairs (X, x), where X is an object of $\mathfrak{G}$, (meaning, an element of G_0) and x is a morphism $x : f_0(X) \to 0$ in $\mathfrak{H}$, (meaning an element $x \in H_1$ such that $f_0(X) = \beta(x)$). This means exactly that the pair (X, x) belongs to Pb. Thus, it is an object of $\mathfrak{K}$. This proves the expected isomorphism of the level of objects. Morphisms are handled similarly. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 1.4.10. *Let $\alpha : G_1 \to G_0$ be a homomorphism of abelian groups and $\mathfrak{G}$ its associated cat-group. Denote the associated cat-groups of $G_1 \to 0$ and $G_0 \to 0$ by $\mathfrak{G}_1$ and $\mathfrak{G}_0$ respectively. There is an equivalence of cat-groups*

$$2\text{-}\ker(\mathfrak{G}_1 \to \mathfrak{G}_0) \xrightarrow{\sim} \mathfrak{G}.$$

Chapter 2

On the group of separable quadratic algebras and stacks

Overview

This chapter is a collection of the results of paper [53], which deals with the group of isomorphism classes of torsors of finite flat group schemes of rank 2 over a commutative ring R. It will heavily use the results and definitions of the previous chapter, as we will work with 2-categories.

A special, but important, case of our results, will concern the group of quadratic algebras. Theorem 2.0.1 will give a new exact sequence for this class.

The group of separable quadratic algebras was introduced by Bass in [4, Section 4.3] and it was extensively studied by several authors (see for example [26], [65] and references therein). The aim of this chpater is to study this group using methods of 2-dimensional categorical algebra, based on the fact that stackification is an exact 2-functor.

In more detail, for a commutative ring R, we denote the group of isomorphism classes of separable quadratic R-algebras by $\mathsf{Qu}(R)$ [26, Ch. 12], and by $\mathsf{Dis}(R)$ the group of isomorphism classes of nonsingular symmetric bilinear R-modules, which are finitely generated and projective of rank 1 [26, Ch. 4C]. According to [26, Statement 12.6, p.178], one has an exact sequence of abelian groups

$$\mathsf{Qu}(R) \to \mathsf{Dis}(R) \to \mathsf{Dis}(R/4R).$$

Our goal is to extend this important exact sequence on the left hand side. It follows from our results that the following is true:

Theorem 2.0.1. *If 2 is not a zero-divisor in R, one has an exact sequence of abelian groups*

$$0 \to \mathbb{Z}_2(R) \to \mu_2(R) \to \mathsf{Im}(\mu_2(R/4R) \to \mu_2(R/2R)) \to \mathsf{Qu}(R) \to \mathsf{Dis}(R) \to \mathsf{Dis}(R/4R).$$

Here

$$\mu_2(R) = \{r \in R \mid r^2 = 1\} \quad \text{and} \quad \mathbb{Z}_2(R) = \{r \in R \mid r^2 = r\}.$$

The operation in $\mathbb{Z}_2$ is given by $r +_1 s = r + s - 2rs$ (see [26, p.202]). The first non-trivial map is given by $r \mapsto 1 - 2r$.

In order to obtain the above, we are going to define a symmetric categorical group $\mathfrak{Qu}(R)$ such that the groups $\mathrm{Qu}(R)$ and $\mathbb{Z}_2(R)$ are its π_0 and π_1 respectively (see Section 1.4.4 for symmetric categorical groups). Similarly, behind of the groups $\mathrm{Dis}(R)$ and $\mu_2(R)$, there is a symmetric categorical group $\mathfrak{Dis}(R)$, while the groups $\mathrm{Dis}(R/4R)$ and $\mathrm{Im}(\mu_2(R/4R) \to \mu_2(R/2R))$ are obtained from the symmetric categorical group $\mathfrak{Dis}^{-2,1}(R)$. Hence, it suffices to show that $\mathfrak{Qu}(R)$, $\mathfrak{Dis}(R)$ and $\mathfrak{Dis}^{-2,1}(R)$ fit in an exact sequence of symmetric categorical groups, and then apply the classical six-term exact sequence [21, p.84] involving π_i, $i = 0, 1$. The symmetric categorical groups $\mathfrak{Qu}(R)$ and $\mathfrak{Dis}(R)$ are classical and they appear (at least implicitly) in many papers, for example in [26]. However, $\mathfrak{Dis}^{-2,1}(R)$ is new. The next observation is that, by varying R, these symmetric categorical groups form stacks in the Zariski topology.

As it turns out, they can be seen as the stackifications of the analogue prestacks of symmetric categorical groups, based on free modules instead of projective ones. It is a fairly easy exercise to show that the 'free' versions of the above symmetric categorical groups fit in an exact sequence.

2.1 Preliminaries on Hopf algebras and Galois algebras over them

Actually, we will work in a more general setting than that of separable quadratic algebras. We will fix a Hopf algebra J, which is free of rank two as an R-module, and as such, commutative and cocommutative. We will then consider the group of isomorphism classes of J-Galois algebras. We will therefore first fix the corresponding terminology.

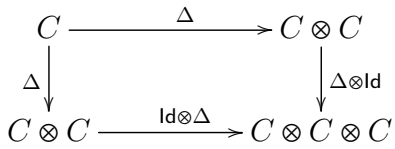
2.1.1 Coalgebras and comodules

Let R be a commutative algebra. As we remarked in Remark 1.4.3, the monoid objects and their actions in the symmetric monoidal category of R-modules are just associative R-algebras with unit and modules over them.

The tensor product also defines the structure of a symmetric monoidal category on the opposite category of R-modules. The corresponding monoids and actions are known as coalgebras and comodules. We aim to describe them in more detail below:

Coassociative Coalgebra A coassociative coalgebra C over R is an R-module C, together with an R-module homomorphism $\Delta : C \to C \otimes C$ (the tensor product is taken over R), called

the *comultiplication*, such that the following diagram



commutes.

We will say that a coassociative coalgebra has a *counit*, if there is given an R-module homomorphism $\epsilon : C \to R$, for which the following diagrams



commute.

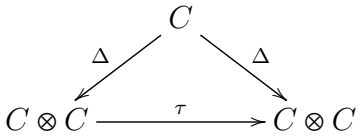
Above, we use the identification $C = C \otimes R$ and $C = R \otimes C$.

Recall that if A and A' are associative R-algebras, then $A \otimes A'$ is also an associative algebra for which $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. In terms of diagrams, the map

$$(A \otimes A') \otimes (A \otimes A') \to A \otimes A'$$

is the composition $(\mu \otimes \mu') \circ (\mathrm{Id}_A \otimes \tau \otimes \mathrm{Id}_{A'})$, where $\tau : A' \otimes A \to A \otimes A'$ is the twist. Quite similarly, if C and C' are coalgebras, then $C \otimes C'$ is also a coalgebra with counit $\epsilon\epsilon'$ and coproduct $(\mathrm{Id}_C \otimes \tau \otimes \mathrm{Id}_{C'}) \circ (\Delta \otimes \Delta')$.

Commutative Coalgebra A coassociative coalgebra C is *cocommutative* if the following diagram



commutes.

Comodules over Coalgebras A left *comodule* over a coalgebra (C, Δ, ϵ) is an R-module A, together with an R-module homomorphism $\delta : A \to C \otimes A$, for which the following diagrams



commute.

Right C-comodules are defined in an analogous way.

Let S be an associative algebra, with M and N left and right S-modules respectively. One can define $M \otimes_S N$ as the coequaliser of the diagram

$$M \otimes S \otimes N \rightrightarrows M \otimes N.$$

The two arrows are induced by the left and right S-module structures of M and N. We can define the cotensor product by reversing the arrows.

More precisely, assume A and B are left and right comodules over a coalgebra (C, Δ, ϵ). The cotensor product $B \boxtimes_C A$ is defined as the equaliser of the diagram

$$B \otimes A \rightrightarrows B \otimes C \otimes A.$$

As expected, the maps are induced by the left and right C-comodule structures on A and B.

If C is a cocommutative coalgebra, any right comodule can be considered as a left comodule (and vice-versa) in the obvious way. In particular, the cotensor product of two left or right comodules makes sense.

2.1.2 Bialgebras

Definition 2.1.1. *A bialgebra over R is an R-module B, which is equipped with both the structure of an associative algebra with unit and a coassociative coalgebra structure with counit. Further, the multiplication $\mu : B \otimes B \to B, \mu(a \otimes b) = ab$ and the unit $\nu : R \to B$, $\nu(1) = 1$ are morphisms of coalgebras with counit. In terms of diagrams, these conditions mean the commutativity of the following diagrams:*

$$\begin{array}{ccccc} B \otimes B & \xrightarrow{\mu} & B & \xrightarrow{\Delta} & B \otimes B \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \uparrow{\scriptstyle \mu\otimes\mu} \\ B \otimes B \otimes B \otimes B & & \xrightarrow{\mathrm{Id}\otimes\tau\otimes\mathrm{Id}} & & B \otimes B \otimes B \otimes B \end{array}$$

$$\begin{array}{ccc} B \otimes B & \xrightarrow{\mu} & B \\ & {\scriptstyle \epsilon\otimes\epsilon} \searrow \quad \swarrow {\scriptstyle \epsilon} & \\ & R & \end{array}$$

$$\begin{array}{ccc} B \otimes B & \xleftarrow{\Delta} & B \\ & {\scriptstyle \eta\otimes\eta} \nwarrow \quad \nearrow {\scriptstyle \eta} & \\ & R & \end{array}$$

$$\begin{array}{ccc} & B & \\ {\scriptstyle \eta} \nearrow & & \searrow {\scriptstyle \epsilon} \\ R & \xrightarrow{\mathrm{Id}} & R. \end{array}$$

For example, the last condition means that $\epsilon(1) = 1$.

We could have also required in the definition that the comultiplication and the counits were morphisms of algebras with unit. It would be an equivalent definition.

2.1.3 Hopf Algebras

Definition 2.1.2. *A* Hopf algebra *is a bialgebra H, together with a morphism of R-modules $S : H \to H$ (called the* antipode*) such that the following diagrams commute:*

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow{\scriptstyle \epsilon\circ\eta} & & \downarrow{\scriptstyle S\otimes \mathrm{Id}} \\ H & \xleftarrow{\mu} & H \otimes H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow{\scriptstyle \epsilon\circ\eta} & & \downarrow{\scriptstyle \mathrm{Id}\otimes S} \\ H & \xleftarrow{\mu} & H \otimes H. \end{array}$$

Examples If G is a group, the group algebra $R[G]$ has a canonical Hopf algebra structure, where the coproduct is induced by $g \to g \otimes g$, $g \in G$, and the counit is given by $\epsilon(g) = 1$. The antipode is induced by $S(g) = g^{-1}, g \in G$.

If G is finite, there is an other Hopf algebra we can define: $\mathrm{Maps}(G, R)$. It can be thought of as the "dual" of the previous one. As an R-algebra, it is the algebra of all functions $G \to R$, with pointwise addition and multiplication. The counit $\epsilon : \mathrm{Maps}(G, R) \to R$ is given by

$$\epsilon(f) = f(1), \quad f \in \mathrm{Maps}(G, R).$$

In order to describe the comultiplication, observe that $\mathrm{Maps}(G, R)$ is a free R-module, with basis $\delta_g, g \in G$. Here

$$\delta_g(x) = \begin{cases} 0, & x \neq g \\ g & x = g. \end{cases}$$

We then put

$$\Delta(\delta_g) = \sum_{xy=g} \delta_x \otimes \delta_y, \quad S(\delta_g) = \delta_{g^{-1}}.$$

2.1.4 H-Galois Algebras

Let H be a Hopf algebra over a commutative ring R. Assume that H is finitely generated and projective as an R-module. Recall that an R-module A is called faithful if $ra = 0$ for all $a \in A$ implies $r = 0$.

We are now in a position to define H-Galois algebras, following [38].

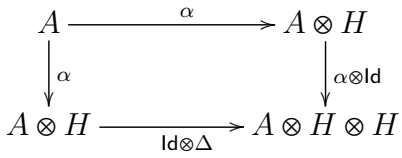
Definition 2.1.3. *An R-algebra A is called H-Galois if the following conditions hold:*

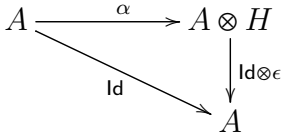
i) A is faithful, finitely generated and projective as an R-module.

ii) There exists an R-algebra homomorphism

$$\alpha : A \to A \otimes H$$

such that the following diagrams





commute.

iii) The composite morphism

$$A \otimes A \xrightarrow{\mathrm{Id}\otimes\alpha} A \otimes A \otimes H \xrightarrow{\mu\otimes\mathrm{Id}} A \otimes H$$

is an isomorphism of R-modules, where, as usual, μ is the multiplication $A \otimes A \to A$.

To put the definition in a more understandable terms, the second condition may be summarised as saying that α is a homomorphism of R-algebras, with respect to which A becomes a right counital, comodule over H. The last condition means that (assuming H and A are commutative), the affine scheme Spec(A) is a torsor over the group scheme Spec(H).

It is well-known that H-Galois algebras form a cat-group, where morphisms are isomorphisms of H-Galois algebras and the monoidal structure is induced by the cotensor product [38].

2.2 The Hopf Algebra $J^{p,q}$

Having considered the general case, we now wish to specialise to what can be considered the "quadratic case". For this, we will introduce the Hopf algebras $J^{p,q}$, the Galois algebras over which will be our objects of interest. They will generalise the separable quadratic algebras. This will be true in both the free and the projective setting, which we will introduce in a little while.

Algebra and Coalgebra Structure Consider an R-module J, which is freely generated by the symbols 1 and x. Let us fix an elements p in R. We can define a commutative algebra structure on J by declaring 1 to be the unit and

$$x^2 = qx.$$

We wish to extend it to a Hopf algebra structure. To this end, we fix an other element $q \in R$ and put

$$\epsilon(x) = 0, \epsilon(1) = 1, \Delta(1) = 1 \otimes 1 \quad \text{and} \quad \Delta(x) = 1 \otimes x + x \otimes 1 + p(x \otimes x).$$

We claim that in this way, we obtain a cocommutative and coassociative coalgebra. To verify this claim, one would have to check the commutativity of the diagrams in the definition of coalgebras (2.1.1). A typical computation would look like this:

$$\begin{aligned}(\mathrm{Id}\otimes\epsilon)\Delta(x) &= (\mathrm{Id}\otimes\epsilon)(x\otimes 1+1\otimes x+px\otimes x)\\ &= x\epsilon(1)+1\epsilon(x)+px\epsilon(x)\\ &= x,\end{aligned}$$

which shows that $(\mathrm{Id}\otimes\epsilon)\Delta$ and Id take on the save value on x. The fact that they agree on 1 is easier to see. This implies the commutativity of the second diagram in Paragraph 2.1.1. The other diagrams are checked in a similar way.

Bialgebra Structure So far, p and q were arbitrary elements in R. The question we wish to explore is, whether we obtain a bialgebra or not. One can check that all the diagrams in Definition 2.1.1, except for the first one, commute. Moreover, we have

$$\Delta(\mu(x\otimes x)) = \Delta(x^2) = \Delta(qx) = qx\otimes 1+q1\otimes x+pqx\otimes x.$$

On the other hand, the value of $(\mu\otimes\mu)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})\circ(\Delta\otimes\Delta)$ on $x\otimes x$ is equal to the following longer expression:

$$\begin{aligned}(\mu\otimes\mu)(&1\otimes 1\otimes x\otimes x+1\otimes x\otimes x\otimes 1+p1\otimes x\otimes x\otimes x\\ &+x\otimes 1\otimes 1\otimes x+x\otimes x\otimes 1\otimes 1+px\otimes x\otimes 1\otimes x\\ &+pqx\otimes 1\otimes x\otimes x+px\otimes x\otimes x\otimes 1+p^2x\otimes x\otimes x\otimes x).\end{aligned}$$

This is the same as

$$\begin{aligned}&1\otimes qx+x\otimes x+pqx\otimes x\\ &+x\otimes x+qx\otimes 1+pqx\otimes x\\ &+px\otimes x+pqx\otimes x+p^2q^2x\otimes x.\end{aligned}$$

Thus, in order to get a bialgebra, we must have

$$pq = 1+pq+1+pq+pq+pq+p^2q^2.$$

One can easily show that if this holds, we really obtain a bialgebra. This condition can be rewritten as

$$(pq+2)(pq+1) = 0. \tag{2.1}$$

It is well-known [38, p. 354] that any free bialgebra of rank two is of this form.

Hopf Algebra Structure Let $Sx = kx$, where $k \in R$. We wish to investigate under which condition we obtain a Hopf algebra structure. We have

$$\mu(S \otimes \mathrm{Id})\Delta(x) = \mu(1 \otimes x + kx \otimes 1 + pkx \otimes x) = (1 + k + kpq)x.$$

On the other hand,

$$\eta\epsilon(x) = 0.$$

Thus, in order to get a Hopf algebra, we need to have

$$1 + k(1 + pq) = 0. \tag{2.2}$$

We now have two different conditions (2.1) and (2.2), both of which need to be satisfied. The second condition implies that $1 + pq$ is invertible. Thus, (2.1) yields

$$pq + 2 = 0, \quad k = 1.$$

Though our assumption on the antipode ($S(x) = kx$) seemed to be a special case, it is well known that any Hopf algebra of rank two is of this form [38, Theorem 1.2]. In particular, any such Hopf algebra is commutative and cocommutative. We will denote it by $J^{p,q}$. In this Hopf algebra, we have

$$x^2 = qx, \quad \Delta(x) = 1 \otimes x + x \otimes 1 + px \otimes x, \quad \eta(x) = 0, \quad S(x) = x.$$

Denote by $\mathfrak{Qu}^{pq}(R)$ the cat-group of H-Galois algebras, where $H = J^{p,q}$. The morphisms of $\mathfrak{Qu}^{pq}(R)$ are defined to be isomorphisms of H-Galois algebras and the monoidal structure is induced by the cotensor product over H. We denote the isomorphism classes of the groupoid $\mathfrak{Qu}^{pq}(R)$ by $\mathrm{Qu}^{pq}(R)$.

Example As an example, take $G = \mathbb{Z}/2\mathbb{Z}$. Denote by t the generator of G and put $x = 1 - t$. Since $t^2 = 1$, we have

$$x^2 = -2x.$$

We also have

$$\begin{aligned}\Delta(x) - 1 \otimes x - x \otimes 1 &= \Delta(t-1) - (t-1) \otimes 1 - 1 \otimes (t-1) \\ &= t \otimes t - 1 \otimes 1 - t \otimes 1 + 1 \otimes 1 - 1 \otimes t + 1 \otimes 1 \\ &= (t-1) \otimes (t-1) = x \otimes x.\end{aligned}$$

Thus, $R[\mathbb{Z}2/2\mathbb{Z}]$ and $J^{1,-2}$ are isomorphic as Hopf algebras. Quite similarly, we have an isomorphism of Hopf algebras

$$\mathrm{Maps}(\mathbb{Z}/2\mathbb{Z}, R) \simeq J^{-2,1}.$$

2.2.1 Free $J^{p,q}$-Galois Algebras

In what follows, $J^{p,q}$ is used interchangeably with H. For $a, b \in R$, we denote by $A_{[a,b]}$ the free R-module of rank two, with basis $1, v$. We define the multiplicative structure by

$$v^2 = aqv + b.$$

Of course, 1 will be the unit of $A_{[a,b]}$. We used the notation b for $b \cdot 1$. Define

$$\alpha : A_{[a,b]} \to A_{[a,b]} \otimes J^{p,q}$$

by

$$\alpha(1) = 1 \otimes 1, \quad \alpha(v) = a \cdot 1 \otimes x + v \otimes 1 + pv \otimes x. \tag{2.3}$$

Recall also that $S : J^{p,q} \to J^{p,q}$ is the identity map for $H = J^{p,g}$.

Lemma 2.2.1. *i) The map α, defined above, is an algebra homomorphism.*

ii) One has

$$(S \otimes \epsilon) \circ \alpha = \mathsf{Id}_S.$$

iii) We have

$$(\alpha \otimes \mathsf{Id}) \circ \alpha = (\mathsf{Id} \otimes \Delta) \circ \alpha.$$

Proof. i) We have to show that α respects the algebra structures. That is to say, we have to show that

$$(\alpha(v))^2 = aq\alpha(v) + b\alpha(1).$$

The left hand side is equal to

$$\begin{aligned}(\alpha \cdot 1 \otimes x + v \otimes 1 + p \cdot v \otimes x)^2 &= (\alpha^2 q + bp^2 q + 2bq) \cdot 1 \otimes x + aq \cdot v \otimes 1 \\ &+ b \cdot 1 \otimes 1 + (p^2 q^2 + 2apq + 2apq) \cdot v \otimes x,\end{aligned}$$

while the right hand side is equal to

$$a^2 q \cdot 1 \otimes x + aq \cdot v \otimes 1 + apq \cdot v \otimes x + b \cdot 1 \otimes 1.$$

Since $pq = -2$, these expressions equal and hence, α is an R-algebra homomorphisms.
ii) The proof is straightforward.
iii) This is an equality of maps $A_{[a,b]} \to A_{[a,b]}$. Obviously, both maps are R-linear and have the same value at 1. Thus, it suffices to show that they have the save values at v. We have

$$\begin{aligned}(\alpha \otimes \mathsf{Id}) \circ \alpha(v) &= \alpha \otimes \mathsf{Id}(a \cdot 1 \otimes x + v \otimes 1 + p \cdot v \otimes x) \\ &= a \cdot 1 \otimes 1 \otimes x + a \cdot 1 \otimes x \otimes 1 + v \otimes 1 \otimes 1 + p \cdot v \otimes x \otimes 1 \\ &+ pa \cdot 1 \otimes x \otimes x + p \cdot v \otimes 1 \otimes x + p^2 \cdot v \otimes x \otimes x.\end{aligned}$$

Similarly

$$\begin{aligned}(\mathrm{Id}\otimes\Delta)\circ\alpha(v) &= (\mathrm{Id}\otimes\Delta)(a\cdot 1\otimes x + v\otimes 1 + p\cdot v\otimes x)\\ &= a\cdot 1\otimes 1\otimes x + a\cdot 1\otimes x\otimes 1 + ap\cdot 1\otimes x\otimes x\\ &+ v\otimes 1\otimes 1 + p\cdot v\otimes 1\otimes x + p\cdot v\otimes x\otimes 1 + p^2 v\otimes x\otimes x,\end{aligned}$$

and part iii) follows. $\mathcal{Q.E.D}$

Proposition 2.2.2. *The algebra $A_{[a,b]}$, together with the morphism α, is an H-Galois algebra if and only if $a^2 + p^2 b$ is invertible in R.*

Proof. Condition ii) of Definition 2.1.3 holds, as shown in Lemma 2.2.1. Condition i) of the same definition obviously holds, because $A_{[a,b]}$ is free as an R-module. It remains to investigate condition iii). The elements $1\otimes 1, v\otimes 1, 1\otimes v$ and $v\otimes v$ form a basis of $A_{[a,b]}\otimes A_{[a,b]}$, while $1\otimes 1, v\otimes 1, 1\otimes x$ and $v\otimes x$ form a basis of $A_{[a,b]}\otimes H$. In these bases, the map

$$(\mu\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\alpha) : A_{[a,b]}\otimes A_{[a,b]}\to A_{[a,b]}\otimes H$$

is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & b\\ 0 & 1 & 1 & ap\\ 0 & 0 & a & bp\\ 0 & 0 & p & a+apq\end{pmatrix}.$$

Since $pq = -2$, we see that this matrix is invertible if and only if $a^2 + p^2 b$ is invertible in R, as desired. $\mathcal{Q.E.D}$

Let

$$(A_{[c,d]}, \beta : A_{[c,d]}\to A_{[c,d]}\otimes H)$$

be another H-Galois algebra, spanned by 1 and w with

$$w^2 = cqw + d \quad \text{and} \quad \beta(w) = c\cdot 1\otimes x + w\otimes 1 + p\cdot w\otimes x.$$

We have the following lemma:

Lemma 2.2.3. *Let $r\in R$ and $s\in R^*$ be invertible. A map*

$$f_{(r,s)} : A_{[c,d]}\to A_{[a,b]},$$

given by

$$f_{(r,s)}(1) = 1, \quad f_{(r,s)}(w) = r + sv,$$

is an isomorphism of H-Galois algebras if and only if

$$c = as - pr \quad \text{and} \quad d = s^2 b - qrsa - r^2.$$

Conversely, any isomorphism of H-Galois algebras $A_{[c,d]}\to A_{[a,b]}$ has this form.

Proof. Since any homomorphism of algebras sends 1 to 1, any homomorphism $A_{[c,d]} \to A_{[a,b]}$ must be of the form $\begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix}$. Moreover, such a map is a linear isomorphism if and only if s is invertible.

We next need to determine under which condition $f = f_{(r,s)}$ respects the coaction and multiplication.
Compatibility with the coaction means

$$(f \cdot \mathrm{Id}_H) \circ \beta = \alpha \circ f.$$

Clearly, both have the same value at 1. Next, we examine their values at x. We have

$$(f \cdot \mathrm{Id}_H) \circ \beta(w) = r \cdot 1 \otimes 1 + sa \cdot 1 \otimes x + s \cdot v \otimes 1 + ps \cdot v \otimes x$$

and

$$\alpha \circ f(w) = r \cdot 1 \otimes 1 + (c + pr) \cdot 1 \otimes x + s \cdot v \otimes 1 + ps \cdot v \otimes x.$$

Thus, f is compatible with the coaction of H if and only if $sa = c + pr$. The map f respects multiplication if and only if

$$(r + sr)^2 = cq(r + sv) + d.$$

This is equivalent to

$$r^2 + s^2 = cqr + d \quad \text{and} \quad 2rs + s^2aq = cqs.$$

The first one is a consequence of $sa = c + pr$ and $pq + 2 = 0$, while for the second one, we have

$$d^2 = r^2 + sb^2 - cqr = r^2 + s^2b - asr + pqr^2 = -r^2 + s^2 - asrq.$$

The result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Proposition 2.2.4. *One has an isomorphism of $J^{p,q}$-Galois algebras*

$$A_{[a,b]} \boxtimes_{J^{p,q}} A_{[c,d]} \simeq A_{[ac,a^2d+bc^2+p^2cd]}.$$

2.3 The cat-group $\mathfrak{Qu}_f^{pq}(R)$

Let R be a commutative ring and p,q two elements of R such that $pq + 2 = 0$. These elements will be fixed throughout this section. Of special interest is the case $p = -2$, $q = 1$, as this yields the *free quadratic group* introduced in [26, p. 31], see our discussion below.

We define the groupoid $\mathfrak{Qu}_f^{pq}(R)$ as follows: Objects are pairs $[a, b]$, where $a, b \in R$ and $a^2 + p^2b \in R^*$ (see Proposition 2.2.2). Morphisms $[c, d] \to [a, b]$ are pairs (u, r) such that $r \in R, u \in R^*$, and the following relations hold:

$$c = au - pr \quad \text{and} \quad d = u^2b - qrua - r^2 \tag{2.4}$$

(see Lemma 2.2.3). The composition of morphisms is defined by

$$(v,s)\circ(u,r)=(vu,rv+s).$$

The morphism $(1,0):[a,b]\to[a,b]$ is the identity morphism of $[a,b]$. We define the functor

$$\star:\mathfrak{Qu}_f^{pq}(R)\times\mathfrak{Qu}_f^{pq}(R)\to\mathfrak{Qu}_f^{pq}(R)$$

as follows (see Proposition 2.2.4): On objects, we set

$$[a_1,b_1]\star[a_2,b_2]=[a_1a_2,a_1^2b_2+a_2^2b_1+p^2b_1b_2],$$

where we write $[a_1,b_1]\star[a_2,b_2]$ instead of $\star([a_1,b_1],[a_2,b_2])$, and on morphisms $\star$ is given by

$$(u_1,r_1)\star(u_2,r_2)=(u_1u_2,-pr_1r_2+r_1u_2a_2+u_1a_1r_2).$$

One checks that $\star$ defines a symmetric monoidal structure on $\mathfrak{Qu}_f^{pq}(R)$, whose associativity and commutativity constraints are identities. The neutral object is $[1,0]$. Since the pair (a^2+p^2b,pb) defines an isomorphism $[a,b]\star[a,b]\to[1,0]$, the symmetric monoidal groupoid $\mathfrak{Qu}_f^{pq}(R)$ is in fact a cat-group. Moreover, every object has order two, up to isomorphism.

Our aim is to understand the groups $\pi_i(\mathfrak{Qu}_f^{pq}(R)), i=0,1$. The case $i=1$ is significantly easier than $i=0$, and the answer is given here:

Lemma 2.3.1. *i) The rule $r+_1 s:=r+s+prs$ induces the structure of an abelian group on*

$$\mathbb{Z}_{pq}(R)=\{r\in R\mid r^2=qr\}.$$

ii) The map $r\mapsto(1+pr,r)$ yields an isomorphism of groups

$$\mathbb{Z}_{pq}(R)\xrightarrow{\simeq}\pi_1(\mathfrak{Qu}_f^{pq}(R)).$$

Proof. i) First, we have to show that $\mathbb{Z}_{pq}(R)$ is closed under this operation: Let $r,s\in\mathbb{Z}_{pq}$, meaning $r^2=qr$ and $s^2=qs$. We have

$$\begin{aligned}(r+s+prs)^2 &= r^2+s^2+p^2r^2s^2+2rs+2pr^2s+2prs^2\\ &= qr+qs+p^2q^2rs+2rs+2pqrs+2pqrs\\ &= qr+qs-2rs\\ &= q(r+s+prs),\end{aligned}$$

hence, $r+_1 s\in\mathbb{Z}_{pq}$. Associativity is straightforward to check and clearly, 0 is the neutral element. The inverse is the element itself, meaning

$$r+_1 r=r+r+pr^2=2r+pqr=0,$$

as $p,q\in R$ were explicitly chosen such that $pq+2=0$.

ii) By definition, $\pi_1(\mathfrak{Qu}_f^{pq}(R))$ is the collection of all morphisms $[1,0]\to[1,0]$. Any such

morphism is given by pairs $(u,r), r \in R, u \in R^*$ such that

$$1 = u - pr, \quad \text{and} \quad 0 = -qru - r^2,$$

see the equations (2.4). This yields $u = pr + 1$, which in turn gives us $0 = -qrpr - qr - r^2$. Since $pq = -2$, we obtain $r^2 = qr$. Thus, $r \in \mathbb{Z}_{pq}(R)$ and $(u,r) \mapsto r$ defines the map

$$\pi_1(\mathfrak{Qu}_f^{pq}(R)) \mapsto \mathbb{Z}_{pq}(R).$$

This is a bijection with the inverse being defined by $r \mapsto (pq+1, r)$. The only think to check is the invertibility of $pq+1$:

$$(pr+1)(pr+1) = p^2r^2 + 2pr + 1 = p^2qr + 2pq + 1 = 1.$$

To check that this bijection respects the operations is also straightforward. $\mathcal{Q.E.D}$

We let $\mathrm{Qu}_f^{pq}(R)$ denote the group of isomorphism classes of $\mathfrak{Qu}_f^{pq}(R)$, that is

$$\mathrm{Qu}_f^{pq}(R) = \pi_0(\mathfrak{Qu}_f^{pq}(R)).$$

Observe that if $p = -2$ and $q = 1$, the relations in (2.4) are exactly the same as in [26, pp. 29-30]. Hence, $\mathrm{Qu}_f^{-2,1}(R)$ is the same as the *free quadratic group* introduced in [26, p. 31], where it is denoted by $\mathrm{Qu}_f(R)$. Recall that the group $\mathrm{Qu}_f(R)$ is the group of isomorphism classes of free separable R-algebras of rank two [26, Ch. 3A]. It is a direct implication of Proposition 2.4.2 that the groups $\mathrm{Qu}^{pq}(R)$ have a similar interpretation for all (p,q). See [26] for an extensive study of $\mathrm{Qu}_f(R)$. The group $\mathbb{Z}_{-2,1}(R)$ is the group of idempotents of R and it appears in many places, see for example [26, p. 202], where it is denoted by $\mathbb{Z}_2(R)$.

Proposition 2.3.2 (An isomorphism of categories)**.** *Let (p,q) be a pair of elements in R such that $pq + 2 = 0$. For any invertible element $t \in R^*$, there is an isomorphism of categories*

$$t_* : \mathfrak{Qu}_f^{pq}(R) \to \mathfrak{Qu}_f^{pt,t^{-1}q}(R).$$

Proof. As $(pt)(t^{-1}q) + 2 = pq + 2 = 0$, both categories are well-defined. Let $[a,b] \in \mathfrak{Qu}_f^{pq}(R)$, meaning $a^2 + p^2b^2 \in R^*$. Then $(at)^2 + (pt)^2b^2 = t^2(a^2 + p^2b^2)$ is also invertible. Therefore, $[at,b]$ is also an object of $\mathfrak{Qu}_f^{pt,t^{-1}q}(R)$. This gives us a map t_* from the objects of $\mathfrak{Qu}_f^{pq}(R)$ to the objects of $\mathfrak{Qu}_f^{pt,t^{-1}q}(R)$, given by

$$t_*([a,b]) = [at,b].$$

This map is clearly bijective, with the inverse being given by $[a',b'] \mapsto [a't^{-1}, b']$.

Let $(u,r) : [c,d] \to [a,b]$ be a morphism in $\mathfrak{Qu}_f^{pq}(R)$. This means that u and r are elements of R such that $u \in R^*$ and

$$c = au - pr, \quad d = u^2b - qrua - r^2.$$

Since

$$ct = (at)(u) - (pt)r \qquad \text{and} \qquad d = u^2b - (b^{-1}q)u(ta) - r^2,$$

we see that the same pair (u, r) can be considered as a morphism

$$t_*([c, d]) \to t_*([a, b]).$$

Hence, we can extend our map t_* to a functor by putting $t_*(u, r) = (u, r)$. This is an isomorphism of categories, and indeed of cat-groups. $\mathcal{Q.E.D}$

2.3.1 The Cat-group $\mathfrak{G}(R)$ and the symmetric monoidal functor $\alpha_f : \mathfrak{Qu}_f^{pq}(R) \to \mathfrak{G}(R)$

Consider the homomorphism of multiplicative abelian groups

$$R^* \xrightarrow{\mathsf{sq}} R^*, \quad \mathsf{sq}(r) = r^2.$$

Denote its associated cat-group by $\mathfrak{G}(R)$. We have

$$\pi_0(\mathfrak{G}(R)) = \mathsf{U}_2(R) \quad \text{and} \quad \pi_1(\mathfrak{G}(R)) = \mu_2(R),$$

where

$$\mathsf{U}_2(R) = R^*/(R^*)^2 \quad \text{and} \quad \mu_2(R) = \{r \in R \mid r^2 = 1\}.$$

See our discussion on page 50. Define the morphism of categorical groups

$$\alpha_f : \mathfrak{Qu}_f^{pq}(R) \to \mathfrak{G}(R)$$

as follows: For an object $[a, b]$ of $\mathfrak{Qu}_f^{pq}(R)$, put

$$\alpha_f([a, b]) = a^2 + p^2b.$$

If $(u, r) : [c, d] \to [a, b]$ is a morphism in $\mathfrak{Qu}_f(R)$, we have

$$c^2 + p^2d = (au - pr)^2 + p^2(u^2b - qrua - r^2) = u^2(a^2 + p^2b),$$

thanks to the equalities in (2.4). Thus, u can be considered as a morphism $c^2 + p^2d \to a^2 + p^2b$ in $\mathfrak{G}(R)$, and we put

$$\alpha_f(u, r) = u : \alpha([c, d]) \to \alpha([a, b]).$$

The morphism of cat-groups $\alpha_f : \mathfrak{Qu}_f^{pq}(R) \to \mathfrak{G}(R)$ induces the homomorphisms of abelian groups

$$\mathbb{Z}_{pq}(R) = \pi_1(\mathfrak{Qu}_f(R)) \to \pi_1(\mathfrak{G}(R)) = \mu_2(R)$$

and

$$\mathsf{Qu}_f^{pq}(R) = \pi_0(\mathfrak{Qu}_f(R)) \to \pi_0(\mathfrak{G}(R)) = \mathsf{U}_2(R).$$

We note that the first homomorphism is given by $a \mapsto 1 + pa$. It is well known that for

$p = -2, q = 1$, the second homomorphism fits in the exact sequence (see [26, (3.6) on p.32])

$$\mathsf{Qu}_f(R) \to \mathsf{U}_2(R) \to \mathsf{U}_2(R/4R). \tag{2.5}$$

We wish to extend this sequence on the left hand side (for arbitrary (p, q)), as well as lift it to the level of cat-groups. This will then yield an analogous exact sequence after stackification. Observe that the obvious candidate

$$\mathfrak{Qu}_f^{pq}(R) \to \mathfrak{G}(R) \to \mathfrak{G}(R/p^2R)$$

does not work, as the composite functor $\mathfrak{Qu}_f^{pq}(R) \to \mathfrak{G}(R/p^2R)$ is not naturally isomorphic to the trivial one, not even when $p = -2, q = 1$. As such, we have to modify the last cat-group.

2.3.2 The Cat-group $\mathfrak{G}^p(R)$ and the morphism $\beta_f : \mathfrak{G}(R) \to \mathfrak{G}^p(R)$

We will need the following easy and well-known fact.

Lemma 2.3.3. *Let R be a commutative ring and $r \in (R/pR)^*$. If $\tilde{r} \in R/p^2R$ is a lifting of r, then $\tilde{r} \in (R/p^2R)^*$. Moreover, there is a well-defined homomorphism*

$$\mathsf{Sq} : (R/pR)^* \to (R/p^2R)^*,$$

given by $\mathsf{Sq}(r) = \tilde{r}^2$.

Proof. If r is invertible and $s = r^{-1}$ in R/pR, then $\tilde{r}\tilde{s} = 1 + px$ for some $x \in R/p^2R$. Here, $\tilde{s}$ is a lifting of s in R/p^2R, and we obtain $\tilde{r}\tilde{s}(1 - px) = 1$ by multiplying both sides with $(1 - px)$. It follows that $\tilde{r}$ is invertible. To show the last assertion, observe that for any $x \in R/p^2R$, one has $(\tilde{r} + px)^2 = \tilde{r}^2 + 2prx = \tilde{r}^2$. This is because $2 = -pq$ and $p^2 = 0$ in R/p^2. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Let $\mathfrak{G}^p(R)$ be the cat-group corresponding to the homomorphism

$$\mathsf{Sq} : (R/pR)^* \to (R/p^2R)^*.$$

Lemma 2.3.4. *We have*

$$\pi_0(\mathfrak{G}^p(R)) = \mathsf{U}_2(R/p^2R)$$

and

$$\pi_1(\mathfrak{G}^p(R)) = \ker((R/pR)^* \xrightarrow{\mathsf{Sq}} (R/p^2R)^*) \cong \mathsf{Im}(\mu_2(R/p^2R) \to \mu_2(R/pR)).$$

Proof. To see the first isomorphism, it suffices to note that

$$\mathsf{Im}(\mathsf{Sq}) = \mathsf{Im}((R/p^2R)^* \xrightarrow{\mathsf{sq}} (R/p^2R)^*).$$

The second statement is obvious. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Since the following diagram of abelian groups

$$\begin{array}{ccc} R^* & \longrightarrow & (R/pR)^* \\ {\scriptstyle \mathsf{sq}}\downarrow & & \downarrow{\scriptstyle \mathsf{Sq}} \\ R^* & \longrightarrow & (R/p^2R)^* \end{array}$$

commutes, it gives rise to a morphism of categorical groups

$$\beta_f : \mathfrak{G}(R) \to \mathfrak{G}^p(R).$$

2.3.3 The natural isomorphism $\delta_f : \beta_f\alpha_f \Rightarrow 1$

Take an object $[a,b]$ in $\mathfrak{Qu}_f(R)$. Since $a^2 + p^2b \in R^*$, the element $a \mod pR$ is invertible in R/pR. As such, it can be considered as a morphism $a^2 \mod p^2R \to 1$ in $\mathfrak{G}^p(R)$. We denote this map by $\delta_f([a,b])$ and obtain $\delta_f([a,b]) : (\beta_f \circ \alpha_f)([a,b]) \to 1$. Moreover, if $(u,r) : [c,d] \to [a,b]$ is a morphism of $\mathfrak{Qu}_f^{pq}(R)$, one has $c = ua \mod pR$, thanks to the equations in (2.4) on page 63. This means that the following diagram

$$\begin{array}{ccc} 1 & \overset{\delta_f([c,d])}{\underset{\delta_f([a,b])}{\rightleftarrows}} & c^2 \mod p^2R \\ & & \downarrow{\scriptstyle u \mod pR} \\ & & a^2 \mod p^2R \end{array}$$

commutes in $\mathfrak{G}^p(R)$. Thus, $\delta_f : \beta_f\alpha_f \Rightarrow 1$ is a natural transformation from $\beta_f \circ \alpha_f$ to the trivial functor.

2.3.4 $\mathfrak{Qu}_f^{pq}(R)$ as a 2-kernel

We can now formulate the main result of this section.

Theorem 2.3.5. *i) The sequence of cat-groups*

$$\mathfrak{Qu}_f^{pq}(R) \xrightarrow[\alpha_f]{} \mathfrak{G}(R) \xrightarrow[\beta_f]{} \mathfrak{G}^p(R), \qquad 1 \overset{\delta_f \Uparrow}{}$$

is exact at $\mathfrak{G}(R)$. That is, the induced functor

$$\gamma_f : \mathfrak{Qu}_f^{pq}(R) \to 2\text{-}\ker(\beta_f)$$

is essentially surjective.

ii) If p is not a zero-divisor in R, then γ_f is an equivalence of cat-groups.

Proof. i) According to Proposition 1.4.9, the cat-group $2\text{-}\ker(\mathfrak{G}(R) \to \mathfrak{G}^p(R))$ is equivalent to the cat-group $\mathfrak{G}_h$ associated to the following homomorphism of abelian groups:

$$R^* \xrightarrow{h} \{(x,s) \mid x \in R^*, s \in (R/pR)^*, x \equiv \tilde{s}^2 \mod p^2R\}.$$

Here, $\tilde{s} \in R/p^2R$ is a lifting of s and

$$h(u) = (u^2, u \mod pR).$$

Hence, the induced functor

$$\gamma_f : \mathfrak{Qu}_f(R) \to \mathfrak{G}_h$$

is given by

$$\begin{cases} \gamma_f([a,b]) = (a^2 + p^2b, a \mod pR) & \text{on objects} \\ \gamma_f(u,r) = u & \text{on morphisms.} \end{cases}$$

Take an object (x,s) in $\mathfrak{G}_h$. By assumption, $x \in R^*$ and there are elements $a, b \in R$ such that $x = a^2 + p^2b$ and $s = a \mod pR$. Thus $\gamma_f([a,b]) = (x,s)$, and so, γ_f is surjective on objects.

ii) Thanks to part i), we only need to show that the functor γ_f is full and faithful. Take two objects $[a,b], [c,d]$ in $\mathfrak{Qu}_f^{pq}(R)$ and let $(u,r), (u',r')$ be two morphisms $[c,d] \to [a,b]$. In particular, $c = ua - pr = u'a - pr'$. If $\gamma_f(u,r) = \gamma_f(u',r')$, then $u = u'$, and so $pr = pr'$. Thus $r = r'$, implying that γ_f is faithful. Take a morphism $u : \gamma_f([a,b]) \to \gamma_f([c,d])$ in $\mathfrak{G}_h$. By definition, $u \in R^*$ and the following two conditions hold:

$$c^2 + p^2d = u^2(a^2 + p^2b) \quad \text{and} \quad c \equiv ua \mod pR.$$

The last condition implies that there exist an element $r \in R$ such that $c = ua + pr$. We have

$$p^2d = -c^2 + u^2(a^2 + p^2b) = -(ua - pr)^2 + u^2(a^2 + p^2b) = p^2(u^2b - qrua - r^2).$$

Since p is not a zero-divisor in R, we see that $d = u^2b - qrua - r^2$. This, together with the equality $c = ua + pr$, shows that the pair (u,r) is a morphism $[c,d] \to [a,b]$ in $\mathfrak{Qu}_f(R)$, implying that γ_f is full. $\mathcal{Q.E.D}$

Corollary 2.3.6. *i) For any commutative ring R, one has an exact sequence of abelian groups*

$$\mathrm{Qu}_f^{pq}(R) \to \mathrm{U}_2(R) \to \mathrm{U}_2(R/p^2R). \tag{2.6}$$

ii) If p is not a zero divisor in R, one has an exact sequence of abelian groups

$$0 \to \mathbb{Z}_{pq}(R) \to \mu_2(R) \to \mathrm{Im}(\mu_2(R/p^2R) \to \mu_2(R/pR)) \to \mathrm{Qu}_f(R) \to \mathrm{U}_2(R) \to \mathrm{U}_2(R/p^2R). \tag{2.7}$$

iii) For any commutative ring R, the functor $\mathfrak{Qu}_f^{1,-2}(R) \to \mathfrak{G}(R)$ is an equivalence of cat-groups. In particular, $\mathrm{Qu}_f^{1,-2}(R) \cong \mathrm{U}_2(R)$.

Proof. i) By applying π_0 to the exact sequence of cat-groups constructed in Theorem 2.3.5 i), one obtains the exact sequence (2.6). Similarly, the exact sequence (2.6) is a specialization of

the above, thanks to Theorem 2.3.5 ii). If $p = 1$, p is not a zero-divisor and as such, we can use Theorem 2.3.5 ii). The result now follows since $\mathfrak{G}^{1,-2}(R)$ is the trivial cat-group. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Remark 2.3.7. *i) The exact sequence (2.6) in Corollary 2.3.6, in the case $p = -2, q = 1$, is well-known (see the statement [26, (3.6) on p.32])). The second exacts sequence (2.7), however, is new, even in the case $p = -2, q = 1$.*

ii) If 2 is invertible in R, the categories $\mathfrak{Qu}_f^{-2,1}(R)$ and $\mathfrak{Qu}_f^{1,-2}(R)$ are equivalent (see Proposition 2.3.2). In this case, we recover the well-known isomorphism $\mathrm{Qu}_f^{-2,1}(R) \cong \mathrm{U}_2(R)$, see [26, Statement (3.4) i). on p.32].

2.3.5 An exact sequence

The aim of this section is to generalize the result [26, Exercises 6-10, p.41] to arbitrary $\mathrm{Qu}^{pq}(R)$. To this end, we define the set

$$\mathcal{V}_1(R) = \{r \in R \mid 1 + pr \in R^*\},$$

and equip it with the operation $+_1$, by declaring

$$r +_1 s := r + s + prs.$$

One easily sees that, under this operation, $\mathcal{V}_1(R)$ is an abelian group (0 is also the zero with respect to $+_1$). We also need the following set

$$\mathcal{V}_2(R) = \{x \in R \mid 1 + p^2x \in R^*\},$$

which is also an abelian group under the operation $+_2$, where

$$x +_2 y := x + y + p^2xy.$$

It is straightforward to check that the map

$$\zeta : \mathcal{V}_1(R) \to \mathcal{V}_2(R), \quad \zeta(r) = r^2 - qr$$

is a group homomorphism and as such, defines a cat-group, which we denote by $\mathfrak{V}^{pq}(R)$. We have

$$\pi_1(\mathfrak{V}^{pq}(R)) = \{r \in R \mid r^2 = qr \quad \text{and} \quad 1 + pr \in R^*\},$$

whose group structure agrees with $+_1$. We also set

$$\mathrm{V}^{pq}(R) := \pi_0(\mathfrak{V}^{pq}(R)).$$

In the case when $p = -2$ and $q = 1$, this group is the same group as one defined in [26, Exercise 6, p.41]. We construct the symmetric monoidal functor

$$\rho : \mathfrak{V}^{pq}(R) \to \mathfrak{Qu}_f^{pq}(R)$$

as follows: On objects, it is given by $\rho(x) = [1, x]$. If r is a morphism from y to x in $\mathfrak{V}^{pq}(R)$ (that is, $y = x + (-qr + r^2)(1 + p^2x)$), then $\rho(r) = (1 + pr, r)$, which is considered as a morphism $[1, y] \to [1, x]$ in $\mathfrak{Qu}_f^{pq}(R)$.

We also need a cat-group $\mathfrak{S}^{pq}(R)$. The objects of $\mathfrak{S}^{pq}(R)$ are elements $a \in R$ for which there exist $b \in R$ such that $a^2 + p^2b \in R^*$. Let a and c be objects of $\mathfrak{S}^{pq}(R)$. A morphism $c \to a$ is an element $u \in R^*$ for which there exist an element $r \in R$ such that $c = au - pr$. The composition of morphisms and the symmetric monoidal structure is induced by the multiplication in R. Its associativity and commutativity constraints are identity maps. For any object a of $\mathfrak{S}^{pq}(R)$, the element $v = a^2 + p^2b$ is invertible in R. This shows that v defines a morphism $a^2 \to 1$. Hence, objects have multiplicative weak inverses and so, $\mathfrak{S}^{pq}(R)$ is a cat-group. One sets

$$\mathsf{S}^{pq}(R) := \pi_0(\mathfrak{S}^{pq}(R)).$$

Define a symmetric monoidal functor $\omega : \mathfrak{Qu}_f^{pq}(R) \to \mathfrak{S}^{pq}(R)$ as follows: On objects, we set $\omega([a, b]) = a$. If $(u, r) : [c, d] \to [a, b]$ is a morphism of $\mathfrak{Qu}_f^{pq}(R)$, we set $\omega(u, r) = u$.

As already mentioned, the following result is well-known when $p = -2$ and $q = 1$ [26, Exerecises 9 and 10, p. 41].

Proposition 2.3.8. *i) One has a short exact sequence of abelian groups*

$$0 \to \mathsf{V}^{pq}(R) \xrightarrow{\rho_*} \mathsf{Qu}_f^{pq}(R) \xrightarrow{\omega_*} \mathsf{S}^{pq}(R) \to 0, \tag{2.8}$$

where ρ_* and ω_* are induced by the functors ρ and ω.

ii) If R is a local ring, then $\mathsf{S}^{pq}(R) = 0$.

iii) If $p \in \mathsf{Rad}(R)$, then $\mathsf{S}^{pq}(R) = 0$. Here, $\mathsf{Rad}(R)$ is the intersection of all maximal ideals of R. In particular, if $p = 0$, then

$$\mathsf{Qu}_f^{0q}(R) = R/R_0, \quad \text{where} \quad R_0 = \{r^2 + qr | r \in R\} \subset R.$$

Proof. i) Since $1 + p^2x \in R^*$, it is straightforward that $\omega_* \circ \rho_*(x) = 1$, hence Equation (2.8) is a chain complex.

Let us show that ω_* is an epimorphism. Take an object a of $\mathfrak{S}^{pq}(R)$. By definition, there exist $b \in R$ such that $a^2 + p^2b \in R^*$. Thus, $[a, b] \in \mathfrak{Qu}_f^{pq}(R)$ and $\omega([a, b]) = a$.

Next, we will show the exactness at $\mathsf{Qu}_f^{pq}(R)$. Take an object $[a, b]$ of $\mathfrak{Qu}_f^{pq}(R)$ such that $\omega_*([a, b]) = 0$. This means that there exist $u \in R^*$ and $r \in R$ such that $1 = au - pr$. We set $d = u^2b - qrua - r^2$. Then (u, r) is a morphism $(1, d) \to (a, b)$. Thus, the class of $[a, b]$ in $\mathsf{Q}_f^{pq}(R)$ equals the class of $[1, d] = \rho(d)$, and exactness at $\mathsf{Qu}_f^{pq}(R)$ follows.

It remains to show that ρ_* is a monomorphism. Assume $x \in R$ is an element satisfying $1 + p^2x \in R^*$, for which there exists a morphism $[1, 0] \to [1, x]$ in $\mathsf{Qu}_f^{pq}(R)$. This means that there exist $u \in R^*$ and $r \in R$ such that $1 = u - pr$ and $0 = u^2x - qru - r^2$. It follows that $u = 1 + pr \in R^*$ and $0 = (1 + pr)^2x - qr(1 + pr) - r^2$. The last equation is equivalent to $0 = x + (r^2 - rq)(1 + p^2x)$, or $0 = x +_2 \zeta(r)$. So, r defines a morphism from $0 \to x$ in $\mathfrak{V}^{pq}(R)$. As such, the class of x is zero in $\mathsf{V}^{pq}(R)$, proving i).

ii) Assume R is a local ring with maximal ideal $\mathfrak{m}$. Take an object a of $\mathfrak{S}^{pq}(R)$. We have $a^2 + p^2b = v \in R^*$, for some $b \in R$. If $p \in \mathfrak{m}$, then $v^{-1}a^2 = 1 - p^2bv^{-1} \notin \mathfrak{m}$, showing that $v^{-1}a^2$

is invertible. This implies that a is invertible and a^{-1} defines a morphism $1 \to a$ in $\mathfrak{S}^{pq}(R)$. Therefore, the class of a is zero in $\mathsf{S}^{pq}(R)$. If $p \notin \mathfrak{m}$, p is invertible in R. If a is invertible, the same argument works as before, so assume $a \in \mathfrak{m}$. Then $1+a$ is invertible in R. Since

$$a = 1 \cdot (1+a) - p\frac{1}{p},$$

we see that the pair $(1+a, \frac{1}{p})$ defines a morphism $a \to 1$ in $\mathfrak{S}^{pq}(R)$. Hence, the class of a is zero in $\mathsf{S}^{pq}(R)$, implying that $\mathsf{S}^{pq}(R) = 0$.

iii) Take an object a of $\mathfrak{S}^{pq}(R)$, that is we have $a^2 + p^2 b = v \in R^*$ for some $b \in R$. Thus, $v^{-1}a^2 = 1 - p^2 b v^{-1}$ and by Nakayama, $1 + \mathsf{Rad}(R) \subseteq R^*$. Hence, a is invertible and a^{-1} defines a morphism $1 \to a$ in $\mathfrak{S}^{pq}(R)$, so the class of a is zero in $\mathsf{S}^{pq}(R)$, and $\mathsf{S}^{pq}(R) = 0$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

2.4 Projective J-Galois Algebras

2.4.1 Remarks on stacks

Recall our discussions on the theory of stacks in Subsection 1.3.2. One easily observes that if $\alpha : \mathcal{S}_1 \to \mathcal{S}_2$ is a morphism of stacks, then $2\text{-}\ker(\alpha)$ is also a stack. Furthermore, the stackification preserves 2-kernels. Recall also that if $\mathcal{S}$ is a stack of cat-groups, then $\pi_1(\mathcal{S})$ is a sheaf, but $\pi_0(\mathcal{S})$ need not be a sheaf in general.

Our underlying site is the Zariski site of affine schemes over $\mathsf{Spec}(R)$. However, we prefer to work with commutative R-algebras instead of affine schemes. Accordingly, a covariant pseudofunctor from commutative R-algebras to the 2-category of cat-groups is called a stack, if the corresponding contravariant pseudofunctor on affine schemes over $\mathsf{Spec}(R)$ is one. It is clear that the assignments $A \mapsto \mathfrak{Qu}_f^{pq}(A), \mathfrak{G}(A), \mathfrak{G}^p(A)$, where A is an R-algebra, define fibred categories. Indeed, all of these are prestacks in the sense of [45], because π_1 of the corresponding cat-groups are sheaves. This is clear for Z_{pq} and μ_2, because they are representable functors. For $\pi_1(\mathfrak{G}^p(A))$, this follows from Lemma 2.3.4. We will identify the stackifications of these prestacks and then use Theorem 2.3.5 to obtain an exact sequence in Theorem 2.4.8. The case $p = -2$, $q = 1$ yields exactly Theorem 2.0.1, which involves the well studied ([26], [65]) group $\mathsf{Qu}(R)$.

2.4.2 The stack $\mathfrak{Qu}^{pq}$

Let p and q be two elements in R such that $pq + 2 = 0$ and $J = J^{pq}$ be a commutative and cocommutative Hopf algebra, which is freely generated as an R-module by 1 and x, with

$$x^2 = qx \quad \text{and} \quad \Delta(x) = x \otimes 1 + 1 \otimes x + p \cdot x \otimes x.$$

As already mentioned on page 60, it is well known [38, Theorem 1.2] that any Hopf algebra, which is free of rank two as an R-module, is of this form. Recall also the definition of J-Galois algebra, which was a commutative and associative algebra with unit A, being additionally faithful, finitely generated and projective as an R-module. Further, there is given an algebra

homomorphism $\eta : A \to A \otimes J$, which makes A a right J-comodule and the composition

$$A \otimes A \xrightarrow{id\otimes\eta} A \otimes A \otimes J \xrightarrow{\mu\otimes id} A \otimes J$$

needs to be an isomorphism, where μ is the multiplication in A [38]. As J^{pq} is given generated by two elements, it follows that the rank of A is 2.

Denote by $\mathfrak{Qu}^{pq}(R)$ the cat-group of J-Galois algebras. Morphisms of $\mathfrak{Qu}^{pq}(R)$ are isomorphism of J-Galois algebras. The monoidal structure is induced by the cotensor product of J. We denote the isomorphism classes of the groupoid $\mathfrak{Qu}^{pq}(R)$ by $\mathrm{Qu}^{pq}(R)$ (the same group is denoted by $A(J)$ in [38]).

Definition 2.4.1. *Let $\mathfrak{F} : \mathfrak{A} \to \mathfrak{B}$ be a functor and consider*

$$\{B \in \mathfrak{B} \mid \textit{there exists } A \in \mathfrak{A}, \textit{ together with an isomorphism } \mathfrak{F}(A) \xrightarrow{\sim} B.\}$$

We call this the essential image *of $\mathfrak{F}$.*

We can define a symmetric monoidal functor

$$\mathfrak{Qu}_f^{pq}(R) \to \mathfrak{Qu}^{pq}(R),$$

which sends an objects $[a, b]$ to (A, η). Here $A = R[t]/(t^2 - aqt - b)$, while the algebra homomorphism $\eta : A \to A \otimes J$ is given by $\eta(v) = a \cdot 1 \otimes x + 1 \cdot v \otimes 1 + p \cdot v \otimes x$, where v denotes the class of t in A, see (2.3). Moreover, if $(u, r) : [c, d] \to [a, b]$ is a morphism in $\mathfrak{Qu}_f^{pq}(R)$, the corresponding morphism $B = R[s]/(s^2 - cqs - d) \to A$ is induced by $s \mapsto ut + r$, see Lemma 2.2.3.

Proposition 2.4.2. *i) The functor $\mathfrak{Qu}_f^{pq}(R) \to \mathfrak{Qu}^{pq}(R)$ is full and faithful.*

ii) The essential image of $\mathfrak{Qu}_f^{pq}(R) \to \mathfrak{Qu}^{pq}(R)$ consists of J-Galois algebras which are free as R-modules.

Proof. We will prove only part ii), as part i) is straightforward to check. Take a J-Galois algebra A. Thanks to [38, Corollary 2.5], the submodule spanned by 1 is a free R-module, which is a direct summand of M as an R-module. Hence, $A = R \cdot 1 \oplus V$, where V is projective of rank 1. If A is free as an R-module, then V is a free R-module of rank 1 (see [38, Lemma 1.1]). Denote a generator of V by v. It follows that

$$v^2 = mv + b, \quad \eta(v) = l_0 \cdot 1 \otimes 1 + l_1 \cdot v \otimes 1 + a \cdot 1 \otimes x + l_3 \cdot v \otimes x \quad \text{and} \quad \eta(1) = 1 \otimes 1,$$

for some $a, b, m, l_0, l_1, l_3 \in R$. Since $(\mathrm{Id} \otimes \epsilon) \circ \eta(v) = v$, where $\epsilon : A \to R$ is the counit of J and $\epsilon(1) = 1, \epsilon(v) = 0$, one obtains $l_0 = 0$ and $l_1 = 1$. Thus

$$\eta(v) = v \otimes 1 + a \cdot 1 \otimes x + l_3 \cdot v \otimes x. \tag{2.9}$$

The condition $(\mathrm{Id} \otimes \Delta) \circ \eta(v) = (\eta \otimes \mathrm{Id}) \circ \eta(v)$ implies that

$$al_3 = ap \quad \text{and} \quad l_3^2 = l_3 p. \tag{2.10}$$

Next, consider $1 \otimes 1$, $1 \otimes v$, $v \otimes 1$, $v \otimes v$ as a basis of $A \otimes A$, and $1 \otimes 1$, $v \otimes 1$, $1 \otimes x$, $v \otimes x$ as a basis of $A \otimes J$. One easily computes that the matrix of $(\mu \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \eta) : A \otimes A \to A \otimes J$ in the chosen bases is

$$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 1 & m \\ 0 & 0 & a & l_3 b \\ 0 & 0 & l_3 & a + l_3 m \end{pmatrix}.$$

Hence, $(\mu \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \eta) : A \otimes A \to A \otimes J$ is an isomorphism if and only if

$$a^2 + a l_3 m - l_3^2 b \in R^*. \tag{2.11}$$

It follows from the equations (2.10) that

$$(l_3 - p)(a^2 + a l_3 m - l_3^2 b) = (l_3 - p)a \cdot a + (l_3 - p)a \cdot l_3 m - (l_3 - p) l_3 \cdot l_3 b = 0,$$

and by the condition (2.11), that $l_3 = p$. Thus, we obtain

$$\eta(v) = v \otimes 1 + a \cdot 1 \otimes x + p \cdot v \otimes x \quad \text{and} \quad a^2 + pam - p^2 b \in R^*. \tag{2.12}$$

As, η is an algebra homomorphism, we have

$$m \cdot v \otimes 1 + ma \cdot 1 \otimes x + mp \cdot v \otimes x + b \cdot 1 \otimes 1 = \eta(mv + b) = \eta(v^2) = (v \otimes 1 + a \cdot 1 \otimes x + p \cdot v \otimes x)^2,$$

from which we obtain

$$am = a^2 q \quad \text{and} \quad mp = -2a.$$

Lastly, we have

$$(m - aq)(a^2 + pam - p^2 b) = 0,$$

and thus $m = aq$. We conclude $v^2 = aqv + b$ and $a^2 + pam - p^2 b = -(a^2 + p^2 b) \in R^*$ and hence, A is in the image of the functor $\mathfrak{Qu}_f^{pq}(R) \to \mathfrak{Qu}^{pq}(R)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 2.4.3. *The stackification of $R \mapsto \mathfrak{Qu}_f^{pq}(R)$ is $R \mapsto \mathfrak{Qu}^{pq}(R)$.*

Proof. The descent theory of projective modules implies that the assignment $R \mapsto \mathfrak{Qu}^{pq}(R)$ is a stack, which, by abuse of notation, we denote by $\mathfrak{Qu}^{pq}$. Proposition 2.4.2 shows that the canonical functor $\mathfrak{Qu}_f^{pq} \to \mathfrak{Qu}^{pq}$ is a weak equivalence (full and faithful and locally essentially surjective) in the sense of [45, Definition 2.3] and hence, the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

2.4.3 The stack $\mathfrak{Pic}$

Let R be a commutative ring. We let $\mathfrak{Pic}(R)$ be the following cat-group: The objects of $\mathfrak{Pic}(R)$ are finitely generated projective R-modules of rank 1, while its morphisms are isomorphisms of R-modules. The tensor product $\otimes_R$ equips $\mathfrak{Pic}(R)$ with the structure of a cat-group. If $f : R \to S$ is a homomorphism of commutative rings, the functor $f_* : \mathfrak{Pic}(R) \to \mathfrak{Pic}(S)$, given by $f_*(M) = M \otimes_R S$, is a morphism of cat-groups.

It follows from descent theory that $R \mapsto \mathfrak{Pic}(R)$ is a stack, which we denote by $\mathfrak{Pic}$. Actually, it is the stackification of the prestack which assigns to R the cat-group corresponding to the homomorphism $R^* \to \{1\}$ (see [45, Example (ii), p. 11]). It is well-known that

$$\pi_1(\mathfrak{Pic}(R)) = R^* \quad \text{and} \quad \pi_0(\mathfrak{Pic}(R)) = \mathrm{Pic}(R).$$

2.4.4 The stack $\mathfrak{Dis}$

Objects of the category $\mathfrak{Dis}(R)$ are pairs (M, μ), where M is a finitely generated projective R-module of rank 1 and μ is a non-singular, symmetric, bilinear form on M. Morphisms $(M, \mu) \to (M', \mu')$ are isomorphisms of R-modules $h : M \to M'$ such that $\mu(x, y) = \mu'(h(x), h(y))$. Define the symmetric monoidal structure on $\mathfrak{Dis}(R)$ by

$$(M', \mu') \otimes (M'', \mu'') = (M' \otimes M'', \mu),$$

where

$$\mu(m'_1 \otimes m''_1, m'_2 \otimes m''_2) = \mu'(m'_1, m'_2)\mu''(m''_1, m''_2).$$

The neutral object is (R, μ), where $\mu(r_1, r_2) = r_1 r_2$.

Lemma 2.4.4. *There is an equivalence of cat-groups*

$$\mathfrak{Dis}(R) \cong 2\text{-}\ker(\mathsf{sq}_R : \mathfrak{Pic}(R) \to \mathfrak{Pic}(R)),$$

where sq_R *is given by* $\mathsf{sq}_R(M) = M \otimes_R M$, $M \in \mathfrak{Pic}(R)$. *In particular,* $\mathfrak{Dis}$ *is a stack, which is the stack associated to the prestack* $R \mapsto \mathfrak{G}(R)$.

Proof. Let μ be a non-singular, bilinear form on a projective module M of rank 1. Then $\mu : M \otimes_R M \to R$ is an isomorphism, where R is seen as the unit object of the cat-group $\mathfrak{Pic}(R)$. Hence, it defines an object of $2\text{-}\ker(\mathsf{sq}_R : \mathfrak{Pic}(R) \to \mathfrak{Pic}(R))$. One easily checks that this construction yields an equivalence of categories

$$\mathfrak{Dis}(R) \cong 2\text{-}\ker(\mathsf{sq}_R : \mathfrak{Pic}(R) \to \mathfrak{Pic}(R)).$$

We use Corollary 1.4.10 to show the last statement. It says that the cat-group $\mathfrak{G}(R)$ is the 2-kernel of the morphism of cat-groups (associated to the vertical arrows), corresponding to the commutative diagram of abelian groups

$$\begin{array}{ccc} R^* & \xrightarrow{\mathsf{sq}} & R^* \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1. \end{array}$$

We see that the stackification of $R \mapsto \mathfrak{G}(R)$ is the 2-kernel of $\mathsf{sq} : \mathfrak{Pic} \to \mathfrak{Pic}$, which is $\mathfrak{Dis}$, since stackification preserves 2-kernels. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Note that equivalent prestacks give rise to equivalent stackifications and recall the exact sequence (1.2) on page 50. Denote $\mathrm{Dis}(R) := \pi_0(\mathfrak{Dis}(R))$. We have the following results:

Corollary 2.4.5. *i) There is an exact sequence ([26, (12.4) on p. 176])*

$$0 \to \mathsf{U}_2(R) \to \mathsf{Dis}(R) \to {}_2\mathsf{Pic}(R) \to 0,$$

where $\mathsf{U}_2(R) = R^*/(R^*)^2$ *and* ${}_2\mathsf{Pic}(R) = \{r \in R \mid 2r = 0\}$.

ii) We have an isomorphism of abelian groups

$$\pi_1(\mathfrak{Dis}(R)) \cong \mu_2(R) = \{x \in R \mid x^2 = 1\}.$$

iii) There is an equivalence of cat-groups

$$t_*^+ : \mathfrak{Qu}^{pq}(R) \to \mathfrak{Qu}^{pt,t^{-1}q}(R).$$

iv) We have an equivalence of cat-groups $\mathfrak{Qu}^{1,-2} \cong \mathfrak{Dis}$. *If* 2 *is invertible in* R, *we also have*

$$\mathfrak{Qu}^{-2,1}(R) \cong \mathfrak{Dis}(R).$$

In particular,

$$\mathsf{Qu}(R) \cong \mathsf{Dis}(R).$$

Proof. Parts i) and ii) are direct consequences of Lemma 2.4.4 and the fact that a short exact sequence of cat-groups induces a long exact sequence. See Formula (1.2) on page 50.

Part iii) follows from Propositions 2.3.2 and 2.4.2. We get the first equivalence of part iv) immediately from Corollary 2.3.6, part iii). The second equivalence of cat-groups is now just Remark 70. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

2.4.5 The stack $\mathfrak{Dis}^{pq}$

Proposition 2.4.6. *There is a morphism of cat-groups*

$$\mathsf{Sq} : \mathfrak{Pic}(R/pR) \to \mathfrak{Pic}(R/p^2R),$$

for which the following diagram

$$\begin{array}{ccc} \mathfrak{Pic}(R/p^2R) & \longrightarrow & \mathfrak{Pic}(R/pR) \\ {\scriptstyle \mathsf{sq}}\downarrow & \swarrow_{\mathsf{Sq}} & \downarrow{\scriptstyle \mathsf{sq}} \\ \mathfrak{Pic}(R/p^2R) & \longrightarrow & \mathfrak{Pic}(R/pR) \end{array}$$

commutes up to natural isomorphisms. Here, we dropped the index from the vertical arrows and the horizontal arrows are induced by the obvious ring homomorphism $R/p^2R \to R/pR$.

Proof. Since the kernel of the ring homomorphism $R/p^2R \to R/p$ is a nilpotent ideal, the functor $\mathfrak{Pic}(R/p^2R) \to \mathfrak{Pic}(R/pR)$ is full, essentially surjective and reflects isomorphisms (see for example [5, Proposition 2.12]). Hence, it suffices to show that for two morphisms $f, g : M \to N$ in $\mathfrak{Pic}(R/p^2R)$, whose induced maps $f_*, g_* : M/pM \to N/pN$ coincide,

$\mathsf{sq}(f) = \mathsf{sq}(g) : \mathsf{sq}(M) \to \mathsf{sq}(N)$. This can be checked locally and so, we may assume that $M = N = R/p^2R$. The result follows from Lemma 2.3.3. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

We put

$$\mathfrak{Dis}^{pq}(R) = 2\text{-}\ker(\mathsf{Sq} : \mathfrak{Pic}(R/pR) \to \mathfrak{Pic}(R/p^2R)).$$

Since the 2-kernel of a morphism of stacks is again a stack, we see that $\mathfrak{Dis}^{pq}$ is the stack associated to the prestack $R \mapsto \mathfrak{G}^p(R)$.

Lemma 2.4.7. *One has natural isomorphisms*

$$\pi_0(\mathfrak{Dis}^{pq}(R)) = \mathsf{Dis}(R/p^2R) \quad \text{and} \quad \pi_1(\mathfrak{Dis}^{pq}(R)) \cong \ker(\mathsf{Sq} : (R/pR)^* \to (R/p^2R)^*).$$

Proof. By our construction, we have a diagram of cat-groups

$$\begin{array}{ccc} \mathfrak{Pic}(R/p^2R) & \xrightarrow{\mathsf{sq}} & \mathfrak{Pic}(R/p^2R) \\ \downarrow & & \downarrow id \\ \mathfrak{Pic}(R/pR) & \xrightarrow[\mathsf{Sq}]{} & \mathfrak{Pic}(R/p^2R), \end{array}$$

which commutes up to a natural isomorphism. We apply the exact sequence (1.2), p.50 to the horizontal morphisms. It follows from the discussion in [21, p. 88], that one obtains a commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mu_2(R/p^2R) & \longrightarrow & (R/p^2R)^* & \xrightarrow{\mathsf{sq}} & (R/p^2R)^* & \longrightarrow & \mathsf{Dis}(R/p^2R) & \longrightarrow & \mathsf{Pic}(R/p^2R) \xrightarrow{\mathsf{sq}} \mathsf{Pic}(R/p^2R) \\ & & \downarrow & & \downarrow & & \downarrow id & & \downarrow & & \downarrow & & \downarrow id \\ 0 & \to & \pi_1(\mathfrak{Dis}^{pq}(R)) & \to & (R/pR)^* & \xrightarrow[\mathsf{Sq}]{} & (R/p^2R)^* & \to & \pi_0(\mathfrak{Dis}^{pq}(R)) & \to & \mathsf{Pic}(R/pR) \xrightarrow[\mathsf{Sq}]{} \mathsf{Pic}(R/p^2R). \end{array}$$

This already implies the result for $\pi_1(\mathfrak{Dis}^{pq}(R))$. Since the kernel of $R/p^2R \to R/pR$ is a nilpotent ideal, the vertical map $\mathsf{Pic}(R/p^2R) \to \mathsf{Pic}(R/pR)$ is an isomorphism and $(R/p^2R)^* \to (R/pR)^*$ is an epimorphism ([5, Proposition IX.3.4]). Hence, the result is also true for $\pi_0(\mathfrak{Dis}^{pq}(R))$, thanks to the 5-lemma. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

2.4.6 The main result

After stackification of the prestacks involved in the diagram described in Theorem 2.3.5, one obtains the diagram of stacks

$$\mathfrak{Qu}^{pq} \xrightarrow[\alpha]{} \mathfrak{Dis} \xrightarrow[\beta]{} \mathfrak{Dis}^{pq}, \qquad 1 \overset{\delta \Uparrow}{\longrightarrow} \beta\alpha$$

By the general properties of stackification, Theorem 2.3.5 i) implies that the induced functor

$$\gamma : \mathfrak{Qu}^{pq} \to 2\text{-}\ker(\beta)$$

is locally essentially surjective.

Assume that p is not a zero-divisor in R. Then p is not a zero-divisor in any of the localization rings R_r, where $r \in R$. It follows that p is not a zero divisor for any of the rings involved in the process of stackification. By Theorem 2.3.5 ii), the induced morphism

$$\gamma(R) : \mathfrak{Qu}^{pq}(R) \to 2\text{-}\ker\left(\mathfrak{Dis}(R) \xrightarrow{\beta(R)} \mathfrak{Dis}^{pq}(R)\right)$$

is an equivalence of cat-groups. The following is now a specialization of the exact sequence (1.2):

Theorem 2.4.8. *If p is not a zero divisor of R, one has an exact sequence of abelian groups*

$$0 \to \mathbb{Z}_{pq}(R) \to \mu_2(R) \to \mathsf{Im}(\mu_2(R/p^2R) \to \mu_2(R/pR)) \to \mathsf{Qu}^{pq}(R) \to \mathsf{Dis}(R) \to \mathsf{Dis}(R/p^2R).$$

Chapter 3

Comparison of the Colimit and the 2-Colimit

The following chapter is mainly composed of the results written in Paper [55]. There are, however, some results that already appear in my PhD thesis, for example Proposition 3.2.17.

The aim is to compare the colimit and the 2-colimit of strict 2-functors in the 2-category of groupoids. Of main interest is the case when the colimit is taken over certain types of posets. These posets are of special importance, as they correspond to the coverings of topological spaces. The main result of this chapter, Theorem 3.2.6, gives a condition on the 2-functor $\mathfrak{F}$, for which $2\text{-colim}\,\mathfrak{F} \simeq \text{colim}\,\mathfrak{F}$.

To be more explicit, let $\mathfrak{P}$ be a boolean lattice associated to an open covering, seen as a (strict) 2-category, and $\mathfrak{P} : \mathfrak{P} \to \mathfrak{Grpd}$ a strict 2-functor in small groupoids. Theorem 3.2.6 states that the colimit and 2-colimit of $\mathfrak{P}$ are equivalent, if a certain injectivity condition is satisfied. Though this condition seems fairly restrictive, it should be pointed out that any 2-functor can be deformed into one satisfying it. This is due to the fact that injectivity is only required on objects and it is an obvious and well-known fact that any functor $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ between groupoids (or categories) can easily be replaced with a functor $\mathcal{F}' : \mathcal{G} \to \mathcal{H}'$ that is injective on objects. As such, this is only a theoretical restriction that will not be a great hindrance in practice.

Calculating the 2-colimit of a functor over a poset coming from a covering is especially relevant for this thesis, due to its application later in Chapter 5, which deals with a joint paper with Prof. Brenner. In it, we give a method of calculating the fundamental groupoid of so called real binoid varieties. However, the method introduced in this chapter is more general and should have many other applications. The following is a very brief summary on how it can be used to calculate the fundamental groupoid:

Let X be a topological space and $\mathcal{U} = \{U_i\}_{i\in I} \in \text{Cov}(U)$ an open covering of an open subset $U \subseteq X$. We have shown in [50] that, under some small assumptions on X, there are equivalences of categories

$$\Pi_1(U) \simeq 2\text{-}\operatorname*{colim}_{\mathcal{U}} \Pi_1 \simeq \text{colim}_{\mathcal{U}}\, \Pi_1.$$

Of course, that $\Pi_1(U)$ is isomorphic to $\text{colim}_{\mathcal{U}}\, \Pi_1$ is an easy consequence of the famous re-

formulation of the classical Seifert-van Kampen theorem by Brown and Salleh [11]. However, showing that $\Pi_1(U)$ is equivalent to the 2-colimit, rather than the colimit, has some advantages. For one, this allows us to axiomatise the fundamental groupoid, as shown in [50]. The main argument used there is showing that it is the 2-terminal costack. Additionally, it enables us to replace the participating groupoids $\Pi_1(U_i)$ of the 2-colimit with equivalent ones, not just isomorphic ones. This is very helpful for calculations, as we shall show in Chapter 5, specifically in Section 5.6. We can usually replace groupoids consisting of uncountably many objects with groupoids having only finitely many objects. The downside is that, in general, calculating 2-colimits is harder than calculating colimits. The aim of this chapter is to alleviate this.

We will also give a small application of this result in calculating the fundamental groupoid of a topological space in Example 3.2.9.

Lastly, we will show that it is possible to truncate the 2-colimit at "depth 3". That is, we can define a new poset $\mathfrak{P}'$ from $\mathfrak{P}$, which only consists of the elements with codimension at most 3. We have an equivalence of categories

$$2\text{-colim}\,\mathfrak{F} \simeq 2\text{-colim}\,\mathfrak{F}',$$

where $\mathfrak{F}' : \mathfrak{P}' \to \mathfrak{Grpd}$ is just the restriction of $\mathfrak{F} : \mathfrak{P} \to \mathfrak{Grpd}$. This result is an analogue of the classical result for colimits, where we can restrict ourselves to codimension at most 2.

3.1 Construction of limits and colimits in $\mathfrak{Grpd}$

We recall our conversation about (co)limits and 2-(co)limits in Section 1.2. These constructions simplify a little for the case when our source 2-category comes from a poset and our target 2-category is the 2-category of groupoids, as we will discuss in this section. In addition, this section serves to fix the notations we will use later.

Let I be a poset, regarded as a 2-category, and $\Phi : I^{op} \to \mathfrak{Grpd}$ a strict contravariant 2-functor to the 2-category of groupoids. It is well-known [21] that the category $\mathfrak{Grpd}$ has limits and colimits, as well as 2-limits and 2-colimits.

3.1.1 Construction of limits in $\mathfrak{Grpd}$

For $i \leq j$, let $\phi_{ij} : \Phi(j) \to \Phi(i)$ be the induced functor. Objects of the groupoid $\lim_i \Phi$ (or simply $\lim \Phi$) are families (x_i), such that $\phi_{ij}(x_j) = x_i$ for any $i \leq j$. Here, x_i is an object of the category $\Phi(i)$. A morphism $(x_i) \to (y_i)$ is a family (f_i) such that $\phi_{ij}(f_j) = f_i$ for any $i \leq j$. Here, $f_i : x_i \to y_i$ is a morphism of $\Phi(i)$.

3.1.2 Construction of colimits in $\mathfrak{Grpd}$

A sketch of the construction of $\operatorname{colim}_i \Phi(i)$ for a strict 2-functor $\Phi : I \to \mathfrak{Grpd}$ is given as follows: (see [21, pp.4-5, p.11]). The set of objects of the groupoid $\operatorname{colim}_i \Phi(i)$ is the colimit of $i \mapsto \mathbf{Ob}(\Phi(i))$ in the category of sets, where $\mathbf{Ob}(\Phi(i))$ is the set of objects

of $\Phi(i)$. Likewise, $\mathbf{Mor}(\Phi(i))$ denotes the set of morphisms, and we take the colimit of the functor $i \mapsto \mathbf{Mor}(\Phi(i))$ in the category of sets. The domain and codomain functions $\mathbf{Mor}(\Phi(i)) \rightrightarrows \mathbf{Ob}(\Phi(i))$ induce maps between the colimits

$$\operatorname*{colim}_i \mathbf{Mor}(\Phi(i)) \rightrightarrows \operatorname*{colim}_i \mathbf{Ob}(\Phi(i)).$$

This yields a directed graph (*diagram scheme* in the terminology of [21]). Denote this directed graph by X and let $\mathbf{Pa}(X)$ (sometimes called path category) be the free category generated by X. The category $\operatorname*{colim}_i \Phi(i)$ is the maximal quotient category for which the composite

$$\Phi(i) \to \mathbf{Pa}(X) \to \operatorname*{colim}_i \Phi(i)$$

is a functor for all $i \in I$. It can be checked that the category $\operatorname*{colim}_i \Phi(i)$ is a groupoid, which is the colimit of $\Phi(i)$ in the 2-category $\mathfrak{Grpd}$.

3.1.3 Construction of 2-limits in $\mathfrak{Grpd}$

To construct the 2-limit $2\text{-}\operatorname*{lim}_i \Phi(i)$, we proceed as follows: Objects of $2\text{-}\operatorname*{lim}_i \Phi(i)$ (or simply $2\text{-}\lim \Phi$) are collections (x_i, ξ_{ij}), where x_i is an object of $\Phi(i)$, while $\xi_{ij} : \phi_{ij}(x_j) \to x_i$ for $i \leq j$ is an isomorphism of the category $\Phi(j)$. It satisfies the so called *1-cocycle condition*: For any $i \leq j \leq k$, one has

$$\xi_{ik} = \xi_{ij} \circ \phi_{ij}(\xi_{jk}).$$

A morphism from (x_i, ξ_{ij}) to (y_i, η_{ij}) is a collection (f_i), where $f_i : x_i \to y_i$ is a morphism of $\Phi(i)$. Furthermore, the following

$$\begin{array}{ccc} \phi_{ij}(x_j) & \xrightarrow{\xi_{ij}} & x_i \\ {\scriptstyle \phi_{ij}(f_j)}\downarrow & & \downarrow{\scriptstyle f_i} \\ \phi_{ij}(y_j) & \xrightarrow[\eta_{ij}]{} & y_i \end{array}$$

is a commutative diagram for any $i \leq j$.

3.1.4 Construction of 2-colimits in $\mathfrak{Grpd}$ - The Grothendieck construction

3.1.4.1 The Grothendieck construction

Let $\Phi : I \to \mathfrak{Grpd}$ be a covariant 2-functor. We regard $\mathfrak{Grpd}$ as a natural 2-category.

Let us recall the Grothendieck construction $\int_I \Phi$: Objects of the category $\int_I \Phi$ are pairs (i, x), where i is an object of I and x is an object of $\Phi(i)$. If $i \leq j$, morphisms $(i, x) \to (j, y)$ are exactly the morphisms $\alpha : \Phi_{i,j}(x) \to y$ of the category $\Phi(j)$. Of course, if $i > j$, there are no morphisms $(i, x) \to (j, y)$.

Let (i,x), (j,y) and (k,z) be objects of $\int_I \Phi$. Here $i \leq j \leq k$, with $\alpha : \Phi_{i,j}(x) \to y$ and $\beta : \Phi_{j,k}(y) \to z$ defining the morphisms $(i,x) \to (j,y) \to (k,z)$ in $\int_I \Phi$. The composite morphism $(i,x) \to (k,z)$ is given by

$$\beta \circ \phi_{j,k}(\alpha) : \phi_{i,k}(x) \to z.$$

We fix some additional notations. For $i \in I$. We define a functor

$$L_i : \Phi(i) \to \int_I \Phi.$$

It sends an object $x \in \mathbf{Ob}(\Phi(i))$ to the pair (i,x), and a morphism $\alpha : x \to y$ of $\Phi(i)$ to the morphism $(i,x) \to (i,y)$, determined by α.

Next, let $i \leq j$. There is a natural transformation

$$\lambda_{ij} : L_i \Rightarrow L_j \circ \Phi_{i,j},$$

defined by $\lambda_{ij}(x) = \mathrm{Id}_{\Phi_{ij}(x)}$. Here, $\mathrm{Id}_{\Phi_{ij}(x)}$ is considered as a morphism $(i,x) \to (j, \Phi_{i,j}(x))$. We observe that one has

$$\lambda_{i,k} = (\lambda_{j,k} * \Phi_{jk}) \circ \lambda_{i,j} \tag{3.1}$$

for any $i \leq j \leq k$, where $*$ is the vertical composition of natural transformations.

3.1.4.2 The Grothendieck construction related to the 2-limit

Equation (3.1) allows us to define the functor

$$K : \mathrm{Hom}_{\mathfrak{Cat}}\left(\int_I \Phi, \mathcal{G}\right) \to 2\text{-}\varprojlim_i \mathrm{Hom}_{\mathfrak{Grpd}}(\Phi(i), \mathcal{G})$$

for any groupoid $\mathcal{G}$, by

$$K(\theta) = (\theta \circ L_i, \theta \circ \lambda_{ij}).$$

Here, $\mathfrak{Cat}$ is the 2-category of small categories, $\theta : \int_I \Phi \to \mathcal{G}$ is a functor and $\mathrm{Hom}_{\mathfrak{Cat}}$ denotes the category of functors.

The functor K has an inverse

$$J : 2\text{-}\varprojlim_i \mathrm{Hom}_{\mathfrak{Grpd}}(\Phi(i), \mathcal{G}) \to \mathrm{Hom}_{\mathfrak{Cat}}\left(\int_I \Phi, \mathcal{G}\right),$$

which is constructed as follows: Recall that an object of $2\text{-}\varprojlim_i \mathrm{Hom}_{\mathfrak{Grpd}}(\Phi(i), \mathcal{G})$ is a collection (X_i, A_{ij}), where $X_i : \Phi(i) \to \mathcal{G}$ is a functor and for $i \leq j$, A_{ij} is a natural transformation $X_i \to X_j \circ \Phi_{i,j}$ satisfying the 1-cocycle condition.

Define the functor $J(X_i, A_{ij}) : \int_I \Phi \to \mathcal{G}$ on objects by

$$(i,x) \mapsto X_i(x)$$

and on morphisms by

$$\alpha \mapsto \alpha : X_j(\alpha) \circ A_{ij}(x).$$

Here, $\alpha : (i,x) \to (j,y)$ is a morphism in $\int_I \Phi$, induced by $\alpha : \Phi_{ij}(x_i) \to x_j$. This makes K an equivalence of categories.

3.1.4.3 The Grothendieck construction related to the 2-colimit

The equivalence constructed in Subsection 3.1.4.2 means that the Grothendieck construction can be considered a model of the 2-colimit in $\mathfrak{Cat}$, the 2-colimit of categories. However, even for a 2-functor $\Phi : I \to \mathfrak{Grpd}$, the Grothendieck construction $\int_I \Phi$ is not a groupoid in general. Hence, it can not be the 2-colimit in $\mathfrak{Grpd}$. This can be remedied as follows:

Let $\iota : \mathfrak{Cat} \to \mathfrak{Grpd}$ be the left adjoint of the inclusion 2-functor $\mathfrak{Grpd} \subseteq \mathfrak{Cat}$. Recall that $\iota(\mathcal{C})$ is obtained from $\mathcal{C}$ by inverting every morphisms in $\mathcal{C}$ (see, [21, Section 1.5.4]). In the case when $\mathcal{C} = \int_I \Phi$, it suffices to invert the morphisms $(i,x) \to (j, \Phi_{ij}(x_i))$ defined by $\mathrm{Id}_{\Phi_{ij}(x)}$. It follows from our discussion in Paragraph 3.1.4.2, that the groupoid $\iota(\int_I \Phi)$ is the 2-colimit of Φ.

3.1.4.4 Comparison functor

The 2-categorical version of the Yoneda Lemma and the functor $\gamma : \lim \Phi \to 2\text{-lim}\, \Phi$ defined in Lemma 1.2.6 allow us to define a comparison functor

$$\delta : 2\text{-colim}\, \Phi \to \text{colim}\, \Phi.$$

This represents the following composition of functors:

$$\mathrm{Hom}_{\mathfrak{Grpd}}(\text{colim}\, \Phi, \mathcal{G}) \cong \lim \mathrm{Hom}_{\mathfrak{Grpd}}(\Phi, \mathcal{G}) \xrightarrow{\gamma} 2\text{-lim}\, \mathrm{Hom}_{\mathfrak{Grpd}}(\Phi, \mathcal{G}) \cong \mathrm{Hom}_{\mathfrak{Grpd}}(2\text{-colim}\, \Phi, \mathcal{G}).$$

3.2 On the equivalence of $\delta : 2\text{-colim} \to \text{colim}$

Our aim is to understand under what conditions on Φ, the above constructed functor δ, is an equivalence of groupoids. This is not always the case, of course. However, as we shall show, any functor $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ can be deformed to an equivalent functor $\mathcal{F}' : \mathcal{G}' \to \mathcal{H}$, for which the colimit and 2-colimit are equivalent.

3.2.1 Injectivity on objects

We wish to fix some terminology. Let $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ be a functor between small groupoids. This induces a map $\mathbf{Ob}(\mathcal{F}) : \mathbf{Ob}(\mathcal{G}) \to \mathbf{Ob}(\mathcal{H})$ from the set of objects of $\mathcal{G}$ to the set of objects of $\mathcal{H}$. We say that the functor $\mathcal{F}$ is *injective on objects* if the map $\mathbf{Ob}(\mathcal{F})$ is injective.

3.2.1.1 The initial step

Lemma 3.2.1 (Deformation Lemma). *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i_1} & \mathcal{B} \\ {\scriptstyle i_2}\downarrow & \swarrow_{\lambda} & \downarrow{\scriptstyle j_1} \\ \mathcal{C} & \xrightarrow[j_2]{} & \mathcal{D} \end{array}$$

be a 2-diagram of groupoids. That is, $\lambda : j_1 i_1 \Rightarrow j_2 i_2$ is a natural isomorphism. If the functor i_1 is injective on objects, there exist a functor $j_1' : \mathcal{B} \to \mathcal{D}$ and a natural transformation $\kappa : j_1 \Rightarrow j_1'$, for which the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i_1} & \mathcal{B} \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1'} \\ \mathcal{C} & \xrightarrow[j_2]{} & \mathcal{D} \end{array}$$

commutes and λ coincides with $\kappa \star i_1 : j_1 i_1 \Rightarrow j_1' i_1 = j_2 i_2$.

Proof. Define j_1' on objects by

$$j_1'(b) = \begin{cases} j_1(b) & \text{if } \ b \notin \mathrm{Im}(i_1) \\ j_2 i_2(a) & \text{if } \ b = i_1(a). \end{cases}$$

To define j_1' on morphisms, we proceed as follows. Let $\beta : b_1 \to b_2$ be a morphism in B. To define

$$j_1'(\beta) : j_1'(b_1) \to j_1'(b_2),$$

we have to consider five different cases.

Case 1. $b_1 = i_1(a_1)$ and $b_2 \notin \mathrm{Im}(i_1)$: One defines $j_1'(\beta)$ to be the composite

$$j_1'(b_1) = j_2 i_2(a_1) \xrightarrow{\lambda^{-1}(a_1)} j_1 i_1(a_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1'(b_2).$$

Case 2. $b_1 = i_1(a_1)$, $b_2 = i_1(a_2)$, but $\beta \notin \mathrm{Im}(i_1)$: One defines $j_1'(\beta)$ to be the composite:

$$j_1'(b_1) = j_2 i_2(a_1) \xrightarrow{\lambda^{-1}(a_1)} j_1 i_1(a_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1 i_1(a_2) \xrightarrow{\lambda(a_2)} j_2 i_2(a_2) = j_1'(b_2).$$

Case 3. $\beta \in \mathrm{Im}(j_1)$: We choose $\alpha : a_1 \to a_2$ in A such that $\beta = i_1(\alpha)$. One defines $j_1'(\beta)$ by

$$j_1'(b_1) = j_2 i_2(a_1) \xrightarrow{j_2 i_2(\alpha)} j_2 i_2(a_2) = j_1'(b_2).$$

To check that this is independent of the choice of α, consider $\alpha' : a_1 \to a_2$ with the property

$i_1(\alpha') = \beta$. Since λ is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} j_1 i_1(a_1) & \xrightarrow{\lambda(a_1)} & j_2 i_2(a_1) \\ {\scriptstyle j_1 i_1(\alpha)}\downarrow & = & \downarrow{\scriptstyle j_1 i_1(\alpha')} \qquad {\scriptstyle j_2 i_2(\alpha)}\downarrow \quad \downarrow{\scriptstyle j_2 i_2(\alpha')} \\ j_1 i_1(a_2) & \xrightarrow[\lambda(a_2)]{} & j_2 i_2(a_2) \end{array}$$

Since the left vertical arrows are equal and the horizontal ones are isomorphisms, it follows from the commutativity, that the right vertical arrows are also equal. Hence, $j_1'(\beta)$ is well-defined in this case.

Case 4. $b_1 \notin \mathrm{Im}(i_1)$ and $b_2 = i_1(a_2)$. One defines $j_1'(\beta)$ to be the composite

$$j_1'(b_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1 i_1(a_2) \xrightarrow{\lambda(a_2)} j_2 i_2(a_2) = j_1'(b_2).$$

Case 5. $b_1 \notin \mathrm{Im}(i_1)$ and $b_2 \notin \mathrm{Im}(i_1)$. One defines $j_1'(\beta)$ as

$$j_1'(b_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1'(b_2).$$

Checking case by case shows that j_1' is really a functor, with $j_1' i_1 = j_2 i_2$. Define κ by

$$\kappa(b) = \begin{cases} \mathrm{Id}_{j_1(b)}, & \text{if } \ b \notin \mathrm{Im}(i_1) \\ \lambda(a), & \text{if } \ b = i_1(a). \end{cases}$$

One easily sees that j_1' and κ satisfy the assertions of the Lemma. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

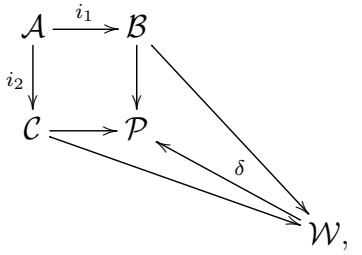
This theorem is Proposition 4.2 of [50].

Theorem 3.2.2. *Let*

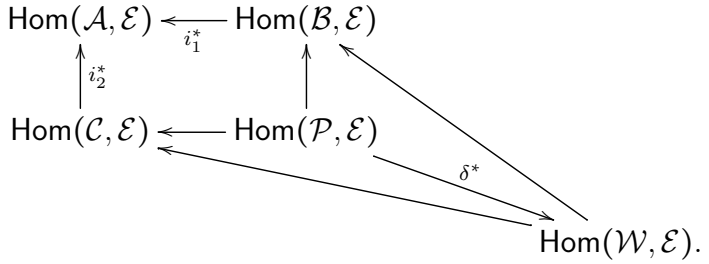
$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i_1} & \mathcal{B} \\ {\scriptstyle i_2}\downarrow & & \\ \mathcal{C} & & \end{array}$$

be a diagram of groupoids, where i_1 is injective on objects. The colimit and the 2-colimit are equivalent.

Proof. Consider the diagram



where $\mathcal{P}$ is the pushout and $\mathcal{W}$ is the 2-pushout of $\mathcal{C} \xleftarrow{i_2} \mathcal{A} \xrightarrow{i_1} \mathcal{B}$. Let $\mathcal{E}$ be any groupoid and consider



Since $\mathsf{Hom}(-,\mathcal{E})$ maps pushouts and 2-pushouts to pullbacks and 2-pullbacks, we have that $\mathsf{Hom}(\mathcal{P},\mathcal{E})$ and $\mathsf{Hom}(\mathcal{W},\mathcal{E})$ are the pullback and 2-pullback of the diagram

$$\mathsf{Hom}(\mathcal{B},\mathcal{E}) \xrightarrow{i_1^*} \mathsf{Hom}(\mathcal{A},\mathcal{E}) \xleftarrow{i_2^*} \mathsf{Hom}(\mathcal{C},\mathcal{E})$$

respectively. Using Lemma 1.2.6, we know that δ^* is full and faithful. To see that it's an equivalence of categories, we only need to show that it's essentially surjective.

Since, as mentioned, $\mathsf{Hom}(\mathcal{W},\mathcal{E})$ is the 2-pullback, taking an object there is the same as taking objects in $\mathsf{Hom}(\mathcal{B},\mathcal{E})$ and $\mathsf{Hom}(\mathcal{C},\mathcal{E})$, and an equivalence between these objects in $\mathsf{Hom}(\mathcal{A},\mathcal{E})$. That is to say, we have a 2-commutative diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i_1} & \mathcal{B} \\ {\scriptstyle i_2}\downarrow & \swarrow & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{E}. \end{array}$$

From this, we immediately see that the Deformation Lemma (3.2.1) shows the essential surjectivity of δ^*. This, in turn, using the Yoneda lemma for 2-categories [43, Lemma 2.3] shows that $\delta : \mathcal{W} \to \mathcal{P}$ is an equivalence of categories, completing the proof. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

As a direct consequence, we have the following Lemma:

Corollary 3.2.3. *The coproduct and the 2-coproduct of groupoids are equivalent.*

3.2.2 The Poset $\mathfrak{B}(\mathbf{n})$

The poset of all proper subset of a set V is denoted by $\mathfrak{B}(V)$. By proper we mean that the subsetset should not equal V itself. More generally, for a proper subset $U \subsetneq V$, we set

$$\mathfrak{B}(V : U) = \{X \in \mathfrak{B}(V) \mid U \subseteq X\}.$$

Thus, $\mathfrak{B}(V : \varnothing) = \mathfrak{B}(V)$ and $\mathfrak{B}(V : V) = \varnothing$. If $V \subseteq W$, then $\mathfrak{B}(V : U) \subseteq \mathfrak{B}(W : U)$. Hence, we can and we will identify the elements of $\mathfrak{B}(V : U)$ with the corresponding elements of $\mathfrak{B}(W : U)$.

For a natural number $n \geq 1$, denote by $\mathbf{n}$ the set $\{1, 2, \cdots, n\}$. Accordingly, we write $\mathfrak{B}(\mathbf{n})$ instead of $\mathfrak{B}(\{1, \cdots, n\})$. If $k < n$, we considder $\mathfrak{B}(\mathbf{k})$ to be the subposet of $\mathfrak{B}(\mathbf{n})$ consisting

of elements $X \subsetneq \mathbf{n}$ such that $k+1, \cdots, n \notin X$ and $X \neq \{1, \cdots, k\}$. As such, $\mathfrak{B}(\mathbf{n-1})$ will be identified with the subposet of $\mathfrak{B}(\mathbf{n})$ consisting of subsets $X \subsetneq \{1, \ldots, n-1\} \subseteq \mathbf{n}$ such that $n \notin X$ and $X \neq \{1, \cdots, n-1\}$. We can also immediately see that

$$\mathfrak{B}(\mathbf{n} : \mathbf{m}) = \{X \mid \{1, \ldots, m\} \subseteq X \subsetneq \{1, \ldots, n\}\} \simeq \mathfrak{B}(\mathbf{n-m}).$$

Parallel to this, we consider the subposet $\mathfrak{B}'(\mathbf{n-1})$, defined as $\mathfrak{B}(\mathbf{n} : \{n\})$. It consists of subsets $Y \subsetneq \mathbf{n}$ such that $n \in Y$. This gives us a partition of posets

$$\mathfrak{B}(\mathbf{n}) = \mathfrak{B}(\mathbf{n-1}) \coprod \mathfrak{B}'(\mathbf{n-1}) \coprod \{\mathbf{n-1}\}.$$

Obviously, the map $\alpha : \mathfrak{B}(\mathbf{n-1}) \to \mathfrak{B}'(\mathbf{n-1})$, given by $X \mapsto X \coprod \{n\}$, defines an isomorphism of posets.

Definition 3.2.4. *Let U be a proper subset of $V \in \mathfrak{B}(\mathbf{n})$. We will say that a strict 2-functor $\Phi : \mathfrak{B}(\mathbf{n}) \to \mathfrak{Grpd}$ satisfies the condition A_U^V if the canonical functor*

$$\text{colim}_{\mathfrak{B}(V:U)} \Phi \to \Phi(V)$$

is injective on objects.

Definition 3.2.5. *Let $n \geq 2$ be an integer. We will say that Φ satisfies the condition $A(n)$, if for any proper subset $1 \in V \subsetneq \mathbf{n}$, the condition A_U^V holds. Here, U is constructed from V as follows: Let k be the minimal integer such that $k+1 \notin V$. We set $U = V \setminus \mathbf{k}$.*

Theorem 3.2.6 ([55], Cor. 5.4)**.** *Let $n \geq 2$ be a natural number and $\Phi : \mathfrak{B}(n) \to \mathfrak{Grpd}$ a covariant strict 2-functor. The comparison functor*

$$\delta_{\mathfrak{B}(n)} : 2\text{-}\operatorname*{colim}_{\mathfrak{B}(n)} \Phi \to \operatorname*{colim}_{\mathfrak{B}(n)} \Phi \tag{3.2}$$

is an equivalence of categories if condition $A(n)$ holds.

Proof. We proceed by induction. Assume $n = 2$. Then $V = \{1\}$ and thus, $U = \varnothing$. By our assertion, the canonical functor $\Phi(\varnothing) \to \Phi(\mathbf{1})$ is injective on objects. Hence, the case $n = 2$ corresponds to Theorem 3.2.2. The general case follows from $n = 2$ and Corollaries 3.2.11 and 3.2.13, which are proven below.

To write it out explicitly, assume $n > 2$. Thanks to Corollary 3.2.13, we have a 2-pushout of groupoids,

$$\begin{array}{ccc} 2\text{-colim}_{\mathfrak{B}(\mathbf{n-1})} \Phi & \longrightarrow & 2\text{-colim}_{\mathfrak{B}'(\mathbf{n-1})} \Phi \\ \downarrow & & \downarrow \\ \Phi(\mathbf{n-1}) & \longrightarrow & 2\text{-colim}_{\mathfrak{B}(\mathbf{n})} \Phi. \end{array}$$

By Corollary 3.2.11, we have a pushout diagram

$$\begin{array}{ccc} \text{colim}_{\mathfrak{B}(\mathbf{n-1})}\,\Phi & \longrightarrow & \text{colim}_{\mathfrak{B}'(\mathbf{n-1})}\,\Phi \\ \downarrow & & \downarrow \\ \Phi(\mathbf{n-1}) & \longrightarrow & \text{colim}_{\mathfrak{B}(\mathbf{n})}\,\Phi. \end{array}$$

Observe that δ (3.2) maps the first square to the second square. The left vertical map in the bottom diagram corresponds to our condition when $V = \mathbf{n-1}$. Hence, it is injective on objects. It follows again from Theorem 3.2.2, that the second square is also a 2-pushout diagram. To show that $\delta_{\mathfrak{B}(\mathbf{n})}$ is an equivalence of groupoids, it suffices to show that $\delta_{\mathfrak{B}(\mathbf{n-1})}$ and $\delta_{\mathfrak{B}'(\mathbf{n-1})}$ are equivalences.

One can easily see that Condition $A(n)$ implies $A(n-1)$. Our theorem thus also hold for $\mathfrak{B}(\mathbf{n-1})$. Hence, the functor $\delta_{\mathfrak{B}(\mathbf{n-1})}$ is an equivalence of categories by our induction assumption.

The map $\alpha : \mathfrak{B}(\mathbf{n-1}) \to \mathfrak{B}'(\mathbf{n-1})$ is an isomorphism of posets. We wish to apply the induction assumption to the poset $\mathfrak{B}'(\mathbf{n-1})$. For this, we have to translate Condition $A(n-1)$ to Condition $A'(n-1)$, using the bijection α.

The restated condition becomes $A^{V'}_{U'}$, where $1, n \subseteq V' \subsetneq \mathbf{n}$. It is clear that our initial assumption encompasses this. The reason being that for such V', we can write $V' = V \cup \{n\}$, $1 \in V$. If k is the minimal integer for which $k+1 \notin V$, the same must also hold for V'. Hence $U' = U \cup \{n\} = V' \setminus \mathbf{k}$. This proves that $\delta_{\mathfrak{B}'(\mathbf{n-1})}$ is also an equivalence of categories and we are done. $\mathcal{Q.E.D}$

While the condition in the above theorem seem fairly strict, injectivity on objects is actually not a big hurdle. Any functor can be replaced with an equivalent one for which the conditions of the theorem are satisfied. We call this *stretching* and will discuss it below.

Remark 3.2.7. *The conditions of Theorem 3.2.6 can be simplified if the functors $\mathcal{F}(U_\alpha) \to \mathcal{F}(U_\beta), \alpha \leq \beta$ are surjective on objects. In such a case, we only need to ensure that the functors $\mathcal{F}(U_\alpha) \to \mathcal{F}(U_\beta)$ are injective on objects. This will imply bijectivity on objects of the functors $\mathcal{F}(U_\alpha) \to \mathcal{F}(U_\beta)$, and as such,* $\text{colim}_{\mathfrak{B}^c(I:J)}\,\Phi \to \Phi(I^c)$ *will also be bijective on objects.*

Stretching Let $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ be a functor. Assume there exist $x \neq x' \in \mathcal{G}$ such that $\mathcal{F}(x) = \mathcal{F}(x') = y$. We define $\mathcal{H}'$ to be the category generated by adding a new object y' to $\mathcal{H}$ and an isomorphism $\alpha : y \to y'$. Let z, z' be objects of $\mathcal{H}'$. We have

$$\text{Hom}'_{\mathcal{H}}(z,z') = \begin{cases} \text{Hom}_{\mathcal{H}}(z,z'), & z \neq y' \neq z' \\ \text{Hom}_{\mathcal{H}}(y,z')\alpha^{-1}, & z = y' \neq z' \\ \alpha\,\text{Hom}_{\mathcal{H}}(z,y), & z \neq y' = z' \\ \alpha\,\text{Hom}_{\mathcal{H}}(y,y)\alpha^{-1}, & z = y' = z'. \end{cases}$$

We also define a new functor $\mathcal{F}' : \mathcal{G} \to \mathcal{H}'$ as follows. On objects, we set

$$\mathcal{F}'(u) = \begin{cases} \mathcal{F}(u), & u \neq x' \\ y, & u = x'. \end{cases}$$

Here, u is an object of $\mathcal{H}$. On morphisms, $\mathcal{F}'$ is defined by

$$\mathcal{F}'(u \xrightarrow{\beta} v) = \begin{cases} \mathcal{F}(\beta), & u \neq x' \neq v \\ \mathcal{F}(\beta)\alpha^{-1}, & u = x' \neq v' \\ \alpha\mathcal{F}(\beta), & u \neq x' = v' \\ \alpha\mathcal{F}(\beta)\alpha^{-1}, & u = x' = v'. \end{cases}$$

The two functors $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic via $\theta : \mathcal{F} \Rightarrow \mathcal{F}'$, where

$$\theta(u) = \begin{cases} \mathsf{Id}_{\mathcal{F}(u)}, & u \neq x' \\ \alpha, & u = x'. \end{cases}$$

We call this construction *stretching*.

A trivial consequence of our main theorem says that if the condition A_U^V holds for all subsets $U \subsetneq V \subsetneq \mathbf{n}$, the canonical map $\delta_{\mathfrak{B}(n)} : 2\text{-colim}_{\mathfrak{B}(n)} \Phi \to \text{colim}_{\mathfrak{B}(n)} \Phi$ is an equivalence of categories. As such, we do not loose much by restating the theorem in a slightly less general, but more intuitive way:

Corollary 3.2.8. *Let $n \geq 1$ be a natural number and $\Phi : \mathfrak{B}(\mathbf{n}) \to \mathfrak{Grpd}$ a covariant strict 2-functor. Assume for all $S \in \mathfrak{B}(\mathbf{n})$, the functor $\text{colim}_{\mathfrak{B}(S)} \Phi \to \Phi(S)$ is injective on objects. The canonical functor*

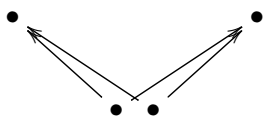
$$2\text{-}\operatorname{colim}_{\mathfrak{B}(\mathbf{n})} \Phi \to \operatorname{colim}_{\mathfrak{B}(\mathbf{n})} \Phi$$

is an equivalence of categories.

We can see the application of the above theorem (in conjunction with [50, Thm 3.2]) already in the following simple example:

Example 3.2.9. *Consider the unit circle S^1. Let $x, y \in S^1$ be two distinct points. The circle can be covered by the two open subsets $U_x := S^1 \setminus \{x\}$ and $U_y := S^1 \setminus \{y\}$. We set $U_{xy} := U_x \cap U_y$. Clearly $\Pi_1(U_x) \simeq \bullet$, $\Pi_1(U_y) \simeq \bullet$ and $\Pi_1(U_{xy}) \simeq \bullet \quad \bullet$ are equivalences of groupoids. Here, $\bullet = \mathbf{1}$ is the groupoid with one object and only the identity morphism, and $\bullet \quad \bullet = \mathbf{1} \coprod \mathbf{1}$ is the groupoid with two objects and only the two identity morphisms.*

As such, $\Pi_1(S^1)$ is the 2-colimit of the following diagram in $\mathfrak{Grpd}$ by [50, Thm 3.2]:

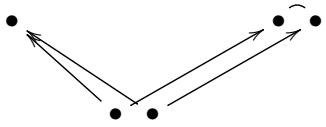


As we the only morphisms of $\Pi_1(U_{xy})$ are identities, the functors $\Pi_1(U_{xy}) \to \Pi_1(U_x)$ and $\Pi_1(U_{xy}) \to \Pi_1(U_y)$ need only be defined on objects: The arrows above do exactly that.

Denote by $\bullet \frown \bullet := \bullet \leftrightarrows \bullet$ *the groupoid with two objects and a single isomorphism between them. Clearly, this is equivalent to the groupoid* $\bullet$*. We can replace groupoids with equivalent ones in a diagram without changing the 2-colimit.*

As $n = 2$*, the condition of Theorem 3.2.6 only requires for the functor* $\Pi_1(U_{xy}) \to \Pi_1(U_x)$ *to be injective on objects.*

As such, $\Pi_1(S^1)$ *is equivalent to the colimit of the following diagram:*



which consists of a groupoid with one object and automorphism group $\mathbb{Z}$*.*

3.2.3 Induction principle for colimits

The aim of this section is to prove Corollary 3.2.11 below, which we used in our proof of Theorem 3.2.6. Let $\Phi : \mathfrak{B}(\mathbf{n})^{op} \to \text{Set}$ be a strict contravariant 2-functor. According to the definition, elements of $\lim_{\mathfrak{B}(\mathbf{n})} \Phi$ are families (a_W), satisfying the compatibility condition

$$\Phi_{V,W}(a_W) = a_V, \quad V \subseteq W \subsetneq \mathbf{n}.$$

Here, W and V are running through all the proper subsets of $\mathbf{n}$.

We would like to have an inductive procedure for studying such limits. Consider the restriction of Φ on $\mathfrak{B}(\mathbf{n}-\mathbf{1})$ and on $\mathfrak{B}'(\mathbf{n}-\mathbf{1})$. By abuse of notation, we denote these functors by Φ as well. Consider the limit of Φ over these posets. This yields the diagram

$$\begin{array}{ccc} & & \lim_{\mathfrak{B}'(\mathbf{n-1})} \Phi \\ & & \downarrow \eta \\ \Phi(\mathbf{n}-\mathbf{1}) & \xrightarrow{\delta} & \lim_{\mathfrak{B}(\mathbf{n-1})} \Phi. \end{array}$$

Here, δ sends $a_{\mathbf{n-1}} \in \Phi(\mathbf{n}-\mathbf{1})$ to the family $(\Phi_{U,\mathbf{n-1}}(a_{\mathbf{n-1}}))$, where U is a proper subset of $\mathbf{n}-\mathbf{1}$. In order to describe η, we set

$$U_+ = \{n\} \cup U$$

for a proper subset $U \subsetneq \mathbf{n}-\mathbf{1}$. The functor η sends

$$(a_{U_+}) \in \lim_{\mathfrak{B}'(\mathbf{n-1})} \Phi \quad \text{to} \quad (\Phi_{U,U_+}(a_{U_+})) \in \lim_{\mathfrak{B}(\mathbf{n-1})} \Phi.$$

Lemma 3.2.10. *The following diagram is a pullback in the category of sets:*

$$\begin{array}{ccc} \lim_{\mathfrak{B}(\mathbf{n})} \Phi & \longrightarrow & \lim_{\mathfrak{B}'(\mathbf{n-1})} \Phi \\ \downarrow & & \downarrow \\ \Phi(\mathbf{n-1}) & \longrightarrow & \lim_{\mathfrak{B}(\mathbf{n-1})} \Phi. \end{array}$$

Proof. Denote the pullback of the diagram

$$\begin{array}{ccc} & & \lim_{\mathfrak{B}'(\mathbf{n-1})} \Phi \\ & & \downarrow \beta \\ \Phi(\mathbf{n-1}) & \xrightarrow{\alpha} & \lim_{\mathfrak{B}(\mathbf{n-1})} \Phi \end{array}$$

by P. By definition, elements of P are pairs $(b_{\mathbf{n-1}}, B)$ satisfying the condition

$$\delta(b_{\mathbf{n-1}}) = \eta(B).$$

Here, $b_{\mathbf{n-1}}$ denotes an element in $\Phi(\mathbf{n-1})$, while $B \in \lim_{\mathfrak{B}'(\mathbf{n-1})} \Phi$. That is to say, B is itself a compatible family (b_{U_+}), where U is a proper subset of $\mathbf{n-1}$ and b_{U_+} is an object of $\Phi(U_+)$. These objects satisfy the compatibility condition

$$\Phi_{U_+,V_+}(b_{V_+}) = b_{U_+},$$

where $U \subseteq V$ are proper subsets of $\mathbf{n-1}$. The condition $\delta(b_{\mathbf{n-1}}) = \eta(B)$ means that one has

$$\Phi_{U,\mathbf{n-1}}(b_{\mathbf{n-1}}) = \Phi_{U,U_+}(b_{U_+})$$

for any proper subset $U \subseteq \mathbf{n-1}$. Define $\xi : \lim_{\mathfrak{B}(\mathbf{n})} \Phi \to P$ by

$$(a_W) \mapsto (a_{\mathbf{n-1}}, A); \quad A = (a_{U_+}).$$

Here, W (resp. U) is running through the set of proper subset of $\mathbf{n}$ (resp. $\mathbf{n-1}$). We need to show that ξ is bijective. To this end, we define the inverse map $\theta : P \to \lim_{\mathfrak{B}(\mathbf{n})} \Phi$ by

$$(b_{\mathbf{n-1}}, B) \mapsto (a_W).$$

Here, we denote

$$a_W = \begin{cases} b_W & \text{if } W \in \mathfrak{B}'(\mathbf{n-1}) \\ b_{\mathbf{n-1}} & \text{if } W = \mathbf{n-1} \\ \Phi_{W,W_+}(b_{W_+}) & \text{if } W \subseteq \mathfrak{B}(\mathbf{n-1}). \end{cases}$$

One readily shows that $(a_W) \in \lim_{\mathfrak{B}(\mathbf{n})} \Phi$ and that θ is inverse to ξ. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Yoneda's lemma yields the following statement for 2-colimits:

Corollary 3.2.11. *Let $\mathcal{C}$ be a category with finite limits (resp. colimits) and $\Phi : \mathfrak{B}(\mathbf{n})^{op} \to \mathcal{C}$ (resp. $\Phi : \mathfrak{B}(\mathbf{n}) \to \mathcal{C}$) a strict contravariant (resp. covariant) 2-functor. The following are pullback (resp. pushout) diagrams in $\mathcal{C}$:*

$$\begin{array}{ccc} \lim_{\mathfrak{B}(\mathbf{n})} \Phi & \longrightarrow & \lim_{\mathfrak{B}'(\mathbf{n-1})} \Phi \\ \downarrow & & \downarrow \\ \Phi(\mathbf{n-1}) & \longrightarrow & \lim_{\mathfrak{B}(\mathbf{n-1})} \Phi \end{array}$$

and

$$\begin{array}{ccc} \text{colim}_{\mathfrak{B}(\mathbf{n-1})} \Phi & \longrightarrow & \text{colim}_{\mathfrak{B}'(\mathbf{n-1})} \Phi \\ \downarrow & & \downarrow \\ \Phi(\mathbf{n-1}) & \longrightarrow & \text{colim}_{\mathfrak{B}(\mathbf{n})} \Phi. \end{array}$$

3.2.4 Induction principle for 2-colimits

The aim of this section is to prove Corollary 3.2.13, also used in the proof of Theorem 3.2.6. Let $\Phi : I^{op} \to \mathfrak{Grpd}$ be a strict contravariant 2-functor from the category I to the category of groupoids.

Recall the construction of the groupoid $2\text{-}\lim_i \Phi_i$, as discussed in Subsection 3.1. The 2-pullback (which is in particular a 2-limit) of the diagram

$$\begin{array}{ccc} & & \mathcal{G}_1 \\ & & \downarrow{\scriptstyle f_1} \\ \mathcal{G}_2 & \xrightarrow{f_2} & \mathcal{G}_0 \end{array}$$

is the following groupoid: Objects are triples (A_1, A_2, α), where A_i is an object of $\mathcal{G}_i$ and $\alpha : f_1(A_1) \to f_2(A_2)$ is an isomorphism. A morphism $(A_1, A_2, \alpha) \to (B_1, B_2, \beta)$ is a pair (g_1, g_2), where $g_i : A_i \to B_i$ is a morphism in $\mathcal{G}_i$ such that the following diagram

$$\begin{array}{ccc} f_1(A_1) & \xrightarrow{\alpha} & f_2(A_2) \\ {\scriptstyle f_1(g_1)}\downarrow & & \downarrow{\scriptstyle f_2(g_2)} \\ f_1(B_1) & \xrightarrow[\beta]{} & f_2(B_2) \end{array}$$

commutes.

Likewise, objects of the groupoid $2\text{-}\lim_{\mathfrak{B}(\mathbf{n})} \Phi$ are collections

$$(a_U, \alpha_{U,V} : \Phi_{U,V}(a_V) \to a_U),$$

where U and V are proper subsets of $\mathbf{n}$ with $U \subseteq V$. Moreover, a_U (resp. $\alpha_{U,V}$) is an object

(resp. isomorphism) of $\Phi(U)$. One requires that for all $U \subseteq V \subseteq W \subsetneq \mathbf{n}$, the following diagram

$$\begin{array}{ccc} \Phi_{U,W}(a_W) & \xrightarrow{\Phi_{V,W}(\alpha_{VW})} & \Phi_{U,V}(a_V) \\ & {\scriptstyle \alpha_{UW}} \searrow \quad \swarrow {\scriptstyle \alpha_{U,V}} & \\ & a_U & \end{array}$$

commutes. A morphism $(a_U, \alpha_{U,V}) \to (b_U, \beta_{U,V})$ is a collection (g_U), where U is a proper subset of $\mathbf{n}$ and $g_U : a_U \to b_U$ is a morphism of $\Phi(U)$. Further, for all proper subsets $U \subseteq V$ the following diagram

$$\begin{array}{ccc} \Phi_{U,V}(a_V) & \xrightarrow{\alpha_{U,V}} & a_U \\ {\scriptstyle \Phi_{U,V}(g_V)} \downarrow & & \downarrow {\scriptstyle g_U} \\ \Phi_{U,V}(b_V) & \xrightarrow{\beta_{U,V}} & b_U \end{array}$$

has to commute. Let us denote by $\mathcal{P}$ the 2-pullback of the diagram

$$\begin{array}{ccc} & & \underset{\mathfrak{B}'(\mathbf{n-1})}{2\text{-lim}}\ \Phi \\ & & \downarrow \eta \\ \Phi(\mathbf{n-1}) & \xrightarrow[\delta]{} & \underset{\mathfrak{B}(\mathbf{n-1})}{2\text{-lim}}\ \Phi. \end{array}$$

The functor δ sends an object $a_{\mathbf{n-1}}$ of the category $\Phi(\mathbf{n-1})$ to the collection $(a_U, \alpha_{U,V})$, where $U \subseteq V$ are proper subsets of $\mathbf{n-1}$, $a_U = \Phi_{U,\mathbf{n-1}}(a_{\mathbf{n-1}})$ and $\alpha_{U,V} = \mathrm{Id}_{a_U}$. The functor η sends a collection $(a_{U_+}, \alpha_{U_+,V_+})$ to the collection $(\Phi_{U_+,U}(a_{U_+}), \Phi_{U_+,U}(\alpha_{U_+,V_+}))$. Here, $U \subseteq V$ are proper subsets of $\mathbf{n-1}$.

Thus, objects of $\mathcal{P}$ are triples $(b_{\mathbf{n-1}}, B, \zeta : \eta(B) \to \delta(a_{\mathbf{n-1}}))$. Here, $B = (b_{U_+}, \beta_{U_+,V_+})$ is an object of $\underset{\mathfrak{B}'(\mathbf{n-1})}{2\text{-lim}}\ \Phi$, $b_{\mathbf{n-1}}$ is an object of $\Phi(\mathbf{n-1})$ and ζ is a morphism of the category $\underset{\mathfrak{B}(\mathbf{n-1})}{2\text{-lim}}\ \Phi$.

In more detail, the objects of $\mathcal{P}$ are collections

$$(b_{\mathbf{n-1}} \in Ob(\Phi(\mathbf{n-1})), \quad b_{U_+} \in Ob(\Phi(U_+)), \quad \beta_{U_+,V_+}, \quad \zeta_U),$$

where $U \subseteq V$ are proper subsets of $\mathbf{n-1}$, $\beta_{U_+,V_+} : \Phi_{U_+,V_+}(b_{V_+}) \to b_{U_+}$ is a morphism of the category $\Phi(U_+)$, while

$$\zeta_U : \Phi_{U,\mathbf{n-1}}(b_{\mathbf{n-1}}) \to \Phi_{U,U_+}(b_{U_+})$$

is a morphism of the category $\Phi(U)$. All these must satisfy two compatibility conditions: The first conditions says that for any proper subsets $U \subseteq V \subseteq W$ of $\mathbf{n-1}$, one has

$$\beta_{U_+,W_+} = \beta_{U_+,V_+} \circ \Phi_{U_+,V_+}(\beta_{V_+,W_+}).$$

The second condition is that for any proper subsets $U \subseteq V$ of $\mathbf{n-1}$, the diagram

$$\begin{array}{ccc} \Phi_{U,\mathbf{n-1}}(b_{\mathbf{n-1}}) & \xrightarrow{\Phi_{U,V}(\zeta_V)} & \Phi_{U,V_+}(b_{V_+}) \\ & {\scriptstyle \zeta_U} \searrow \quad \swarrow {\scriptstyle \Phi_{U,U_+}(\beta_{U_+,V_+})} & \\ & \Phi_{U,U_+}(b_{U_+}) & \end{array}$$

commutes.

We are now in a position to construct the functors

$$\Gamma : \underset{\mathfrak{B}(\mathbf{n})}{2\text{-lim}}\, \Phi \to \mathcal{P} \quad \text{and} \quad \Delta : \mathcal{P} \to \underset{\mathfrak{B}(\mathbf{n})}{2\text{-lim}}\, \Phi.$$

The functor Γ simply forgets some data. More precisely, take an object $(a_S, \alpha_{S,T})$ of the category $\underset{\mathfrak{B}(\mathbf{n})}{2\text{-lim}}\, \Phi$, where $S \subseteq T$ are proper subsets of $\mathbf{n}$. Then, Γ sends this object to the collection

$$(b_{\mathbf{n-1}} \in Ob(\Phi(\mathbf{n-1})), \quad b_{U_+} \in Ob(\Phi(U_+)), \quad \beta_{U_+,V_+}, \quad \zeta_U),$$

where $b_{\mathbf{n-1}} = a_{\mathbf{n-1}}$, $b_{U_+} = a_{U+}$, $\beta_{U_+,V_+} = \alpha_{U_+,V_+}$ and $\zeta_U = \alpha^{-1}_{U,U_+} \circ \alpha_{U,\mathbf{n-1}}$. Here, $U \subseteq V$ are proper subsets of $\mathbf{n-1}$.

The functor Δ is defined by

$$\Delta(b_{\mathbf{n-1}}, b_{U_+}, \beta_{U_+,V_+}, \zeta_U) = (a_S, \alpha_{S,T}),$$

where

$$a_S = \begin{cases} b_{\mathbf{n-1}} & \text{if} \quad S = \mathbf{n-1}, \\ b_S, & \text{if} \quad S \in \mathfrak{B}'(\mathbf{n-1}), \\ \Phi_{S,\mathbf{n-1}}(b_{\mathbf{n-1}}) & \text{if} \quad S \in \mathfrak{B}(\mathbf{n-1}) \end{cases}$$

and

$$\alpha_{S,T} = \begin{cases} \beta_{S,T} & \text{if} \quad S \in \mathfrak{B}'(\mathbf{n-1}) \\ \mathsf{Id} & \text{if} \quad T \in \mathfrak{B}(\mathbf{n-1}) \\ \mathsf{Id} & \text{if} \quad T = \mathbf{n-1} \\ \zeta_S^{-1} \circ \Phi_{S_+,S}(\beta_{S^+,T}) & \text{if} \quad T \in \mathfrak{B}'(\mathbf{n-1}) \ \ \& \ \ S \in \mathfrak{B}(\mathbf{n-1}). \end{cases}$$

We have used the fact that there are exactly four different cases if $U \subseteq T$ are proper subsets of $\mathbf{n}$.

The composite $\Gamma \circ \Delta$ is the identity functor. On the other hand, the composite $\Delta \circ \Gamma$ sends the collection $(a_S, \alpha_{S,T})$ to the collection $(\tilde{a}_S, \tilde{\alpha}_{S,T})$. Here,

$$\tilde{a}_S = \begin{cases} a_S & \text{if} \quad S = \mathbf{n-1}, \\ \mathsf{Id} & \text{if} \quad T = \mathbf{n-1}, \\ \mathsf{Id} & \text{if} \quad T \in \mathfrak{B}(\mathbf{n-1}) \end{cases}$$

and

$$\tilde{\alpha}_{S,T} = \begin{cases} \alpha_{S,T} & \text{if } S \in \mathfrak{B}'(\mathbf{n}-\mathbf{1}), \\ \mathsf{Id} & \text{if } S \in \mathfrak{B}'(\mathbf{n}-\mathbf{1}), \\ \mathsf{Id} & \text{if } T = \mathbf{n}-\mathbf{1}, \\ \mathsf{Id} & \text{if } T \in \mathfrak{B}'(\mathbf{n}-\mathbf{1}) \ \ \& \ \ S \in \mathfrak{B}(\mathbf{n}-\mathbf{1}). \end{cases}$$

Observe that

$$\xi_S = \begin{cases} \mathsf{Id} & \text{if } S = \mathbf{n}-\mathbf{1} \\ \mathsf{Id} & \text{if } S \in \mathfrak{B}'(\mathbf{n}-\mathbf{1}) \\ \alpha_{S,\mathbf{n}-\mathbf{1}} & \text{if } S \in \mathfrak{B}(\mathbf{n}-\mathbf{1}) \end{cases}$$

defines a natural transformation $\Delta \circ \Gamma \to \mathsf{Id}$. This implies the following proposition:

Proposition 3.2.12. *The following is a 2-pullback diagram in the 2-category of groupoids:*

$$\begin{array}{ccc} \underset{\mathfrak{B}(\mathbf{n})}{2\text{-lim}}\, \Phi & \longrightarrow & \underset{\mathfrak{B}'(\mathbf{n}-\mathbf{1})}{2\text{-lim}}\ \Phi \\ \downarrow & & \downarrow \\ \Phi(\mathbf{n}-\mathbf{1}) & \longrightarrow & \underset{\mathfrak{B}(\mathbf{n}-\mathbf{1})}{2\text{-lim}}\ \Phi. \end{array}$$

Yoneda's 2-lemma yields the following statement for 2-colimits:

Corollary 3.2.13. *The following is a 2-push-out diagram of groupoids*

$$\begin{array}{ccc} 2\text{-colim}_{\mathfrak{B}(\mathbf{n}-\mathbf{1})}\, \Phi & \longrightarrow & 2\text{-colim}_{\mathfrak{B}'(\mathbf{n}-\mathbf{1})}\, \Phi \\ \downarrow & & \downarrow \\ \Phi(\mathbf{n}-\mathbf{1}) & \longrightarrow & 2\text{-colim}_{\mathfrak{B}(\mathbf{n})}\, \Phi. \end{array}$$

3.2.5 2-limits on subsets with codimension 3

Note that the condition of Theorem 3.2.6 actually consists of 2^{n-1} sub-conditions. Each of these conditions itself has powersets in the indices. It is well-known that the latter can be reduced to n^2 many since the colimit over a poset can be reduced to the colimit over its subposet, consisting of elements with codimension 2.

In this section, we wish to simplify the condition of our theorem from something exponential to something of magnitude n^3.

Let $0 \leq k < n$ be a natural number. Denote by $\mathfrak{B}(n,k)$ the elements of $\mathfrak{B}(\mathbf{n})$ which are subsets S of $\mathbf{n}$, with $|S| \geq k$.

We can consider the restriction of any strict 2-functor $\Phi : \mathfrak{B}^{op} \to \mathfrak{Grpd}$ to $\mathfrak{B}(n,k)$, which will again be denoted by Φ.

Proposition 3.2.14. *Consider the obvious forgetful functor*

$$\gamma_k : \underset{\mathfrak{B}(\mathbf{n})}{2\text{-lim}}\, \Phi \to \underset{\mathfrak{B}(n,k)}{2\text{-lim}}\, \Phi. \tag{3.3}$$

i) The functor γ_k is faithful.

ii) The functor γ_k is full and faithful if $k \leq n-2$.

iii) The functor γ_k is an equivalence of categories if $k \leq n-3$.

Proof. i) Take two objects $A = (a_S, \alpha_{S,T})$ and $B = (b_S, \beta_{S,T})$ of $\underset{\mathfrak{B}(\mathbf{n})}{2\text{-lim}}\, \Phi$, where $S \subseteq T$ are proper subsets of $\mathbf{n}$. Assume (f_U) and (f'_U) are two morphisms $A \to B$ such that $f = f'$ in $2\text{-lim}_{\mathfrak{B}(n,k)}\, \Phi$. In particular, $\gamma_k(f_S) = \gamma_k(f'_S)$ if $|S| = n-1$. We have to show that $f_U = f'_U$ for all proper subsets $U \subsetneq \mathbf{n}$. For a given U, choose an S such that $U \subseteq S$ and $|S| = n-1$. We have a commutative diagram

$$\begin{array}{ccc} \phi_{U,S}(a_S) & \xrightarrow{\alpha_{U,S}} & a_U \\ {\scriptstyle \phi_{U,S}(f_S)}\downarrow & & \downarrow{\scriptstyle f_U} \\ \phi_{U,S}(b_S) & \xrightarrow{\beta_{U,S}} & b_U. \end{array}$$

This implies that $f_U = \beta_{U,S} \circ \phi_{U,S}(f_S)\alpha_{U,S}^{-1}$. Likewise, we have $f'_U = \beta_{U,S} \circ \phi_{U,S}(f'_S)\alpha_{U,S}^{-1}$. Since $f_S = f'_S$, we obtain $f_U = f'_U$.

ii) By i), we need to show that γ_k is full. We keep the notation as in i). Take a morphism $\gamma_k(A) \to \gamma_k(B)$. It is given by a collection of morphisms $f_S : a_S \to b_S$, where $|S| \geq k$. Moreover, if $S \subseteq T$ are two such subsets, we have a commutative diagram

$$\begin{array}{ccc} \phi_{S,T}(a_S) & \xrightarrow{\alpha_{S,T}} & a_S \\ {\scriptstyle \phi_{S,T}(f_T)}\downarrow & & \downarrow{\scriptstyle f_S} \\ \phi_{S,T}(b_T) & \xrightarrow{\beta_{S,T}} & b_S. \end{array}$$

We have to extend the definition of f_S to include subsets $S \subseteq \mathbf{n}$ with $|S| < k$, in such a way that the corresponding diagrams still commutes for all $S \subseteq T$. This is done by induction on i, where $i = k - |S|$. If $i = 0$, we already have such maps.

Let $i > 0$ and $U \subsetneq \mathbf{n}$ be a subset with $|U| = |S| + 1$. Then f_U is already defined by the induction assumption. We claim that the map

$$\beta_{S,U} \circ \phi_{S,U}(f_U) \circ \alpha_{S,U}^{-1} : a_S \to b_S$$

is independent of the choice of U. Assume V is another subset of $\mathbf{n}$ such that $|V| = |S| + 1$, $S \subseteq V$ and $V \neq U$. Take $W = V \cup U$. We have $|W| = |S| + 2 \leq n - 1$. By the induction

assumption, we have a commutative diagram

$$\begin{array}{ccc} \phi_{U,W}(a_W) & \xrightarrow{\alpha_{U,W}} & a_U \\ {\scriptstyle \phi_{U,W}(f_W)}\downarrow & & \downarrow{\scriptstyle f_U} \\ \phi_{U,W}(b_W) & \xrightarrow{\beta_{U,W}} & b_U. \end{array}$$

It follows that

$$\begin{aligned} \beta_{S,U} \circ \phi_{S,U}(f_U) \circ \alpha_{S,U}^{-1} &= \beta_{S,U}\phi_{S,U}(\beta_{U,W}\phi_{U,W}(f_W)\alpha_{U,W}^{-1})\alpha_{S,U}^{-1} \\ &= \beta_{S,U}\phi_{S,U}(\beta_{U,W})\phi_{S,U}(\phi_{U,W}(f_W))\phi_{U.W}(\alpha_{U,W}^{-1})\alpha_{S,U}^{-1} \\ &= \beta_{SW}\phi_{S,W}(f_W)\alpha_{S,W}^{-1}. \end{aligned}$$

Quite similarly, we obtain

$$\beta_{S,V} \circ \phi_{S,V}(f_V) \circ \alpha_{S,V}^{-1} = \beta_{SW}\phi_{S,W}(f_W)\alpha_{S,W}^{-1}$$

and the claim follows.

It remains to show that the appropriate diagram commutes for $S \subseteq T$. Choose U such that $S \subseteq U \subseteq T$ and $|U| = |S| + 1$. The corresponding diagram commutes for $U \subseteq T$ by the induction assumption. On the other hand, we also have a commutative diagram for the pair $S \subseteq U$. This is by the definition of f_S. We obtain a commutative diagram for the pair $S \subseteq T$, by gluing the two diagrams we just discussed.

iii) By ii,) we only need to show that γ_k is essentially surjective. Take an object B of $2\text{-lim}_{\mathfrak{B}(\mathbf{n},\mathbf{k})} \Phi$. By definition, B is a collection $(b_S, \beta_{S,T})$, where $|S| \geq k$ and $S \subseteq T$. These data satisfy the compatibility condition, which means that an appropriate diagram commutes, see the diagram in i). The task is to define an object $A = (a_S, \alpha_{S,T})$ of $2\text{-lim}_{\mathfrak{B}(\mathbf{n})} \Phi$ such that $\gamma_k(A) \simeq B$. In fact, we will show $\gamma_k(A) = B$.

If $|S| \geq k$, we set $a_S = b_S$ and $\alpha_{S,T} = \beta_{S,T}$. Let $|S| < k$. We may assume that the object A_U is already defined for all U with $|U| > |S|$. Choose $i \in \mathbf{n}$ such that $i \notin S$. We write $S \cup i$ instead of $S \cup \{i\}$ and set

$$a_S := \phi_{S,S\cup i}(a_{S\cup i}).$$

We wish to define the morphisms $\alpha_{S,T} : \phi_{S,T}(a_T) \to a_S$. There are several cases to consider.

- Let $T = S$. We put $\alpha_{S,S} := \mathrm{Id}$. We are left with $|T| > |S|$.
- Let $i \in T$. We define $\alpha_{S,T} := \phi_{S,S\cup i}(\alpha_{S\cup i,T})$.
- Assume $T \neq \mathbf{n} \setminus \{i\}$. Then $T \cup i$ is also a proper subset and we can put

$$\alpha_{S,T} = \phi_{S,S\cup i}(\alpha_{S\cup i,T\cup i}) \circ \phi_{S,T}(\alpha_{T,T\cup i})^{-1}.$$

This is equivalent to saying that the following diagram commutes:

$$\begin{array}{ccc} \phi_{S,T\cup i}(a_{T\cup i}) & \xrightarrow{\phi_{S,T}(\alpha_{T,T\cup i})} & \phi_{S,T}(a_T) \\ \downarrow{\scriptstyle \phi_{S,S\cup i}(\alpha_{S\cup i,T\cup i})} & & \downarrow{\scriptstyle \alpha_{S,T}} \\ \phi_{S,S\cup i}(a_{S\cup i}) & \xrightarrow[\mathsf{Id}]{} & a_S. \end{array}$$

- It remains to consider the case $T = \mathbf{n} \setminus \{i\}$. Choose $j \in T$ such that $j \notin S$. We set

$$\alpha_{S,T} = \alpha_{S,S\cup j} \circ \phi_{S,S\cup j}(\alpha_{S\cup j,T}).$$

One gets the following commutative diagram

$$\begin{array}{ccc} & & \phi_{S,T}(a_T) \\ & & \downarrow{\scriptstyle \phi_{S,S\cup j}(\alpha_{S\cup j,T})} \\ \phi_{S,S\cup i\cup j}(a_{S\cup i\cup j}) & \xrightarrow{\phi_{S,S\cup j}(\alpha_{S\cup j,S\cup i\cup j})} & \phi_{S,S\cup j}(a_{S\cup j}) \\ \downarrow{\scriptstyle \phi_{S,S\cup i}(\alpha_{S\cup i,S\cup i\cup j})} & & \downarrow{\scriptstyle \phi_{S,S\cup j}(\alpha_{S,S\cup j})} \\ \Phi_{S,S\cup i}(a_{S\cup i}) & \xrightarrow[\mathsf{Id}]{} & a_S \end{array}$$

from the definition of $\alpha_{S,S\cup j}$. This definition is independent on the choice of j. In fact, chose k instead of j. The set $S \cup i \cup j \cup k$ is still a proper subset of $\mathbf{n}$ (since $|S| < k \leq n-3$). One easily sees that both definitions of α_{ST} equal the composite

$$\phi_{S,S\cup i\cup j}(\phi(a_T)) \xrightarrow{\phi_{S,S\cup i\cup j}(\alpha_{S\cup i\cup j})} a_{S\cup i\cup j}.$$

This finishes the definition of A. Obviously $\gamma_k(A) = B$ and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Recall that we denoted the subposet of $\mathfrak{B}(\mathbf{n})$ which is formed by subsets of cardinality $\geq k$ by $\mathfrak{B}(n,k)$.

Corollary 3.2.15. *Let $n \geq 3$ be a natural number and $\Phi : \mathfrak{B}(n) \to \mathfrak{Grpd}$ a covariant strict 2-functor. The canonical functor $2\text{-colim}_{\mathfrak{B}(\mathbf{n},\mathbf{n-3})} \Phi \to 2\text{-colim}_{\mathfrak{B}(\mathbf{n})} \Phi$ is an equivalence categories.*

Proof. For any groupoid $\mathcal{G}$, we have

$$\mathsf{Hom}_{\mathfrak{Grpd}}(2\text{-}\operatorname*{colim}_{\mathfrak{B}(\mathbf{n})} \Phi, \mathcal{G}) \cong 2\text{-}\lim_{\mathfrak{B}(\mathbf{n})} \mathsf{Hom}_{\mathfrak{Grpd}}(\Phi, \mathcal{G}).$$

Thanks to Proposition 3.2.14, we can replace the RHS with the equivalent groupoids

$$2\text{-}\lim_{\mathfrak{B}(\mathbf{n})} \mathsf{Hom}_{\mathfrak{Grpd}}(\Phi, \mathcal{G}) \cong 2\text{-}\lim_{\mathfrak{B}(\mathbf{n},\mathbf{n-3})} \mathsf{Hom}_{\mathfrak{Grpd}}(\Phi, \mathcal{G}) \cong \mathsf{Hom}_{\mathfrak{Grpd}}(2\text{-}\operatorname*{colim}_{\mathfrak{B}(\mathbf{n},\mathbf{n-3})} \Phi, \mathcal{G}),$$

implying the result. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Recall that for subsets $V \subset \mathbf{n}$ and $U \subsetneq V$, condition A^V_U says that the canonical functor

$$\text{colim}_{\mathfrak{B}(V:U)} \Phi \to \Phi(V)$$

is injective on objects. We define the following V's and U's:

$$V_i = \mathbf{n} \setminus \{i\}, \quad U_i = \{i+1, \cdots, n\}, \quad 2 \leq i \leq n$$

and

$$V_{i,j} = \mathbf{n} \setminus \{i, j\}, \quad U_{i,j} = \{i+1, \cdots, n\} \setminus \{j\}, \quad 2 \leq i < j \leq n.$$

We are now in the position to state the following result.

Proposition 3.2.16. *Let $n \geq 3$ and $\Phi : \mathfrak{B}(\mathbf{n}) \to \mathfrak{Grpd}$ be a covariant strict 2-functor for which the following conditions hold: $A^{V_i}_{U_i}$ and $A^{V_{ij}}_{U_{ij}}$ for all $2 \leq i \leq n$ and $2 \leq i < j \leq n$. Then*

$$\underset{\mathfrak{B}(\mathbf{n})}{2\text{-colim}}\, \Phi = \text{colim}_{\mathfrak{B}(\mathbf{n},\mathbf{n-3})}\, \Phi.$$

Proof. Denote by $\overline{\Phi}$ the covariant strict 2-functor $\mathfrak{B}(n) \to \mathfrak{Grpd}$, defined by

$$\overline{\Phi}(V) = \begin{cases} \Phi(V) & \text{if} \quad |V| \geq n-3 \\ \varnothing & \text{if} \quad |V| < n-3. \end{cases}$$

We have

$$\underset{\mathfrak{B}(\mathbf{n})}{2\text{-colim}}\, \Phi = \underset{\mathfrak{B}(\mathbf{n},\mathbf{n-3})}{2\text{-colim}}\, \Phi$$

by Corollary 3.2.15. Next, we have

$$\underset{\mathfrak{B}(\mathbf{n},\mathbf{n-3})}{2\text{-colim}}\, \Phi = \underset{\mathfrak{B}(\mathbf{n},\mathbf{n-3})}{2\text{-colim}}\, \overline{\Phi}$$

because both 2-functors restricted on $\mathfrak{B}(\mathbf{n}, \mathbf{n-3})$ are equal. We can use Corollary 3.2.15 again to obtain

$$\underset{\mathfrak{B}(\mathbf{n})}{2\text{-colim}}\, \Phi = \underset{\mathfrak{B}(\mathbf{n})}{2\text{-colim}}\, \overline{\Phi}.$$

The claim is that $\overline{\Phi}$ satisfies Condition $A(n)$ from Theorem 3.2.6. Assuming this, we obtain

$$\underset{\mathfrak{B}(\mathbf{n})}{2\text{-colim}}\, \Phi = \text{colim}_{\mathfrak{B}(\mathbf{n})}\, \overline{\Phi} = \text{colim}_{\mathfrak{B}(\mathbf{n},\mathbf{n-3})}\, \overline{\Phi} = \text{colim}_{\mathfrak{B}(\mathbf{n},\mathbf{n-3})}\, \Phi.$$

To see the claim, take V, k, U as in Condition $A(n)$. Condition $A(n)$ for $\overline{\Phi}$ holds trivially if $|V| \leq n-3$, because all members in the diagrams are empty categories. Assume $|V| = n-2$. Since $1 \in V$, we see that

$$V = V_{i,j},$$

where $2 \leq i < j \leq n$. In the notation of Theorem 3.2.6, $k = i - 1$. Hence $U = U_{i,j}$. Quite

similarly, if $|V| = n - 1$ and $1 \in V$, we see that $V = V_i$ where $2 \leq i \leq n$. Thus, $k = i - 1$ and $U = U_i$. By our assumption, Condition $A(n)$ holds for $\overline{\Phi}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

3.2.6 The filtered case

The following proposition can already be found in my PhD thesis.

Recall that a poset I is called *filtered* provided for any elements $i, j \in I$ there is an element $k \in I$ such that $i \leq k$ and $j \leq k$.

Proposition 3.2.17. *Let I be a filtered poset and $\mathfrak{F} : I \to \mathfrak{Cat}$ a strict 2-functor. We have an equivalence of categories*

$$\delta : 2\text{-}\operatorname*{colim}_i \mathfrak{F}_i \to \operatorname*{colim}_i \mathfrak{F}_i,$$

Proof. We will start by describing the functor δ. Recall that the objects of the 2-colimit are the disjoint union of the objects of the $\mathfrak{F}_i$-s (Section 1.2.4.1). Since the objects of the colimit of $\mathfrak{F}$ are a quotient of the disjoint union of $\text{obj}(\mathfrak{F}_i)$ (Section 1.2.3.2), we clearly have δ defined on objects. We also immediately see that δ is essentially surjective (indeed it is surjective).

To define δ on morphisms, take $A \in \mathfrak{F}_i$, $B \in \mathfrak{F}_j$ and

$$\varphi \in \text{Hom}_{2\text{-}\operatorname*{colim}_i \mathfrak{F}_i}(a, b) \cong \operatorname*{colim}_{k \in I_{i,j}} \text{Hom}_{\mathfrak{F}_k}(\alpha_*(A), \beta_*(B))$$

(for the result and notation, see Proposition 1.2.17). Pick a representative

$$\mathfrak{F}_k \ni \varphi_k : \alpha_*(A) \to \alpha_*(B)$$

of φ. Since every morphism in every $\mathfrak{F}_k$ defines a morphism in the colimit, clearly, so does φ_k. It is straightforward to see that this map is well defined.

To show that δ is full and faithful, assume $\delta(\varphi) = \delta(\phi)$ in $\text{Hom}_{\operatorname*{colim}_i \mathfrak{F}_i}(\delta(A), \delta(B))$. There exists $k \in I_{i,j}$ such that $\delta(\varphi)_k = \delta(\phi)_k$. This is the exact same condition as for the equality in the 2-colimit. Lastly, to prove surjectivity between the Hom-sets, take $f : A \to B$ in the colimit of $\mathfrak{F}$. Since f is in the colimit, choose a representative f_k of f in some $\mathfrak{F}_k$. Then $[\text{Id}, \text{Id}, f_k] \in \text{Hom}_{2\text{-}\operatorname*{colim}_i \mathfrak{F}_i}(A, B)$ maps to f. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Chapter 4

Preliminaries on Monoids

4.1 Monoid Schemes

Throughout this section, a monoid will be assumed to be commutative. Though we will later consider non-commutative monoids as well, this will only be the case past Chapter 7. Our main references are [15], [18], [19] and [49].

The notion of a monoid scheme was introduced by Deitmar [19] in 2005, following the works of Kato [35], who proved the link between monoid schemes and toric varieties in 1994. This link was further explored in [18]. Indeed, monoid schemes are closely connected with many areas of mathematics, such as algebraic k-theory [18], tropical geometry [47], logarithmic geometry [1], [34], absolute mathematics [39] and most notably $\mathbb{F}_1$-geometry [16].

Monoid schemes have a strong similarity to ring schemes. This is part of their attractiveness, as it allows us to directly use the intuition and even results from algebraic geometry.

4.1.1 Affine Monoid Schemes

The aim of this section is to prove several useful results about commutative monoids, which are exactly the affine monoid schemes introduced shortly.

4.1.1.1 Limits, Colimits and the Tensor Product

The *tensor product* $M \otimes N$ of two monoids M and N is defined to be the commutative monoid generated by the elements $m \otimes n$ with $m \in M, n \in N$, modulo the relations:

- $(m_1 m_2) \otimes n = (m_1 \otimes n)(m_2 \otimes n)$
- $m \otimes (n_1 n_2) = (m \otimes n_1)(m \otimes n_2)$
- $1 \otimes n = m \otimes 1 = 1 \otimes 1$.

Note that the last relation does not follow from the first two, as it does in the case of abelian groups. We immediately see that the exponential property

$$\mathsf{Hom}(M \otimes N, S) \cong \mathsf{Hom}(M, \mathsf{Hom}(N, S))$$

holds, where M, N and S are commutative monoids.

Most of the categorical definitions, like limits and colimits that exist for abelian groups, also exist for monoids and are constructed in the same way.

4.1.1.2 Posets, Semi-lattices and Lattices

The results of this section are well known. See for example [24]. The notations we use however, does not follow it.

Semi-lattices Let $(P, \leq)$ be a poset, meaning partially ordered set and $S \subseteq P$ a subset. We set

$$P_S := \{p \in P \mid p \geq s,\ \forall s \in S\}$$

and

$$P^S := \{p \in P \mid p \leq s,\ \forall s \in S\}.$$

We will usually write P instead of $(P, \leq)$, when the ordering is clear. Likewise, when $S = \{a\}$ or $S = \{a, b\}$, we will write P_a, respectively $P_{a,b}$.

Recall that a poset is called a *join semi-lattice* if for every $a, b \in P$, the set $P_{a,b}$ has a minimal element, denoted $a \vee b$ and called the *join* of a and b. We also require it to have a unit, called the *least* element. This is an element 1 such that $a \vee 1 = 1 \vee a = a$ for any $a \in P$. Its dual object is called a *meet semi-lattice*. More specifically, it is a poset P, where every subposet $P^{a,b}$ has a maximal element (the *meet*) $a \wedge b$, for every $a, b \in P$. In addition, it is required to have a unit under this dual operation, which is called the *maximal* element and is denoted by 0.

We observe that the unit of $\wedge$ is the absorbing element ("zero") of $\vee$ and the unit of $\vee$ is the absorbing element of $\wedge$.

Morphisms between semi-lattices (meet or join) are just structure preserving maps. As the categories of join- and meet semi-lattices are equivalent, we will usually just say semi-lattice and use the notation of a join semi-lattice.

A poset that is simultaneously a join- and a meet semi-lattice is called a *lattice*. It is an easy exercise to verify that a finite join semi-lattice with a least element is a lattice. A lattice is said to be *distributive*, if for all $a, b, c \in P$, we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

It is well known, that this is equivalent to the dual condition

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Of course, not every lattice is distributive. The simplest counterexample is one with 3 generators $\langle a, b, c\rangle$, module the relations $a \vee b = a \vee c = 0$, $b \vee c = b$ and $a \wedge b = a \wedge c = 1$, $b \wedge c = c$. In this case, we have

$$b \wedge (a \vee c) = b \wedge 0 = b,$$

but

$$(b \wedge a) \vee (b \wedge c) = 1 \vee c = c.$$

Galois connections We also recall that a *Galois connection* between posets X and Y is a pair $\gamma = (\alpha, \beta)$, where $\alpha : X \to Y$ and $\beta : Y \to X$ are maps such that

$$(x_1 \leq x_2) \Rightarrow (\alpha(x_1) \geq \alpha(x_2)), \quad (y_1 \leq y_2) \Rightarrow (\beta(y_1) \geq \beta(y_2)),$$

and for any $x \in X$, $y \in Y$, one has

$$(x \leq \beta(y)) \Leftrightarrow (\alpha(x) \leq y).$$

It follows that

$$x \leq \beta\alpha(x) \quad \text{and} \quad y \leq \alpha\beta(y).$$

An element $x \in X$ (resp. $y \in Y$) is called *stable* under the Galois connection if $x = \beta\alpha(x)$ (resp. $y = \alpha\beta(y)$). In this case, α and β induce bijections on stable elements.

Idempotent Monoids It is an easy fact that the category of (join) semi-lattice is equivalent to the category of *idempotent monoids*, i.e. where $m^2 = m$ for every $m \in M$. In fact, starting with a join semi-lattice, the opetaion $\vee$ defines a monoid structure where all elements are idempotents. Conversely, if M is a monoid such that $x^{=}x$ for all $x \in M$, we can define the ordering by $x \leq y$ if $xy = y$. In this way, we obtain an equivalence from the category of (joint) semi-latices to the category of idempotent monoids. We sometimes even identify these categories.

We have a functor from the category of monoids to the category of semilattices

$$\mathsf{sl} : \mathsf{Mn} \to \mathsf{SL},$$

given by $M \mapsto M/a^2 \sim a$, which is the left adjoint of the inclusion functor. Explicitly, this relation is given as follows: [24, Theorem 1.2 of Chapter III]. We have

$$a \sim b \iff \exists m, n \geq 1,\ u, v \in M \quad \text{such that} \quad a^m = ub,\ b^n = va. \tag{4.1}$$

We denote the image of M under sl by M^{sl} and call it the the *associated semi-lattice* or the *booleanisation*. The class of an element $a \in M$ in M^{sl} is written as $[a]$. If we identiy it as a subset of M, $[a] \subseteq M$ is a semigroup since all the elements of M^{sl} are idempotents, and the preimage of a semigroup is a semigroup. We also have the isomorphism $M^{\mathsf{sl}} \simeq M \otimes \mathbb{I}$, where $\mathbb{I} = \{0, 1\}$ under multiplication.

Remark 4.1.1. *It should be noted that a morphism $f : L \to L'$ of finite join semi-lattices with least elements need not be a morphism of lattices. This is despite the fact that both L and L' are going to be lattices, as they are finite (see our discussion above).*

Poset Topology A poset $(P, \leq)$ admits a natural topology, where a subset $U \subseteq P$ is declared open if and only if for any $y \leq x \in P$, $x \in U$ implies $y \in U$. In other words, the subsets of the

form P^a, $a \in P$, form a basis of this topology. Naturally, we could have also taken the dual definition. As in all topologies,

$$\mathfrak{Off}(P) := \{U \subseteq P \mid \text{ such that } \ U \ \text{ is open in P } \}$$

forms a distributive lattice, which is finite if and only if P is finite. According to the Birkhoff's representation theorem, any finite distributive lattice L can be uniquely obtained as $\mathfrak{Off}(\mathsf{Irr}(L))$. Here, $\mathsf{Irr}(L)$ is the subposet of join irreducible elements of L. Recall that an element $a \in L$ is called join irreducible if $a = b \vee c$ implies $a = b$ or $a = c$. In particular, $\mathfrak{Off}(P) \simeq \mathfrak{Off}(Q)$ if and only if $P \simeq Q$.

Recall also the definition of a T_0 *space*. For any two distinct points $x, y \in X$, it implies the existence of an open set U such that, either $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$. Let X be a T_0-space. We write $x \leq y$, $x, y \in X$, provided $\overline{y} \subseteq \overline{x}$. Here, $\overline{x}$ denotes the closure of the point x. Then $\leq$ is a partial order on X, called the *underlying poset* of the topological space X. Moreover, if $f : X \to Y$ is a continuous map of T_0-spaces, then f is monotonic. Thus, we have a forgetful functor from the category of T_0-topological spaces to the category of posets.

Let X be a T_0-space. For any sheaf $\mathcal{F}$ of sets on X, the map $x \mapsto \mathcal{F}_x$ gives rise to a contravariant functor on the poset X. As stated in Lemma 4.1.3 below, the functor $x \mapsto \mathcal{F}_x$ is enough to reconstruct the sheaf $\mathcal{F}$, for X a $\mathfrak{P}$-topological space.

A T_0-topological space is called a $\mathfrak{P}$-*topological space*, provided any intersection of open sets is again open.

The link between $\mathfrak{P}$-topological spaces and poset topological spaces is well known, see [44] and [68]. We state the following well-known facts:

Proposition 4.1.2. *1) Let P be a poset and X_P its associated topological space. For any element $x \in X_P$, the smallest open subset of X_P containing x is $P^x = \{y \mid y \leq x\}$. Further, the open subsets of the type P^x form a basis of the topology.*

2) Let X_P and X_Q be the topological spaces associated to the posets P and Q, respectively. We have a bijection

$$\mathsf{C}(X_P, X_Q) \cong \mathsf{MHom}(P, Q),$$

where MHom *denotes the set of monotonic maps from P to Q, and* C *denotes the set of continuous maps from X_P to X_Q.*

3) The category of $\mathfrak{P}$-topological spaces with continues maps is equivalent to the category of posets with monotonic maps.

Using Proposition 4.1.2, we will identify posets, equipped with the ordering topology given above, with $\mathfrak{P}$-topological spaces. Of course, this can be extend to sheaves. Specifically, for a $\mathfrak{P}$-topological space X and a point $x \in X$, denote by U_x the smallest open subset containing x. This is the intersection of all open subsets containing x.

It is a well known fact that a sheaf $\mathcal{F}$ over a topological space X is uniquely defined by its values on the bases of open sets. Hence, using part 1) of Proposition 4.1.2, it is sufficient to define a sheaf only on the open subsets of the form U_x. That is, it suffices to only define

our sheaf on the stalks of the underlying poset P. If $x \leq y \Rightarrow U_x \subseteq U_y$, which yields a homomorphism $\mathcal{F}_y \to \mathcal{F}_x$, and hence, a contravariant functor on P.

Lemma 4.1.3. *Let X_P be a $\mathfrak{P}$-topological space and $\mathcal{F}$ a sheaf of sets on X_P. The map $\mathcal{F} \mapsto \mathcal{F}_x$ gives rise to an equivalence between the category of sheaves on X and the category of contravariant functors on the poset P.*

Proposition 4.1.4. *To see this result, it suffices to say that the corresponding sheaf $\mathcal{F}$ on X_P is defined by $\mathcal{F}(U) = \lim_{x \in U} \mathcal{F}_x$.*

As finite lattices are self-dual, the above lemma can also be restated to say that covariant functors on a finite poset P are equivalent to sheaves over the corresponding topological space.

4.1.1.3 Prime Ideals and the Zarisiki topology

The results of this subsection can already be found in my PhD thesis.

An *ideal* of a monoid M is a sub-M-set of M. A proper ideal is said to be *prime*, if $ab \in I$ implies $a \in I$ or $b \in I$. It is immediate from this, that the complement of a prime ideal is a submonoid of M. This is vital, as it allows us to localise at prime ideals, which we will mention later. We note that the empty set is a prime ideal.

The set of all prime ideals of a monoid M, called the spectrum (or sometimes Kato-spectrum) and denoted by $\mathrm{Spec}(M)$, has a natural topological structure, where the basis of the topology is given by subsets of the form

$$D(f) = \{\mathfrak{p} \subseteq M \mid f \notin \mathfrak{p}\},$$

see [19, p.92], [35], [41] for more on this. These open subsets are called *principal* and the induced topological space is called the *Zariski topology*. As one would expect, the closed subsets are exactly of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(M) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

There are, however, subtle differences from ring theory, which have a significant impact on the theory, often allowing us to make strong simplifications. One of the most fundamental such differences is the fact that the union of ideals of a monoid is again an ideal and likewise, the union of prime ideals is again prime. As such, the set of prime ideals $\mathrm{Spec}(M)$ of a monoid M is itself a monoid. By taking the union of all prime ideals, we see that we have a unique maximal ideal, being $M \setminus M^*$, where M^* denotes the invertible elements of M.

The complements of prime ideals are likewise satisfying a special condition, which is not the case in the world of commutative rings. A subset $S \subseteq M$ is the complement of a prime ideal $\mathfrak{p}$ if and only if 1) S is a submonoid and 2) if $ab \in S$ implies $a \in S$ and $b \in S$. Submonoids of these types are called *faces*, in accordance to the theory of toric varieties. One can also use the term saturated submonoid, as for commutative rings, these conditions define saturated multiplicative subsets.

As expected, taking the preimage makes Spec : Mn → Mn into a contravariant functor. As shown in [48, Lemma 2.1], this functor is actually representable, being given by

$$\mathsf{Spec}(M) \simeq \mathsf{Hom}(M, \mathbb{I}), \tag{4.2}$$

where, as above, $\mathbb{I} = \{0, 1\}$ under multiplication. This isomorphism is given by sending a prime ideal $\mathfrak{p}$ to the monoid homomorphism $\chi_{\mathfrak{p}} : M \to \{0, 1\}$, defined by

$$\chi_{\mathfrak{p}}(m) = \begin{cases} 0, \text{ if } m \in \mathfrak{p}, \\ 1, \text{ if } m \notin \mathfrak{p}. \end{cases}$$

As we said, the union of prime ideals is again prime ideal. Thus, $\mathsf{Spec}(M)$ has a monoid structure. This allows us to consider Spec as an endofunctor of the category of monoids. This gives sense to the question of what happens under iteration. We will shortly show that the answer has some analogues to vector spaces, with $\mathbb{I}$ playing the role of the base field. This might in some sense fit well with the general philosophy of $\mathbb{F}_1$, but we will not explore it further.

As already mentioned, semi-lattices are equivalent to idempotent monoids. We have defined two important functors Spec : Mn → SL and sl : Mn → SL. These functors are related trough a natural isomorphism

$$\mathsf{Spec} \circ \mathsf{sl} \xRightarrow{\sim} \mathsf{Spec}, \tag{4.3}$$

meaning $\mathsf{Spec}(M^{\mathsf{sl}}) \simeq \mathsf{Spec}(M)$, see [10], [48].

The finitely generated case We have an injective map $\alpha_{\mathsf{sl}} : M^{\mathsf{sl}} \to \mathsf{Spec}(M)$, which is bijective if M is finitely generated [48]. To define the map α_{sl}, we first consider the case when $M = L$ is a join semi-latice. For any $f \in L$, we let $\alpha_{\mathsf{sl}}(f)$ be the the union of all prime ideals not containing f. For a general monoid M, we define α_{sl} as the composition

$$M^{\mathsf{sl}} \to \mathsf{Spec}(M^{\mathsf{sl}}) \cong \mathsf{Spec}(M).$$

According to [48], the map α_{sl} is injective, and it is a bijection if M is finitely generated. In general, α_{sl} is not a "natural map" as sl is covariant, while Spec is contravariant.

Let M be finitely generated. Then α_{sl} it is the order reversion. If we iterate taking the spectrum, we get a natural isomorphism

$$\mathsf{sl} \xRightarrow{\sim} \mathsf{Spec}^2. \tag{4.4}$$

A direct consequence of this is that if M is finitely generated with n generators, then $\mathsf{Spec}(M)$ is finite and bound by 2^n elements. The bound is reached when M is free. As such, our semi-lattice ($\mathsf{Spec}(M)$ or M^{sl}) is finite and hence, a lattice. We will, nonetheless, only regard it as a semi-lattice. The reason was already alluded in Remark 4.1.1: The map $\mathsf{Spec}(N) \to \mathsf{Spec}(M)$, induced by $M \to N$, is only a semi-lattice homomorphism, even in the

finitely generated case.

Isomorphism of Topologies As Spec(M) is a semi-lattice, it has a natural order topology. We mentioned in part 1) of Proposition 4.1.2 that a poset topology admits a basis of the form $P^x = \{y \in P \mid y \leq x\}$. To compare the Zariski topology with this order topology, observe that

$$D(f) = \mathrm{Spec}(M)^{\alpha_{\mathsf{sl}}([f])} = \{\mathfrak{p} \mid \mathfrak{p} \subseteq \alpha_{\mathsf{sl}}([f])\}. \tag{4.5}$$

Recall that $\alpha_{\mathsf{sl}}([f])$ is the union of all prime ideals not containing f. As such, every Zariski open subset is open in the poset topology. The other side is not true in general. This imbalance stems from the fact that, while we can take infinite unions of prime ideals, we can not take infinite products of elements. This is, however, exactly what we would have to do ensure that a subset of the form Spec$(M)^{\mathfrak{p}}$ is open in the Zariski topology, as it "would correspond" to $D(f)$, where f is any lift of $[f] = \prod_{i\in I}[f_i]$, with $f_i \notin \mathfrak{p}$.

As M^{sl} is finite for a finitely generated M, the above product makes sense. Hence, we can say that for a finitely generated M, the order topology and the Zariski topology coincide. We also note that in the order topology, the principal open subsets are in a one-to-one correspondence with the elements of the underlying poset. Hence, the principal open subsets correspond exactly to the prime ideals, with the correspondence being given by $D(f) \mapsto \alpha_{\mathsf{sl}}([f])$.

By declaring the subsets of the form

$$M^{\mathsf{sl}}_{[x]} = \{[y] \in M^{\mathsf{sl}} \mid [y] \geq [x]\}$$

to be open in M^{sl}, we see that α_{sl} respects the topology. Hence, in the finitely generated case, M^{sl} with the opposite order topology $M^{\mathsf{sl}}_{[x]}$, is homeomorphic to Spec(M) with the Zariski topology.

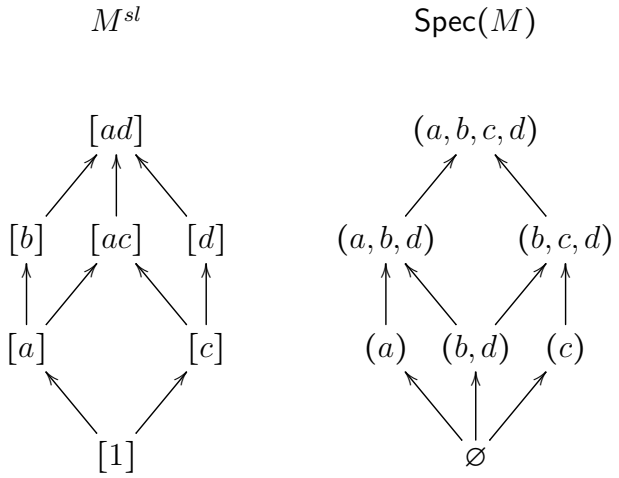
As we will show shortly, this equivalence can be extended to the structure sheaf. Meaning that in the finitely generated case, we can avoid Spec(M) altogether and only work with the simpler M^{sl}.

Indeed, all this also makes sense if only M/M^* is finitely generated, a property we call *almost finitely generated*. We summarise our discussion as follows:

Lemma 4.1.5. *If $M/M^\times$ is finitely generated, the Zariski topology on* Spec(M) *coincides with the order topology.*

Example 4.1.6. *Consider the monoid $M = \langle a, b, c, d\rangle / \{ab = b,\ cd = d,\ bc = ad\}$. Below, we*

give M^{sl} *and* Spec(M) *explicitly, with arrows indicating ordering.*



We observe an obvious, but important fact. While Spec(M) *and* M^{sl} *are bijective with only the order inverted, they are very different in terms of generators and relations. We have* $M^{sl} = \langle a,b,c,d\rangle / \{ab = b, cd = d, bc = ad\}$ *and* Spec$(M) = \langle x,y,z\rangle / \{xz = xyz\}$*, in the category of semi-lattices. In terms of the diagrams above,* $x = (a)$*,* $y = (b,d)$ *and* $z = (c)$*.*

4.1.1.4 Localisation of Monoids

It is an obvious fact that we have the inclusion functor

$$\mathsf{Mn} \to \mathsf{Ab}$$

in the category of abelian groups. This functor has both a left and a right adjoint. The right adjoint maps a monoid to its invertible elements, while the left adjoint is constructed via localisation. As the localisation of rings only requires a multiplicative subset, it is clear that the definition translates verbatim. Specifically, for a submonoid $S \subseteq M$, we can form the *localisation* of M with S. It is denoted by $S^{-1}M$ and elements of it are fractions $\frac{m}{s}$, where $m \in M$ and $s \in S$. By definition, $\frac{m_1}{s_1} = \frac{m_2}{s_2}$ if and only if there is an element $s \in S$ such that $m_1 s s_2 = m_2 s s_1$.

We use the notations M_f, $M_{\mathfrak{p}}$ and M^{Gr} for the following locations respectively: localising M with the submonoid generated by a single element $f \in M$, localising M at the prime ideal $\mathfrak{p}$ (meaning, inverting its complement) and localising M with all of M. The latter is called the Grothendieck Group of M and is the right adjoint of the forgetful functor.

There are several facts about localisations which are true in monoids but not in rings. For instance, all localisations by submonoids are localisations by prime ideals. To see this, take any submonoid S of M and define

$$\hat{S} = \{x \in M | \exists y \in M, xy \in S\}$$

Clearly

$$S^{-1}M \cong \hat{S}^{-1}M.$$

It is also clear that $\hat{S}$ is face and thus, $\mathfrak{p} := M \setminus \hat{S}$ is a prime ideal. This immediately implies, $S^{-1}M \cong M_{\mathfrak{p}}$. See also [18, Lemmas 1.1 & 1.3].

Recall that we denoted the class of an element $f \in M$ in M^{sl} by $[f] \subseteq M$. Since any element in M^{sl} is idempotent, $[f]$ is a semigroup. For $S = \{f^n, n = 0, 1, \cdots\}$ and $S_1 = [f] \cup \{1\}$ we have $\hat{S} = \hat{S}_1$. Hence $M_f \simeq [f]^{-1}M$. When writing $[f]^{-1}$, we mean $[f] = [f] \cup \{1\}$ by abuse of notation. As a direct consequence, we see that if $[f] = [g]$, then $M_f \simeq M_g$. Furthermore, recall that $\alpha_{\mathsf{sl}}([f])$ is the biggest prime ideal not containing f and denote by L_f the complement subset. It is a submonoid (actually a face) and for any submonoid N for which

$$f \in N \subseteq L_f \tag{4.6}$$

we have

$$N^{-1}M \simeq M_f \tag{4.7}$$

4.1.2 Monoid spaces and monoid (co)posets

Recall that a monoid homomorphism $f : M \to N$ is called *local* provided $f^{-1}(N^\times) \subseteq M^\times$. As usual, $M^\times$ denotes the subgroup of invertible elements of M. Equivalently, f is local if it sends non-invertible elements to non-invertible elements.

A *monoid space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of monoids on X. A *morphism of monoid spaces* $f : (X, \mathcal{O}_X) \to Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$, together with a morphism of monoid sheaves $f^\# : \mathcal{O}_Y \to f_\bullet \mathcal{O}_X$. Additionally, the induced morphisms on stalks $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ must be local for all $x \in X$. We will often simply write X instead of $(X, \mathcal{O}_X)$. The category of monoid spaces is denoted by MnSp.

A *monoid poset* is a pair $(P, \mathcal{O}_P)$, where P is a poset and $\mathcal{O}_P$ is a contravariant functor $\mathcal{O}_P : P^{op} \to$ Mn to the category of monoids. A morphism of monoid posets $f : (P, \mathcal{O}_P) \to (Q, \mathcal{O}_Q)$ is given by a morphism of posets $f : P \to Q$, together with a natural transformation of monoid valued functors $f^\# : \mathcal{O}_Q \circ f \to \mathcal{O}_P$ such that for every $x \in P$, the homomorphism $f^\#(x) : \mathcal{O}_Q(f(x)) \to \mathcal{O}_P(x)$ is local. The category of monoid posets is denoted by MnPos.

A *monoid coposet* is a pair $(L, \mathcal{O}_L)$, where L is a poset and $\mathcal{O}_L$ is a covariant functor $\mathcal{O}_L : L \to$ Mn to the category of monoids. A morphism of monoid posets $f : (L, \mathcal{O}_L) \to (L', \mathcal{O}_{L'})$ is given by a morphism of posets $f : L \to L'$, together with a natural transformation of monoid valued functors $f^\# : \mathcal{O}_L \to \mathcal{O}_{L'} \circ f$ such that for all $x \in P$, the homomorphisms $f^\#(x) : \mathcal{O}_L(x) \to \mathcal{O}_{L'}(f(x))$ are local. The category of monoid coposets is denoted by MnCop.

4.1.2.1 The affine case

As in classical algebraic geometry, there is a canonical sheaf of commutative monoids $\mathcal{O}_{\mathsf{Spec}(M)}$ associated to the space $\mathsf{Spec}(M)$, for every commutative monoid M.

The structure sheaf $\mathcal{O}_{\mathsf{Spec}(M)}$ is defined as expected: Define the values on the principal open subsets $D(f)$ to be the localisations M_f. This extends uniquely to a sheaf, which we call

the *structure sheaf* of M. Thus, we obtain monoid spaces $(\mathrm{Spec}(M), \mathcal{O}_{\mathrm{Spec}(M)})$ called the *affine monoid scheme* corresponding to the monoid M. In this way, one obtains a contravariant functor

$$\mathrm{Spec} : \mathrm{Mn}^{op} \to \mathrm{MnSp}.$$

By the same symbol $\mathrm{Spec}(M)$, we denote the poset of prime ideals of M. The assignment $\mathfrak{p} \mapsto M_{\mathfrak{p}}$ defines a structure of a monoid poset, called the *affine monoid poset* corresponding to the monoid M. In this way, one obtains a contravariant functor

$$\mathrm{Spec} : \mathrm{Mn}^{op} \to \mathrm{MnPos}.$$

For a monoid M, we can regard the quotient M^{sl} as a poset (since it is an idempotent monoid). For any class $[f]$, we can consider the localisation $M_{[f]}$ as above. This way, we obtain a monoid coposet called the *affine monoid coposet* corresponding to the monoid M. This defines a covariant functor

$$\mathbf{L} : \mathrm{Mn} \to \mathrm{MnCop}.$$

We have $\mathbf{L}(M) = (M^{sl}, [f] \mapsto M_{[f]})$.

An *affine monoid scheme* is defined to be a monoidal space, isomorphic to $(\mathrm{Spec}(M), \mathcal{O}_M)$. The assignment $M \mapsto (\mathrm{Spec}(M), \mathcal{O}_M)$ yields a contravariant embedding of the category of commutative monoids into the category of monoid schemes. We have analogous constructions for monoid posets and monoid coposets.

4.1.2.2 The case of finitely generated monoids

For a finitely generated monoid M, the space $\mathrm{Spec}(M)$ is finite and hence, it is a $\mathfrak{P}$-topological space. We can use Lemma 4.1.3 to see that the first two approaches to affine schemes, given in the previous section, are essentially the same. This also explains why we used the same notation.

There is a third method, which is also equivalent to previous ones. For this, let $L_f \subseteq M$ be constructed as in Definition (4.6), as the complement of the union of all prime ideals not containing f (which itself is a prime ideal). This prime ideal is the image of $[f] \in M^{\mathsf{sl}}$, under the injective map $\alpha_{\mathsf{sl}} : M^{\mathsf{sl}} \to \mathrm{Spec}(M)$. As such, we have

$$M_f = M_{\alpha_{\mathsf{sl}}([f])}. \tag{4.8}$$

To summarise Equations (4.5) and (4.8): For a finitely generated monoid M, every principal open subset has a unique biggest prime ideal (the union of all prime ideals contained inside). The section of the structure sheaf at said open subset agrees with the stalk of the structure sheaf at the associated prime ideal. Or to put it more direct terms, the stalks fully determine the structure sheaf.

4.1.3 Monoid Schemes

Definition 4.1.7. *A* monoid scheme *is a pair* $(X, \mathcal{O}_X)$*, where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of monoids on* X*, locally isomorphic to* $(\mathsf{Spec}(M), \mathcal{O}_{\mathsf{Spec}(M)})$*, for a monoid* M*.*

See among others [18], [19]. Monoid schemes form a full subcategory of monoidal spaces. We will write X instead of $(X, \mathcal{O}_X)$ when there is no ambiguity. Accordingly, $\mathsf{Spec}(M)$ will also be used synonymously with the affine monoid scheme corresponding to a monoid M.

A *morphism of monoid schemes* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is merely a morphism of monoidal spaces. That is, a morphism of sheaves (of monoids) that is additionally *local*. One has a bijection

$$\mathsf{Hom}(M, N) \cong \mathsf{Hom}_{\mathsf{MnSchm}}(\mathsf{Spec}(N), \mathsf{Spec}(M)),$$

where MnSchm denotes the category of monoid schemes.

Definition 4.1.8. *A monoid poset* $(P, \mathcal{O})$ *is called* monoid poset scheme *if for any* $p \in P$*, the pair* $(P^p, \mathcal{O})$ *is isomorphic to the monoid poset* $\mathsf{Spec}(\mathcal{O}(p))$*.*

Here, $\mathcal{O}$ denotes the structure functor on P, as well its restriction to

$$P^p = \{x \in P | x \leq p\}.$$

Quite similarly, a monoid coposet $(L, \mathcal{O})$ is called *monoid coposet scheme* if for any $a \in L$, the pair $(P_p, \mathcal{O})$ is isomorphic to the monoid coposet $\mathbf{L}(\mathcal{O}(p))$.

A monoid scheme is of *finite type* if it admits a finite open cover by affine monoid schemes associated to finitely generated monoids.

A monoid scheme poset $(P, \mathcal{O})$ is of *finite type* if P is finite and for each $p \in P$, the monoid $\mathcal{O}(p)$ is finitely generated. Similarly for monoid scheme coposets.

Proposition 4.1.9. *If we restrict the categories*

$$\mathsf{MnSchm}, \quad \mathsf{MnPosSchm} \quad \text{and} \quad \mathsf{MnCopSchm}^{op}$$

to the finite type objects, they becomoe equivalent categories.

The fact that the first two categoreis are equivalence is well-known [18, Proposition 2.9]. It is a direct consequence of the equivalence of sheaves and poset functors (see Lemma 4.1.3) for finite spaces. In fact, we have the following lemma:

Lemma 4.1.10. *Let* $(X, \mathcal{O}_X)$ *be a monoid scheme of finite type. The underlying topological space* X *is a finite* $\mathfrak{P}$*-topological space and equivalent to* X_P*, where* P *is a locally finite lattice.*

Proof. Since a subset being open is a local condition, it suffices to consider the case when X is affine. This follows straight from the results of Section 4.1.2.2. $\mathcal{Q.E.D}$

To see the equivalence of the second and third categories in Proposition 4.1.2.2, we use the material from Section 4.1.2.2. More formally, let $(P, \mathcal{O}_P)$ be a monoid poset scheme of

finite type. We assign to it the pair $(L, \mathcal{O}_L)$, where L is P with the reverse order. Another way of saying that is that L is the opposite poset of P. The functor $\mathcal{O}_L$ is the same as $\mathcal{O}_P$, considered now as a covariant functor on the opposite category. This defines a functor

$$\text{MnPosSchm} \to \text{MnCopSchm}$$

on objects. We want to textend it to morphisms. Take $(P, \mathcal{O}_P) \to (P', \mathcal{O}_{P'})$ in MnPosSchm. It is a pair $(f : P \to P', f^{\#} : \mathcal{O}_{P'} \circ f \to \mathcal{O}_P)$. We must define a morphism $(L', \mathcal{O}_{L'}) \to (L, \mathcal{O}_L)$. As a morphism of posets, $L' \to L$ is the left adjoint $f^{\diamond}$ of f. As a transformation of functors, it is the composite

$$\mathcal{O}_{L'} \to \mathcal{O}_{L'} \circ f \circ f^{\diamond} \to \mathcal{O}_L \circ f^{\diamond}.$$

Here, $\mathcal{O}_{L'} \to \mathcal{O}_{L'} \circ f \circ f^{\diamond}$ is induced by the unite of the adjoint pair $(f, f^{\dagger})$. The left adjoint $f^{\diamond}$ exists since the posets involved are finite, [48]. It is given by

$$f^{\diamond}(x') = min\{y | x' \leq f(y)\}.$$

Of course, not every locally finite lattice with a functor in the category of Monoids corresponds to a monoid scheme, even if the induced morphisms are localisations. A simple counterexample is constructed by considering the poset $x \leq y$, $x \neq y$. Associate to y the monoid $\mathbb{N}^2$ and to x the monoid $\mathbb{Z}^2$, with the obvious inclusion as the homomorphism. This is clearly not a monoid scheme, as we are lacking the "in-between" localisations $\mathbb{N} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{N}$.

4.1.3.1 More on Monoid schemes

As one would expect, products exist in the category of monoid schemes and the global section respects it, meaning

$$\Gamma(X \times Y, \mathcal{O}_{X \times Y}) \cong \Gamma(X, \mathcal{O}_X) \times \Gamma(Y, \mathcal{O}_Y).$$

Recall that a monoid scheme X is of finite type it it admits a finite open cover by affine subschemes $U_i = \text{Spec}(M_i)$ such that every M_i is a finitely generated monoid. We will need a more general notion. A monoid M is called *right s-Noetherian* if M satisfies the ascending chain condition on right ideals. Equivalently, if any right ideal of M is finitely generated, see Section **??** below. A monoid scheme is called *s-Noetherian* if it can be covered by a finite number of open affine monoid subschemes $\text{Spec}(M_i)$, where each M_i is an s-Noetherian monoid.

Let $(X, \mathcal{O}_X)$ be a monoid scheme and $U \subseteq X$ an open subset. The restriction of $\mathcal{O}_X$ to U is called an *open subscheme* and is often denoted by $(X, \mathcal{O}_{X|U})$.

Lemma 4.1.11. *An open subscheme of an s-Noetherian monoid scheme X is s-Noetherian.*

Proof. We first consider the case when $X = \text{Spec}(M)$ is affine. Let U be an open subset of X. There is an ideal $\mathfrak{m}$ such that $U = X \setminus V(\mathfrak{m})$, where $V(\mathfrak{m}) = \{\mathfrak{p} \in \text{Spec}(M) \mid \mathfrak{m} \subseteq \mathfrak{p}\}$. Since $\mathfrak{m}$ is finitely generated, there are $f_1, \cdots, f_k \in M$ such that $U = D(f_1) \cup \cdots \cup D(f_k)$. We have $D(f_i) = M_{f_i}$ for all $i \in \{1, \cdots, k\}$, where the M_{f_i}-s are s-Noetherian, by Corollary 7.2.3. This makes U an s-Noetherian monoid scheme.

We have $X = U_1 \cup \cdots \cup U_k$ in the general case, where the U_i-s are open and affine. Let U be an open subscheme of X. We have $U = \cup_{i=1}^{k}(U \cap U_i)$. Each $U \cap U_i$ is an open monoid subscheme of an affine s-Noetherian monoid scheme. As shown above, they can be expressed as a finite union of open and affine s-Noetherian monoid subschemes. Hence, U itself can be expressed in such a way, implying the result. $\mathcal{Q.E.D}$

Definition 4.1.12. *A monoid scheme is said to be* topologically separated *if the intersection of any two affine subsets is again affine.*

4.2 Binoids

Very closely related to monoids are , *binoids*, which can also be referred to as pointed monoids ([16],[18]). In this theory, all monoids, homomorphisms and ideals are pointed. In other words, binoids are monoids with an absorbing element (denoted by 0 in this thesis) and binoid homomorphisms are monoid homomorphisms respecting the 0.

Despite the strong similarity between these two concepts, there are many reasons to be interested in binoids rather than just monoids. Chief amongst them is that they are more general and include many additional important classes, while still being within the scope of the same tools. Some of these classes include the Stanley-Reisner binoids, whose realisations are the Stanley-Reisner rings. Another reason in our case is because it also helps in the proof of one of our main theorems, see Section 5.2.5.

Throughout this thesis, all binoids are assumed to be commutative and finitely generated. The category of commutative binoids is denoted by Bn. It behaves very similarly to the category of commutative and finitely generated monoids, and we are able to define most anything in exactly the same way. Naturally, we are also able to construct binoid schemes much in the same manner we constructed monoid schemes with.

4.2.1 Preliminary Definitions

4.2.1.1 Between Monoids and Binoids

Definition of a Binoid A *binoid* is a monoid B (commutative as usual) with an absorbing element, called zero. That is to say, we have a chosen element $0 \in B$ such that $x0 = 0$ for all $x \in B$. A morphism of binoids $f : B \to B'$ is a monoid homomorphism such that $f(0) = 0$. The category of binoids and binoid homomorphisms is denoted by Bn.

Let B be a binoid and R a ring. The *binoid algebra* $R[B]$ is the quotient of the monoid algebra $R[B]$, by the ideal generated by $0 \in B$. In other words, we identify the two zeros; one coming from the ring R and the other from B. As an R-module, it is free with basis $b \in B$, $b \neq 1$. This construction is the left adjoint functor of the functor Rings $\to$ Bn, which forgets the additive structure of a ring.

4.2.1.2 From Monoids to Binoids

It is clear that the theory of binoids and monoids are very closely linked. If M is an usual monoid, we let M° denote the binoid obtained from M by adding a new element 0.

4.2.1.3 From Binoids to Monoids

We call a binoid *integral* if for all $x, y \in B$, $xy = 0$ implies $x = 0$ or $y = 0$. The non-zero elements $B^\bullet$ of such a binoid B form a monoid. It is clear that for any monoid M, the binoid $M^\circ := M \cup \{0\}$ is integral and any integral binoid B is of this form, for $M = B^\bullet$. Here, M° is just M where we have formally adjoint an absorbing element 0, and the multiplication is defined as $m0 = m$ for all $m \in M$.

While the categories of integral binoids and of monoids are essentially the same on objects, they are not equivalent as categories: To see this, observe that we can have non-zero elements of a binoid going to 0. For example, consider $B = \{0, 1, x\}$ with $x^2 = x$. This is clearly integral and hence $B^\bullet$ makes sense, being $B^\bullet = \{1, x\}$. Note that x is, in a way, the new zero. We readily see that $\mathsf{End}_{\mathsf{Bn}}(B)$ has 3 elements, whereas $\mathsf{End}_{\mathsf{Mn}}(B^\bullet)$ has 2. Nonetheless, they are strongly related, as we shall show in Subsection 4.2.3.

We also introduce the notation of a *separated* Binoid, which we will use later. A binoid is called so if $x = xy$, $x, y \in B$ implies that $x = 0$ or $y = 1$.

4.2.2 Grouplike Binoids

A binoid is said to be *grouplike* if every non-zero element is invertible.

Let G be a group. Then $G^\circ = G \cup \{0\}$ is a grouplike binoid. On the other hand, as grouplike binoids are clearly integral, the set of all non-zero elements of a grouplike binoid defines an abelian group. Unlike integral binoids binoids and commutative monoids, grouplike binoids and abelian groups are equivalent as categories. For this, we mention that for binoid homomorphisms (much as for monoids or rings), invertible elements must map to invertible elements. This is why the hom-sets are going to be equivalent as well.

A direct consequence of this is that free grouplike binoids are of the form $(\mathbb{Z}^{\times^\circ})^n$. We will use the notation $B^{\times^\circ}$ to mean the set of of all invertible elements of a binoid M, together with the absorbing element 0.

4.2.3 Ideals and Prime Ideals of Binoids

Ideals of binoids are defined exactly as for monoids, only that we require 0 to be in the ideal. In other words, a subset $I \subseteq B$ of a binoid B is called an *ideal* if $0 \in I$ and $IB \subseteq I$. An ideal being *prime* is defined exactly as for monoids.

The set of all prime ideals is denoted by $\mathsf{Spec}(B)$. It is a poset with respect to inclusion and is empty if and only if $0 = 1$ in the binoid. In all other cases, it has a unique maximal ideal $\mathfrak{max} = \mathfrak{max}_B$, consisting of all non-invertible elements. If B is integral, the set $\{0\}$ is prime and is clearly the smallest element of $\mathsf{Spec}(B)$. It follows that $\mathsf{Spec}(B)$ has only one element if and only if B is grouplike.

Despite the small differences, most of our results for prime ideals, obtained in Paragraph 4.1.1.3, hold here as well. For example, there is a natural bijection,

$$\mathsf{Spec}(B) \xrightarrow{\sim} \mathsf{Hom}_{\mathbf{Binoids}}(B, \{0, 1\}).$$

If B is additionally integral, we also have and

$$\mathsf{Spec}(B) \xrightarrow{\sim} \mathsf{Spec}(B^{\bullet\mathsf{sl}}).$$

See Equations (4.2) and (4.3) for comparison.

For finitely generated B, $\mathsf{Spec}(B)$ is finite. If moreover M is integral, we have $\mathsf{Spec}(B) \simeq (B^\bullet)^{\mathsf{sl}}$. Hence, Lemma 5.2.1 can be used to estimate the cardinality of $\mathsf{Spec}(B)$. For example, if B is integral and a is not invertible in B, then

$$|\mathsf{Spec}(B)| > |\mathsf{Spec}(B/(a \sim 1))|$$

and if a is not nilpotent, then

$$|\mathsf{Spec}(B)| > |\mathsf{Spec}(B/(a))|.$$

4.2.4 Binoid Schemes

Binoid Schemes are defined much like monoid schemes and most results translate directly. For example,

$$\mathsf{Spec}(B) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Bn}}(B, \{0,1\})$$

readily gives us the analogue of Lemma 4.1.10:

Lemma 4.2.1. *The underlying topological space of a binoid scheme of finite type is a loclaly finite lattice.*

As a reminder, a poset $(P, \leq)$ was called *locally finite lattice*, if for every $p \in P$, the subset $P^p := \{q \in P \mid q \leq p\}$ is a finite lattice. This is called the *underlying poset* of X.

The elements of this poset correspond to prime ideals of some binoid, with $p \leq q$ if and only if $p \subseteq q$. In the finitely generated case, we can replace prime ideals with elements from M^{sl}, see Proposition 4.1.9.

It is easy to verify that the space of monoidal prime ideals of a monoid M and the space of binoidal prime ideals of M° are essentially the same. Since the functor $M \mapsto M^\circ$ is compatible with localisations, it has a unique extension to monoid schemes $(X, \mathcal{O}_X) \mapsto (X^\circ, \mathcal{O}_{X^\circ})$. The resulting functor from the category of monoid schemes to the category of binoid schemes is faithful and preserves finite products. Because of this, the concepts and results from one can often be directly (or with small modifications) applied to the other.

A binoid scheme X is called *topologically separated* if the intersection of affine subschemes is again affine. One can see that this simply means that the connected components of the underlying topological poset of X are globally a meet-semilattices with a maximal element [18, Corollary 3.7]. We remark that there is a small clash of terminology: The authors in the above paper work with monoids with an absorbing element. However, they do not require for a homomorphism to preserve it. Furthermore, their use of the word separated is not to be confused with our notion of a separated binoid. It also differs from our notion of topologically

separatedness, though their notion of a separated binoid scheme implies that it is topologically separated.

Note that the underlying meet semi-lattice of X need not have a unit in general, even if it is connected. Indeed, it will have a unit if and only if it is affine. Note also that the unit of the meet-semilattice is the same thing as the zero or absorbing element of the join semi-lattice.

A sheaf over a binoid scheme X of finite type simply becomes a functor from the underlying poset P_X of X. Note that the functor can be assumed to be either covariant or contravariant, depending on weather we use the isomorphism

$$\mathsf{Spec}(M) \simeq M^{\mathsf{sl}}.$$

We remark that the term "local" will be different depending on our choice, as the topology will either be given by subsets of the form P^x (for $\mathsf{Spec}(M)$) or P_x (for M^{sl}).

Henceforth, unless otherwise stated, a binoid scheme X is assumed to be of finite type and topologically separated.

Chapter 5

Fundamental Groupoid of Binoid Varieties

5.1 Introduction

Let M be a binoid. As a reminder, this is a monoid with an absorbing element. We can consider the set of binoid homomorphisms $\mathsf{Hom}_{\mathsf{Bn}}(M, K)$, where K is any commutative ring, and hence, a natural binoid after forgetting the addition. This set of homomorphisms is clearly a subset of K^n, where n is the number of generators of M. If $M_f \simeq N_g$, then $\mathsf{Hom}_{\mathsf{Bn}}(M_f, K) \simeq \mathsf{Hom}_{\mathsf{Bn}}(N_g, K)$ and hence, this construction respects gluing.

We will focus on the cases $\mathsf{K} = \mathbb{R}$ and $\mathbb{C}$: The real and complex numbers. Our aim is to calculate $\pi_1(\mathsf{K}X, x)$, $x \in \mathsf{K}X$, where the topology of $\mathsf{K}X$ is induced by the topology of K^n. We will actually calculate the more general $\Pi_1(\mathsf{K}X)$, where Π_1 denotes the fundamental groupoid. It can be shown readily that this is equivalent to taking the fundamental group at every connected component simultaneously. As such, it carries the information of π_0 and of π_1 at every connected component.

One of the main advantages of the fundamental groupoid is that the Seifert-van Kampen theorem no longer requires a common point [11], [50], [52]. This allows us to use the affine covering of a binoid scheme X to calculate the fundamental group(oid) of $\mathsf{K}X$.

As such, our focus will become the affine case $\mathsf{KSpec}(M)$. One of our main results of this chapter, see Section 4.2.2, is the reduction of the computation of the homotopy of general binoids to grouplike binoids. For the definition of the latter, see 5.2.7.

This is proven in several steps. We first prove it in the special case when M is integral and separated. Then, removeing these conditions, one after the other, using induction arguments primarily on the number of prime ideals, we will prove the general case.

However, we can simplify things further when $\mathsf{K} = \mathbb{R}$. Theorems 1.3.15 and 3.2.6 allow us to reduce the calculation of the fundamental group(oid) of a topological space X to simple combinatorics, if X admits a discrete covering which we understand in full. As a reminder, by discrete, we mean a covering $\{U_i\}$ of X, where every $\Pi_1(U_i)$, every $\Pi_1(U_{ij})$ and every $\Pi_1(U_{ijk})$ is a discrete groupoid. That is, a groupoid where the automorphism group of every

point is trivial. Equivalently, $\{U_i\}$ is a discrete covering if the fundamental group of every U_i, U_{ij} and U_{ijk} is trivial at every point. Note that, π_0 of these spaces need not be trivial.

As it turns out, that $\Pi_1(\mathbb{R}\mathrm{Spec}(M))$ is a discrete groupoid for every binoid M, by Corollary 5.4.1. We are also able to understand the functors between the fundamental groupoids, induced by the gluing homomorphisms, see Subsection 5.4.1. Hence, every quasi-separable binoid scheme has a natural discrete covering. This enables us to give a method for calculating the fundamental group(oid) of real binoid schemes.

This calculation is especially clear in the case of Stanley-Reisner rings, or more preceisely, Stanley-Reisner binoids, whose realisations are the Stanely-Reisner rings. The aim of our calculations is to give a demonstration of our proposed methods in a relatively simpler case.

Ancillary, this also enables us to simplify the calculation of the homology of $\mathbb{R}X$ in Subsection 5.5.2.2.

Our results are closely connected to the works by V. Uma [72], where the author focuses on the fundamental group of real toric varieties, coming from smooth fans. However, even in that case, our methods are completely different, and the general idea of calculating the fundamental groupoid from a discrete covering could have independent uses.

5.2 The k-Spectrum of finitely generated Binoids

5.2.1 Special Quotients

We will need some special quotients of binoids and monoids. For an ideal I of a binoid (or monoid) M, denote by M/I the *Rees quotient* (also called the *Rees factorisation*), which is defined as

$$M/I := (M \setminus I) \cup \{0\}$$

as a set. The multiplication in M/I is defined in such a way that $q : M \to M/I$ is a homomorphism. Indeed, it is universal with respect to this property. Here, $q(x) = x$ if $x \notin I$ and $q(x) = 0$, if $x \in I$.

We mention two special cases of this factorisation: i) we can take $I = \mathsf{Nil}(M)$ to be the set of all nilpotent elements and ii) we can take I to be the principal ideal generated by an element $a \in M$, meaning $I = aM = (a)$. In the first case, the corresponding quotient is denoted by

$$M_{\mathsf{red}} = M/\mathsf{Nil}(M),$$

while in the second one, we use the notation $M/(a \sim 0) = M/(a)$.

For a submonoid $S \subseteq M$ of M, we can consider the quotient $M/(S \sim 1)$. Two elements $x, y \in M$ are equated under this congruence if and only if there are $s, t \in S$ such that $xs = yt$. This is the smallest congruence where $S \sim 1$, hence our notation.

In particular, for an element a, we can consider the submonoid $\langle a \rangle$ generated by a. In other words, we consider the quotient of M by the congruence which equates x and y in M if and only if $xa^m = ya^n$ for some $m, n \geq 0$. We denote this by $M/(a \sim 1)$ for obvious reasons.

We summarise some properties of these quotients which we will need later. To state them, we introduce the notation $\mathsf{Idem}(M)$ to denote the set of idempotents of a binoid (or monoid)

M.

Lemma 5.2.1. *Let M be a monoid and $a \in M$.*

i) If $f : M \to M_1$ is a surjective homomorphism of monoids, the induced map $f^{sl} : M^{sl} \to M_1^{sl}$ is surjective.

ii) The homomorphism $M^{sl} \to (M/(a))^{sl}$, induced by the natural map $q_0 : M \to M/(a)$, is an isomorphism if and only if a is nilpotent.

iii) The homomorphism $M^{sl} \to (M/(a \sim 1))^{sl}$, induced by the natural map $q_1 : M \to M/(a \sim 1)$, is an isomorphism if and only if a is invertible.

Proof. i) Obvious since M^{sl} is a quotient of M.

ii) Say a is nilpotent. That is, $a^n = 0$ for some n. It follows that $a \sim_{sl} 0$, implying that q_0 is injective. Bijectivity then follows from i).

On the other hand, if q_0^{sl} is injective, it follows that $a \sim_{sl} 0$ in $(M/(a))^{sl}$ since $\alpha_{sl}(a)$ and $\alpha_{sl}(0) = 0$ have the same images in $(M/(a))^{sl}$. Thus $\alpha_{sl}(a) = 0$, showing that a is nilpotent.

iii) The quotient $m \sim m^2$ equates the subgroup of invertible elements to 1. (This can be readily checked by the construction of the quotient $\sim_{sl}$). As such, $M^{sl} \to (M/(a \sim 1))^{sl}$ is injective if a is invertible.

Conversely, assume $M^{sl} \to (M/(a \sim 1))^{sl}$ is injective. The elements a and 1 have the same image in $(M/(a \sim 1))^{sl}$ since they already are the same in $M/(a \sim 1)$. It follows that $\alpha_{sl}(a)$ and $\alpha_{sl}(1)$ have the same image in M^{sl}. By the definition of the congruence $\sim_{sl}$, a is invertible. $\mathcal{Q.E.D}$

Lemma 5.2.2. *Let M be a binoid.*

i) The natural homomorphism $q_0 : M \to M/(a)$ yields a surjective map $\mathrm{Idem}(M) \to \mathrm{Idem}(M/(a))$ for any element $a \in M$. Moreover, the preimage of any non-zero idempotent of $\mathrm{Idem}(M/(a))$ in $\mathrm{Idem}(M)$ is a singleton.

ii) Let $e \in \mathrm{Idem}(M)$ be an idempotent element. The natural homomorphism $q_1 : M \to M/(e \sim 1)$ yields a surjective map $\mathrm{Idem}(M) \to \mathrm{Idem}(M/(e \sim 1))$.

iii) Assume $a \in M$ is invertible. The natural homomorphism $q_1 : M \to M/(a \sim 1)$ yields a surjective map $M^{\times^\circ} \to (M/(a \sim 1))^{\times^\circ}$.

iv) One has $(M_{\mathrm{red}})_{\mathrm{red}} = M_{\mathrm{red}}$. Moreover, the canonical map $M \to M_{\mathrm{red}}$ is bijective on the unit group.

Proof. i) Let $z \in \mathrm{Idem}(M/(a))$. Since q_0 is a surjection, we may assume that $z = q_0(x) \in \mathrm{Idem}(M/(a))$. If $x \in aM$, z is the image of the idempotent 0. If $z \neq 0$, this $x \in \mathrm{Idem}(M)$ is the only idempotent of M, with $z = q_0(x)$, as q_0 restricted to $M \setminus (a)$ is bijective.

ii) Take an element $y \in M$ such that $q_1(y)$ is an idempotent in $M/(e \sim 1)$. Then $ye^m = y^2e^n$ for some $m, n \geq 0$. We have to find an idempotent $u \in M$, with $q_1(u) = q_1(y)$. Since e is an idempotent, we can assume that $0 \leq m, n \leq 1$. There are four cases to consider. If $(m, n) = (0, 0)$, take $u = y$. In all other cases, take $u = ye$.

iii) Let $x \in M$ with $q_1(x) \in (M/(a \sim 1))^{\times^\circ}$. We may assume this since q_1 is surjective. There exists $y \in M$ and $m, n \geq 0$ such that $xya^m = a^n$ since $q_1(x)$ is invertible. It follows that $x \in M^{\times^\circ}$ since a is invertible.

iv) Denote by q the canonical surjection $M \to M_{\mathsf{red}}$. Assume $q(x)$ is nilpotent in M_{red}. There exists $n \in \mathbb{N}$ such that $q(x)^n = 0$. That is, $q(x^n) = q(0)$ and hence, x^n is nilpotent in M. As such, x is also nilpotent and $q(x) = 0$. The last assertion is obvious. $\mathcal{Q.E.D}$

5.2.2 Pivotal elements

Recall that a binoid is called separated if $x = xy$ implies $x = 0$ or $y = 1$. It is called integral if $xy = 0$ implies $x = 0$ or $y = 0$. We use the notation $M^\bullet$ for the monoid of non-zero elements of an integral binoid.

For an element $x \in M$, we set

$$M^x := \{y \in M \mid xy = x\}.$$

It is clear that M^x is a submonoid of M. Call x *pivotal* if $x \neq 0$ (meaning $M^x \neq M$) and $M^x \neq \{1\}$. So, M is separated if and only if M has no pivotal elements.

The set of all pivotal elements is denoted by $\mathsf{Piv}(M)$. We also set

$$\mathsf{P}(M) := \bigcup_{x \in \mathsf{Piv}(M)} M^x.$$

Lemma 5.2.3. *Let M be an integral binoid.*

i) If $x \in \mathsf{Piv}(M)$ and $z \neq 0$, then $xz \in \mathsf{Piv}(M)$. So, $\mathsf{Piv}(M)$ is an ideal of $M^\bullet$.

ii) The set $\mathsf{P}(M)$ is a subsemigroup of $M^\bullet$. Moreover, $\mathsf{P}(M)$ is a submonoid if and only if M is not separated.

iii) Let a be an invertible element of M and $M' = M/(a \sim 1)$. For any element $x' \in \mathsf{Piv}(M')$, there exist $x \in \mathsf{Piv}(M)$ such that $q_1(x) = x'$ and for any $y' \in \mathsf{P}(M')$, there exists $y \in \mathsf{P}(M)$ such that $q_1(y) = y'$. Here, $q_1 : M \to M' = M/(a \sim 1)$ is the canonical projection.

Proof. i) If $1 \neq y \in M^x$, then $y \in M^{xz}$. Hence, $xz \in \mathsf{Piv}(M)$.

ii) If $y_1 \in M^{x_1}$ and $y_2 \in M^{x_2}$, then $y_1 y_2 \in M^{x_1 x_2}$. For the last assertion, observe that $1 \in \mathsf{P}(M)$ if and only if $\mathsf{Piv}(M) \neq \varnothing$.

iii) Assume $x'y' = x'$ holds in M' and $x' \neq 0$, $y' \neq 1$. Choose $x, z \in M$ such that $q_1(x) = x', q(z) = y'$. Then, $xza^m = xa^n$ for some $m, n \geq 0$. We obtain $xy = x$ since a is invertible, where $y = za^{m-n}$ and $q(y) = q(z) = y'$. $\mathcal{Q.E.D}$

5.2.3 The space $\mathsf{KSpec}(M)$

For a binoid M and a commutative ring K, we denote $\mathsf{KSpec}(M) := \mathsf{Hom}_{\mathsf{Bn}}(M, \mathsf{K})$ and call it the K-*spectrum* of M.

Throughout this chapter, we will assume that M is finitely generated. Let $\{e_1, \dots, e_n\}$ be set of generators. Every morphism of binoids is defined by the values on its generators, which gives rise to the following embedding of $\mathsf{KSpec}(M)$ in K^n:

$$\mathsf{KSpec}(M) \ni f \mapsto (f(e_1), \dots, f(e_n)) \in \mathsf{K}^n.$$

Thus, it can be thought of as the 'set of zeros' of the associated binoid ring $\mathsf{K}[M]$. For $\mathsf{K} = \mathbb{R}, \mathbb{C}$, the binoid $\mathsf{KSpec}(M)$ inherits a subspace topology, which is independent of our choice of generators. In what follows, $\mathsf{KSpec}(M)$ is considered with the subspace topology of $\mathbb{R}^n$ and $\mathbb{C}^n$.

Lemma 5.2.4. *Let M be a binoid and* K *a commutative ring.*

i) One has

$$\mathsf{KSpec}(M) = \mathsf{K\text{-}Spec}(\mathsf{K}[M]),$$

where $\mathsf{K\text{-}Spec}(\mathsf{K}[M])$ *is the set of the classical* K*-valued points* $\mathsf{Hom}_{\mathsf{K\text{-}Alg}}(\mathsf{K}[M], \mathsf{K})$.

ii) Let K *be an integral domain* K*. Then*

$$\mathsf{KSpec}(M) = \mathsf{KSpec}(M_{\mathsf{red}}).$$

iii) Let M be integral. Then $\mathsf{KSpec}(M)$ *is a topological monoid.*

Proof. i) This is clear since $M \to \mathsf{K}$ gives rise to a K-algebra homomorphism $\mathsf{K}[M] \to \mathsf{K}$.

ii) Observe that any binoid homomorphism takes nilpotent elements to nilpotent ones. Since K has no nonzero nilpotent element, any binoid homomorphism $M \to \mathsf{K}$ factors through M_{red}, and the result follows.

iii) Since M is integral, one has

$$\mathsf{KSpec}(M) = \mathsf{Hom}_{\mathsf{Mn}}(M^\bullet, \mathsf{K}).$$

The result follows from the fact that the set of all (commutative) monoid homomorphisms has a natural monoid structure, which is compatible with the topology. $\mathcal{Q.E.D}$

Remark 5.2.5. *Recall that a space X is* n-simple*, if $\pi_1(X, x)$ is abelian and acts trivially on $\pi_n(X, x)$, $n \geq 1$. The above lemma shows that* $\mathsf{KSpec}(M)$ *is, in particular, an H-space for an integral binoid M. Hence, it is n-simple for all $n \in \mathbb{N}$.*

This is, however, no longer true in the non-integral case, as the set of binoid homomorphisms need not have a unit.

5.2.4 Decomposition of a binoid by its idempotents

Let M be a commutative binoid. To simplify notation, we write $\mathsf{Adm}(M)$ instead of $\mathsf{Spec}(\mathsf{Idem}(M))$. For an element $\mathfrak{r} \in \mathsf{Adm}(M)$, we set $\mathfrak{r}^c = \mathsf{Idem}(M) \setminus \mathfrak{r}$. Thus, for such $\mathfrak{r}$, one obtains a decomposition

$$\mathsf{Idem}(M) = \mathfrak{r} \sqcup \mathfrak{r}^c.$$

Clearly $0 \in \mathfrak{r}$ and $1 \in \mathfrak{r}^c$. Let $e \in \mathfrak{r}$ and $m \in M$. Assume $e' := me$ is an idempotent. From $ee' = eme = me = e'$ and the fact that $\mathfrak{r}$ is an ideal, we see that $e' \in \mathfrak{r}$ as well. Hence, the only idempotents of the ideal $\mathfrak{r}M \subseteq M$ are elements of $\mathfrak{r}$. Though $\mathfrak{r}$ is a prime ideal of $\mathsf{Idem}(M)$, $\mathfrak{r}M$ need not be a prime ideal.

We let $M(\mathfrak{r})$ be the quotient of M by the relations $\mathfrak{r}M \sim 0$ and $\mathfrak{r}^c \sim 1$. Thus, we can write

$$M(\mathfrak{r}) = M/(\mathfrak{r}M,\ \mathfrak{r}^c \sim 1).$$

The same monoid can be described as a push-out diagram in the category of binoids:

$$\begin{array}{ccc} \mathsf{Idem}(M) & \longrightarrow & M \\ \downarrow{\scriptstyle \chi_{\mathfrak{r}}} & & \vdots \\ \{0,1\} & \dashrightarrow & M(\mathfrak{r}). \end{array}$$

Lemma 5.2.6. *Let M be a finitely generated binoid. One has*

$$\mathsf{KSpec}(M) = \coprod_{\mathfrak{r} \in \mathsf{Adm}(M)} \mathsf{KSpec}(M(\mathfrak{r}))$$

for any integral domain K*, where the binoids $M(\mathfrak{r})$ have only trivial idempotents.*

Proof. Take $\phi : M(\mathfrak{r}) \to \mathsf{K}$, where $\mathfrak{r} \in \mathsf{Adm}(M)$ and pre-compose it with the canonical homomorphism $M \to M(\mathfrak{r})$ to obtain an element in $\mathsf{KSpec}(M)$. Conversely, take an element $\psi : M \to \mathsf{K}$ from the LHS. It sends any idempotent to either 0 or 1. Denote by $\mathfrak{r}$ the set of idempotents $e \in \mathsf{Idem}(M)$, for which $\psi(e) = 0$. Then $\mathfrak{r} \in \mathsf{Adm}(M)$ and ψ factors through $M(r)$. The claim follows.

For the last assertion, observe that one can use Lemma 5.2.2 step by step to show that any idempotent of $M(\mathfrak{r})$ has a lifting to an idempotent of M. The claim follows as any idempotent of M maps to either 0 or 1 in $M(\mathfrak{r})$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

We now consider the functorial behaviour of this decomposition. Let $f : M \to M'$ be a binoid homomorphism. Since Idem (resp. Spec) is a covariant (resp. contravariant) functor, we see that $\mathsf{Adm}(-) := \mathsf{Spec}(\mathsf{Idem}(-))$ is a contravariant functor. Thus, for any $\mathfrak{r}' \in \mathsf{Adm}(M')$, we have

$$Af^{\triangleleft}(\mathfrak{r}') = \{e \in \mathsf{Idem}(M) \mid f(e) \in \mathfrak{r}'\} \in \mathsf{Adm}(M),$$

and there exists a unique homomorphism of binoids $f_{\mathfrak{r}} : M(f^{\triangleleft}(\mathfrak{r}')) \to M'(\mathfrak{r}'))$ for which the diagram

$$\begin{array}{ccc} M & \longrightarrow & M(f^{\triangleleft}(\mathfrak{r}')) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{\mathfrak{r}}} \\ M' & \longrightarrow & M'(\mathfrak{r}') \end{array}$$

commutes. Here, the horizontal arrows are the canonical maps to the quotient. To see this, observe that $M'(\mathfrak{r}') = M'/\mathcal{K}'$, where $\mathcal{K}'$ is the congruence given by $\mathfrak{r} \sim 0$ and $\mathfrak{r}^c \sim 1$. On the other hand, $M(f^{\triangleleft}(\mathfrak{r}')) = M/\mathcal{K}$, where $\mathcal{K}$ is given by $\{e \in \mathsf{Idem}(M) \mid f(e) \in \mathfrak{r}\} \sim 0$ and

$\{e \in \mathsf{Idem}(M) \mid e \in \mathfrak{r}^{\mathsf{c}}\} \sim 1$. The induced diagram

$$\begin{array}{ccc} \mathsf{KSpec}(M') & \xrightarrow{=} & \coprod_{\mathfrak{r}' \in \mathsf{Adm}(M')} \mathsf{KSpec}(M'(\mathfrak{r}')) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*_{\mathfrak{r}}} \\ \mathsf{KSpec}(M) & \xrightarrow{=} & \coprod_{\mathfrak{r} \in \mathsf{Adm}(M)} \mathsf{KSpec}(M(\mathfrak{r})) \end{array}$$

also commutes, where the right vertical morphism maps the component corresponding to $\mathfrak{r}'$ to the component corresponding to $f^{\triangleleft}(\mathfrak{r}')$.

5.2.5 Comparison maps and the main theorem

Recall that $M^{\times^\circ}$ denotes the set of all invertible elements of a binoid M, together with the absorbing element 0. We have the following morphisms in the category Bn of binoids:

$$M^{\times^\circ} \xrightarrow{i} M \xrightarrow{j} M^{\times^\circ}.$$

Here, i is the natural inclusion, while j is the canonical projection onto the quotient $M^{\times^\circ} \simeq M/\mathfrak{m}$. Recall that $\mathfrak{m}$ denotes the unique maximal ideal of M, which consists exactly of the non-invertible elements of M. Concretely, the homomorphism j is given by

$$j(m) = \begin{cases} m, & \text{if } m \in M^{\times^\circ} \\ 0, & \text{if } m \notin M^{\times^\circ}. \end{cases}$$

Since $j \circ i = \mathsf{Id}_{M^{\times^\circ}}$, we have $i^* \circ j^* = \mathsf{Id}_{\mathsf{KSpec}(M^{\times^\circ})}$ for the induced continuous maps

$$\mathsf{KSpec}(M^{\times^\circ}) \xleftarrow{i^*} \mathsf{KSpec}(M) \xleftarrow{j^*} \mathsf{KSpec}(M^{\times^\circ}).$$

This enables us to identify the space $\mathsf{KSpec}(M^{\times^\circ})$ as a subspace of $\mathsf{KSpec}(M)$ via j^*. A point $p : M \to \mathsf{K}$ of the space $\mathsf{KSpec}(M)$ belongs to $\mathsf{KSpec}(M^{\times^\circ})$ under this identification if and only if $p(m) = 0$ for any $m \notin M^{\times^\circ}$.

We can now state the main result of this section.

Theorem 5.2.7. *Let $\mathsf{K} = \mathbb{R}, \mathbb{C}$ and M be a finitely generated binoid. There exists a homotopy equivalence*

$$\mathsf{KSpec}(M) \simeq \coprod_{\mathfrak{r} \in \mathsf{Adm}(M)} \mathsf{KSpec}((M(\mathfrak{r})^{\times^\circ}),$$

which is compatible with the decomposition given in Lemma 5.2.6, and which is induced by $j^*_{\mathfrak{r}} : \mathsf{KSpec}((M(\mathfrak{r}))^{\times^\circ}) \to \mathsf{KSpec}(M(\mathfrak{r}))$ on each summand.

The proof is based on several reductions. We will first check it under the assumption that M is integral and separated in Section 5.2.6. We will then use an induction argument on the number of prime ideals to prove the Theorem in its full generality.

The latter part itself will be done in two independent steps. In the first one, we will get rid of the assumption that M is separated, using another induction argument. Finally, we will also get rid of the assumption that M is integral.

5.2.6 The case of separated and integral binoids

Recall that M is a assumed to be a finitely generated binoid. We will need the following result:

Lemma 5.2.8 ([8], Theorem 5.2.7)**.** *Let M be an integral and separated binoid. There exists a local monoid homomorphism*

$$\delta : M^\bullet \to \mathbb{N},$$

called the grading *of M.*

Recall that a homomorphism is local if non-invertible elements map to non-invertible elements. Hence, $\delta(x) = 0$ if and only if x is invertible in M.

Proof. This was proven in [8, Theorem 5.2.7] for the case when M has no non-trivial invertible elements. As such, all we have to do is compose it with the canonical surjective homomorphism $q : M \to M/(M^{\times^\circ} \sim 1)$. Here, $q(x) = q(y)$ if and only if there exists an invertible element $u \in M$, such that $y = ux$. It is clear that $M/(M^{\times^\circ} \sim 1)$ has no non-trivial invertible elements, reducing it to the cited theorem. $\mathcal{Q.E.D}$

The next result shows that Theorem 5.2.7 is true for integral and separated binoids.

Proposition 5.2.9. *Let* $\mathsf{K} = \mathbb{R}, \mathbb{C}$ *and M be an integral and separated binoid. The topological space* $\mathsf{KSpec}(M^{\times^\circ})$ *is a deformation retract of* $\mathsf{KSpec}(M)$*.*

Proof. We have $\mathsf{KSpec}(M) = \mathsf{Hom}_{\mathsf{Mn}}(M^\bullet, \mathsf{K})$ since M is integral. Consider the monoid homomorphism

$$M^\bullet \to M^\bullet \times \mathbb{N},$$

given by $m \mapsto (m, \delta(m))$, where δ is the grading given in Lemma 5.2.8. As $\mathsf{Hom}(-, K)$ takes products to products, a K-point of $M^\bullet \times \mathbb{N}$ is the same as a K-point p of $M^\bullet$ and a K-point of $\mathbb{N}$. The latter corresponds to an element t of K.

More explicitly, a K-point $(p, t) : M^\bullet \times \mathbb{N} \to \mathsf{K}$ sends (m, a), to $t^a p(m)$, where $m \in M, a \in \mathbb{N}$. Here, we assume that $0^0 = 1$. This induces the continues map

$$\mathsf{KSpec}(M) \times \mathbb{A}^1 \to \mathsf{KSpec}(M),$$

which is defined by sending (p, t) to the monoid homomorphism

$$F(p, t) : M^\bullet \to \mathsf{K},$$

given by

$$F(p, t)(m) = t^{\delta(m)} p(m).$$

We obtain the map

$$F : \mathsf{KSpec}(M) \times [0, 1] \to \mathsf{KSpec}(M).$$

Clearly, $F(p,1) = p$ and hence, $F(-,1) = \mathrm{Id}_{\mathsf{KSpec}(M)}$. On the other hand, if $t = 0$, we have

$$F(p,0)(m) = \begin{cases} 0, & \text{if } \delta(m) \neq 0 \\ p(m), & \text{if } \delta(m) = 0. \end{cases}$$

But, $\delta(m) = 0$ if and only if m is invertible. We rewrite the above as

$$F(p,0)(m) = \begin{cases} 0, & \text{if } m \text{ is not invertible} \\ p(m), & \text{if } m \text{ is invertible.} \end{cases}$$

From this, we can follow that $F(p,0) \in \mathsf{KSpec}(M^{\times^\circ})$. This is due to the fact that if m is a unit, so is $p(m)$. Likewise, if $p \in \mathsf{KSpec}(M^{\times^\circ})$, we have $p(m) = 0$ for all non-invertible m. Thus, $F(-,0) = j^* i^*$. This finishes the proof. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Remark 5.2.10. *It follows from the proof that the map $j^* : \mathsf{KSpec}(M^{\times^\circ}) \to \mathsf{KSpec}(M)$ is a homotopy equivalence. Since $i^* \circ j^* = \mathrm{Id}_{\mathsf{KSpec}(M^{\times^\circ})}$, we see that $i^* : \mathsf{KSpec}(M) \to \mathsf{KSpec}(M^{\times^\circ})$ is also a homotopy equivalence.*

5.2.7 Proof of Theorem 5.2.7

We will use induction on $\ell = |\mathsf{Spec}(M)|$, the number of prime ideals of our binoid M. The base step $\ell = 1$ holds, because in this case, the maximal ideal consists of nilpotent elements only. Hence, the reduction M_{red} of M is grouplike, see Section 4.2.3.

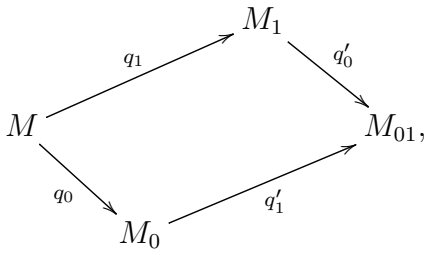
Assume that Theorem 5.2.7 is true for all binoids M, for which $|\mathsf{Spec}(M)| < \ell$. We may assume that 0 is the only nilpotent element of M, and that it only has trivial idempotents, using Lemmas 5.2.4 and 5.2.6.

5.2.7.1 Separatedness

We want to get rid of the assumption that M is separated, but we need to introduce some notations to proceed further. We showed that Theorem 5.2.7 is true if M is separated and integral in Proposition 5.2.9. As such, we can assume that $\mathsf{Piv}(M) \neq \varnothing$. Take any $x \in \mathsf{Piv}(M)$ and $y \neq 1 \in M^x$. Hence, $xy = x$ and $x \neq 0, y \neq 1$. In particular, x is not invertible. Since M has only trivial idempotents, $y \neq x$. Set

$$M_0 := M/(x), \quad M_1 := M/(y \sim 1), \quad M_{01} := M_0/(y \sim 1) \simeq M_1/(x).$$

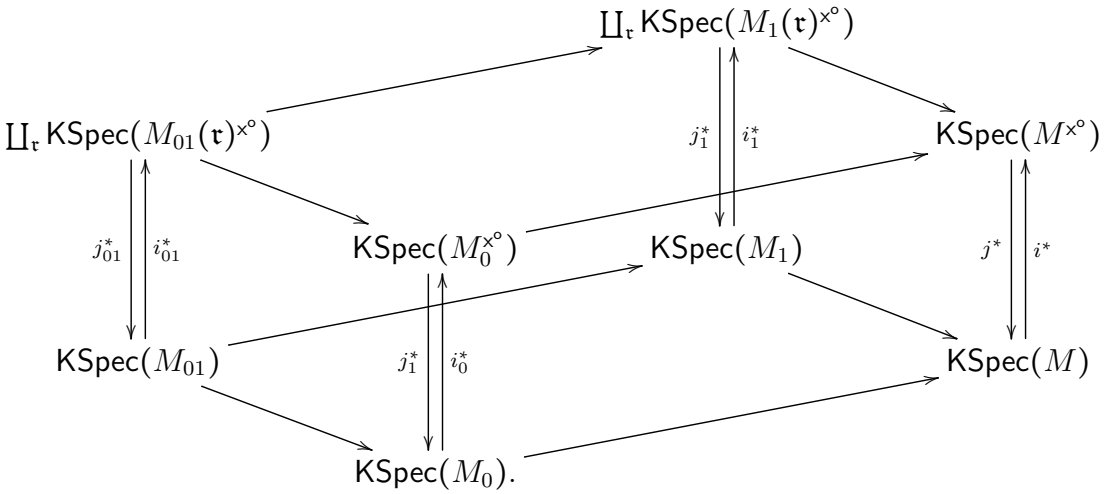
We have the following commutative diagram of binoids



where the arrows are the natural projections onto the quotients. It follows from part i) of Lemma 5.2.2, that M_0 has no non-trivial idempotents in this case.

We claim that the induced map $\mathsf{Idem}(M_1) \to \mathsf{Idem}(M_{01})$ is a bijection. It is surjective by part 1 of Lemma 5.2.2 and all the fibres are singletons, except maybe the preimage of zero. Take an idempotent $q_1(a) \in M_1$ which maps to zero in M_{01}. Then $q_1(a) = q_1(x)q_1(b)$ in M_1, for some $b \in M$. It follows that there are $n, m \geq 0$ such that $x^2b^2y^m = xby^n$. We have $xy^k = x$ for all k as $xy = x$ and so $x^2b^2 = xb$, making xb an idempotent of M. Our assumption implies that either $xb = 0$, or $xb = 1$. The last equality is impossible, because x is not invertible. Hence, $xb = 0$ and so $q_1(a) = 0$, yielding the claim.

Since the assignments $M \mapsto \mathsf{KSpec}(M)$ and $M \mapsto \coprod_{\mathfrak{r}} \mathsf{KSpec}(M(\mathfrak{r})^{\times^\circ})$ are (contravariantly) functorial, one obtains the following commutative diagram:



Here, $\mathfrak{r}$ is running through the set of admissible decompositions of $\mathsf{Idem}(M_1) = \mathsf{Idem}(M_{01})$. One has $|\mathsf{Spec}(M_0)| < \ell$, according to Lemma 5.2.1 and Section 4.2.3. Here $\ell = |\mathsf{Spec}(M)|$. It follows that $|\mathsf{Spec}(M_{01})| \leq |\mathsf{Spec}(M_0)| < \ell$. By the induction assumption, i_0^* and i_{01}^* are homotopy equivalences, with j_0^* and j_{01}^* as their respective homotopy inverses.

Since the maps $M \to M_0$, $M \to M_1$, $M_0 \to M_{01}$ and $M_1 \to M_{01}$ are canonical maps to a quotient, it follows that all maps in the bottom plane are closed embeddings. Since algebraic varieties are CW-complexes, it follows that these inclusions are cofibrations and hence, the push-out diagram is, in particular, a homotopy push-out ([2, Proposition 6.2.6]). One easily

sees that

$$\mathsf{KSpec}(M) = \mathsf{KSpec}(M_0) \cup \mathsf{KSpec}(M_1), \quad \mathsf{KSpec}(M_{01}) = \mathsf{KSpec}(M_0) \cap \mathsf{KSpec}(M_1).$$

Our next observation is that the quotient map $M \to M_0 = M/(x)$ yields an isomorphism $M^{\times^\circ} \to M_0^{\times^\circ}$ and hence, $\mathsf{KSpec}(M_0^{\times^\circ}) \to \mathsf{KSpec}(M^{\times^\circ})$ is a homeomorphism. The map $\mathsf{KSpec}(M_{01}^{\times^\circ}) \to \mathsf{KSpec}(M_1^{\times^\circ})$ is also a homeomorphism, by the same reasoning. Thus, the top plane is also a homotopy push-out.

It follows from [2][Theorem 6.2.8] that the map $i^* : \mathsf{KSpec}(M) \to \mathsf{KSpec}(M^{\times^\circ})$ is a homotopy equivalence, if i_1^* is. So, we have to prove that we can choose x and y as above ($xy = x$ and $x \neq 0, y \neq 1$) for which i_1^* is a homotopy equivalence. For this, we consider the following 2 cases:

Case 1. Assume that there exist a pivotal element $x \in \mathsf{Piv}(M)$, for which M^x is not a group. We can postulate in the above consideration that y is a non-invertible element. One has $|\mathsf{Spec}(M_1)| < \ell$ by Lemma 5.2.1 iii) and so, i_1^* is a homotopy equivalence by the induction assumption. This implies that i^* is also a homotopy equivalence as already observed, proving the induction step in this case.

Case 2. It remains to consider the case, when for all $x \in \mathsf{Piv}(M)$, the monoid M^x is a group. In this case, $|\mathsf{Spec}(M_1)| = \ell$ for any choice of x and y. So we can't use the induction step to analyse i_1^*.

The key idea is to use the monoid $\mathsf{P}(M)$ (see Section 5.2.2). It is a group in this case because $\mathsf{P}(M)$ is a union of subgroups under our assumption. It is furthermore finitely generated since $\mathsf{P}(M)$ is a subgroup of the group of invertible elements of M, which itself is the group of invertible elements of a finitely generated monoid M. Denote by g_M the minimal number of group generators of $\mathsf{P}(M)$. We will use a second induction on g.

Take y to be a generator of $\mathsf{P}(M)$ (thus, $y \neq 1$) and let $x \in \mathsf{Piv}(M)$ such that $y \in M^x$. The group $\mathsf{P}(M_1)$ is a quotient of $\mathsf{P}(M)$, where $M_1 = M/(y \sim 1)$. This is by part iii) of Lemma 5.2.3. Hence, M_1 is either separated (if $g_M = 1$) or $g_{M_1} < g_M$. In both cases, i_1^* is a homotopy equivalence. This finishes the proof.

5.2.7.2 Integrality

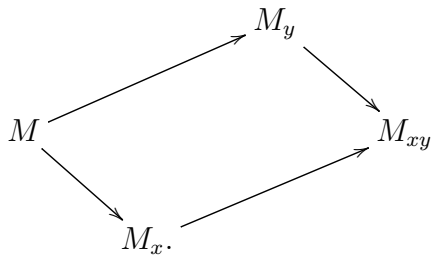
Finally, we want to remove our assumption that M is integral from Proposition 5.2.9. This is done in a similar fashion to the above argument, but is a bit simpler.

We may assume that there are elements $x, y \in M$, with $x \neq 0, y \neq 0$ and $xy = 0$. Further, $x \neq y$ since otherwise x would be nilpotent. Denote

$$M_x := M/(x), M_y := M/(y) \text{ and } M_{xy} := M/(x, y).$$

We will essentially repeat the first half of the above argument. Consider the following com-

mutative diagram



We have

$$\mathsf{KSpec}(M) = \mathsf{KSpec}(M_x) \cup \mathsf{KSpec}(M_y), \quad \mathsf{KSpec}(M_{xy}) = \mathsf{KSpec}(M_x) \cap \mathsf{KSpec}(M_y).$$

We know that M_x, M_y and M_{xy} have less than ℓ prime ideals by Section 4.2.3 and hence, we can use the induction hypothesis. Combining that with part i) of Lemma 5.2.2 shows that we have homotopy equivalences $\mathsf{KSpec}(M_x) \simeq \mathsf{KSpec}(M_x^\times)$, $\mathsf{KSpec}(M_y) \simeq \mathsf{KSpec}(M_y^\times)$ and $\mathsf{KSpec}(M_{xy}) \simeq \mathsf{KSpec}(M_{xy}^\times)$. Using the fact that the induced maps on the K-spectrum are once again cofibrations, as we did in Section 5.2.7.1, yields the proof of Theorem 5.2.7.

5.2.7.3 Examples and some Applications

The next example shows that even if M has no idempotents, they can appear when one considers the admissible decomposition:

Example 5.2.11. *Let $M = \langle x, y, z\rangle / (xy = x, yz^2 = yz)$. It is obvious that M has no idempotents. However, $M_0 = M/x = \langle y, z\rangle / yz^2 = yz$ and $M_1 = \langle x, z\rangle / z^2 = z$. We see that M_1 has z as an idempotent.*

Example 5.2.12. *Let $M = \langle x, y\rangle / x^2y = x$. It has the idempotent $(xy)^2 = xy$, and clearly no other. We have $\mathsf{Adm}(M) = \{(0), (xy)\}$. This yields*

$$\mathsf{KSpec}(M) \simeq \mathsf{KSpec}(M/xy \sim 1)) \sqcup \mathsf{KSpec}(M/x) \simeq \mathsf{KSpec}(\mathbb{Z}^\circ) \sqcup \mathsf{KSpec}(\mathbb{N}^\circ),$$

where $\mathbb{Z}^\circ$ is the free grouplike binoid with one generator and $\mathbb{N}^\circ$ if the free binoid with one generator.

As a consequence of Theorem 5.2.7, we can now prove the following result, which is closely related to [8, Corollary 4.5.8], where the author proves this result for hypersurfaces, but for characteristic zero fields.

Theorem 5.2.13. *Let $M \neq 0$ be a binoid and denote by K_M the number of connected components of the binoid algebra $\mathbb{C}[M]$. We have*

$$K_M = \sum_{\mathfrak{r} \in \mathsf{Adm}} |\mathsf{Tors}(M(\mathfrak{r})^\times)|,$$

where $|\mathsf{Tors}(M(\mathfrak{r})^\times)|$ denotes the number of torsion elements in the subgroup of invertible elements of $M(\mathfrak{r})$.

Proof. For any ring R, the non-existence of non-trivial idempotent elements is equivalent to the connectedness of $\mathsf{Spec}(R)$ in the Zariski topology. For a $\mathbb{C}$ algebra, this is equivalent to the connectedness of $\mathbb{C}\mathsf{Spec}(R)$, with its complex topology. By Theorem 5.2.7, we have

$$\mathbb{C}\mathsf{Spec}(\mathbb{C}[M]) = \mathbb{C}\mathsf{Spec}(M) \simeq \coprod_{\mathfrak{r}\in\mathsf{Adm}(M)} \mathbb{C}\mathsf{Spec}((M(\mathfrak{r})^{\times^\circ}).$$

For every $\mathfrak{r} \in \mathsf{Adm}(M)$, we have

$$\begin{aligned}\mathbb{C}\mathsf{Spec}(M(\mathfrak{r})^{\times^\circ}) &\simeq \mathbb{C}\mathsf{Spec}\left((\mathbb{Z}^\circ)^{n_\mathfrak{r}} \wedge \mathsf{Tors}(M(\mathfrak{r}))^{\times^\circ}\right)\\ &\simeq \mathbb{C}\mathsf{Spec}(\mathbb{Z}^\circ)^{n_\mathfrak{r}} \times \mathbb{C}\mathsf{Spec}(\mathsf{Tors}(M(\mathfrak{r})^{\times^\circ})).\end{aligned}$$

Clearly $\mathbb{C}\mathsf{Spec}(\mathbb{Z}^\circ)$ is connected and so are all of its powers. On the other hand, as $\mathbb{C}$ is algebraically closed, the connected components of $\mathbb{C}\mathsf{Spec}(\mathsf{Tors}(M(\mathfrak{r})^{\times^\circ}))$ agree with the elements of $\mathsf{Tors}(M(\mathfrak{r}))$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 5.2.14. *Let $M \neq 0$ be a binoid with no non-trivial idempotent elements and with a torsion free group of units. Let* K *denote a field of characteristic 0. The binoid algebra* $\mathsf{K}[M]$ *has no non-trivial idempotent elements and* $\mathsf{Spec}\mathsf{K}[M]$ *is connected.*

Proof. We can put all relevant data of K in a subfield of $\mathbb{C}$. Hence, this is a special case of Theorem 5.2.13. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

5.3 Further simplifications: Grouplike binoids

We have reduced the study of the homotopy of $\mathsf{KSpec}(M)$ for finitely generated binoids, to the study of the disjoint union of $\mathsf{KSpec}(G_i)$-s, where the G_i-s are grouplike binoids. Denote the free grouplike binoid with 1 generator by $\mathbb{Z}^\circ$ and the cyclic binoid with n+1 elements (the additional element being the absorbing element) by $C_n^\circ := (\mathbb{Z}/n\mathbb{Z})^\circ = \langle x\rangle / x^n = 1$.

For any nonzero real number r, denote by $\mathsf{sgn}(x)$ the sign of x, that is

$$\mathsf{sgn}(x) := \frac{x}{|x|}.$$

In this way, we obtain a retraction of the inclusion $C_2 = \{\pm 1\} \subseteq \mathbb{R}$.

5.3.1 On spaces

Proposition 5.3.1. *Let G be a finitely generated grouplike binoid. Then* sgn *induces a continuos map*

$$\mathbb{R}\mathsf{Spec}(G) = \mathsf{Hom}_{\mathsf{Bn}}(G, \mathbb{R}) = \mathsf{Hom}(G^\times, \mathbb{R}^\times) \to \mathsf{Hom}(G^\times, C_2)$$

which is simultaneously a group homomorphism and homotopy equivalence.

Proof. Since $\mathsf{Hom}(-, A)$ respects finite product, it suffice to consider the case when $G^\times$ is a cyclic group. If $G^\times$ is infinite, the map is exactly sgn and hence, the statement holds. Assume $G^\times$ is a finite cyclic group of order n. Then the map is an isomorphism, as both groups are trivial if n is odd and both groups are $\{\pm 1\}$ if n is even. $\mathcal{Q.E.D}$

Corollary 5.3.2. *The canonical epimorphism $G^\times \to C_2 \otimes G^\times$ induces a functorial group homomorphism*

$$\mathbb{R}\mathsf{Spec}(G) \to \mathsf{Hom}(C_2 \otimes G^\times, C_2),$$

which is a homotopy equivalence.

We can consider $C_2 \otimes G^\times$ as a vector space over $\mathbb{F}_2$. Denote by $d(G)$, or simply d, the dimension of this vector space. Since $\mathsf{Hom}(C_2 \otimes G^\times, C_2)$ is the dual of $C_2 \otimes G^\times$, we have an isomorphism

$$\mathsf{Hom}(C_2 \otimes G^\times, C_2) \cong C_2^d.$$

This isomorphism can be made explicit: Let

$$G^\times = (\mathbb{Z})^n \times C_{e_1} \times \cdots \times C_{e_l} \times C_{o_1} \times \cdots \times C_{o_m}$$

be a decomposition of $G^\times$, where the e_i-s are even numbers and o_j-s are odd numbers (thus $d = n + l$). Choose generators $(g_1, \cdots, g_n)$ of $\mathbb{Z}^n$ and $(g_{n+1}, \cdots, g_d)$ of $C_{e_1} \times \cdots \times C_{e_l}$. The map

$$\mathsf{Hom}(C_2 \otimes G^\times, C_2) \ni f \mapsto (f(g_1), \cdots, f(g_d)) \in C_2^d$$

is an isomorphism.

Corollary 5.3.3. *Let G be a finitely generated grouplike monoid. Then*

$$\Pi_1(\mathbb{R}\mathsf{Spec}(G)) \cong \mathcal{D}^{d(G)},$$

where $\mathcal{D}^{d(G)}$ is the discrete groupoid with $2^{d(G)}$ objects.

To summarise, we can assume that, up to homotopy, $\mathbb{R}\mathsf{Spec}(G)$ is the discrete space corresponding to the group

$$\pi_0(\mathbb{R}\mathsf{Spec}(G)) \simeq C_2^d.$$

Elements of this discrete group are collections of the form

$$\{(\underbrace{\pm 1, \ldots, \pm 1}_{\times n}, \underbrace{\pm 1, \ldots, \pm 1}_{\times l}, \underbrace{1, \ldots, 1}_{\times m})\}.$$

The group structure is given by the pointwise multiplication. It is isomorphic to d-dimensional vector space over $\mathbb{F}_2$. To a vector $(\delta_1, \cdots, \delta_d)$, $\delta_i \in \mathbb{Z}/2\mathbb{Z}$, the corresponding collection is given by

$$\mathfrak{x}_\delta = ((-1)^{\delta_1}, (-1)^{\delta_2} \ldots, (-1)^{\delta_d}, \underbrace{1, \ldots, 1}_{\times m})).$$

5.3.2 On morphisms

Now we will discus the (non)functorial properties of this correspondence. Let $\alpha : G \to G'$ be a homomorphism of grouplike binoids.It induces a map

$$\alpha_* : \pi_0(\mathbb{R}\mathsf{Spec}(G')) \to \pi_0(\mathbb{R}\mathsf{Spec}(G)).$$

Due to Corollary 5.4.1, we may rewrite this as

$$\alpha_* : (C_2)^{d(G')} \to (C_2)^{d(G)}.$$

We can give this map more explicitly. Let

$$G^\times = (\mathbb{Z})^n \times C_{e_1} \times \cdots \times C_{e_l} \times C_{o_1} \times \cdots \times C_{o_m}$$

and

$$G'^\times = (\mathbb{Z})^{n'} \times C_{e'_1} \times \cdots \times C_{e_{l'}} \times C_{o'_1} \times \cdots \times C_{o_{m'}}$$

be decompositions of $G^\times$ and $G'^\times$ as above. Thus, $d(G) = n + l$ and $d(G') = n' + l'$. For simplicity, we write d and d' for $d(G)$ and $d(G')$. Choose generators $(g_1, \cdots, g_n)$ of $\mathbb{Z}^n$ and $(g_{n+1}, \cdots, g_d)$ of $C_{e_1} \times \cdots \times C_{e_l}$. Similarly for G'. Then α induces the homomorphism $C_2 \otimes G \to C_2 \otimes G'$, which is in fact an $\mathbb{F}_2$-linear map of vector spaces and hence, is completely determined by the matrix $\Delta = (\delta_{ij}) \in \mathsf{Mat}_{d' \times d}(\mathbb{F}_2)$, where

$$\alpha(g_i) \prod_j (g'_j)^{\delta_{ji}} \in (G')^2.$$

The transposition of Δ determines α_* since $\mathsf{Hom}(C_2 \otimes G^\times, C_2)$ is the dual vector space. From this fact, we immediately obtain:

Lemma 5.3.4. *The image of*

$$((-1)^{\epsilon_1}, (-1)^{\epsilon_2} \ldots, (-1)^{\epsilon_{d'}}) \in C_2^{d'}$$

under the map $\alpha_* : (C_2)^{d(G')} \to (C_2)^{d(G)}$ *is*

$$((-1)^{\sum_{i=1}^{d'} \delta_{i1}\epsilon_i}, \ldots, (-1)^{\sum_{i=1}^{d'} \delta_{id}\epsilon_i}) \in C_2^d$$

5.4 Further simplifications: The general binoids

Let M be a finitely generated binoid. Thanks to Theorem 5.2.7, the description of $\mathbb{R}\mathsf{Spec}(M)$ reduces to the case of grouplike binoids. Thus, we obtain the following fact:

Corollary 5.4.1. *Let M be a binoid. There is a homotopy equivalence*

$$\mathbb{R}\mathsf{Spec}(M) \simeq \coprod_{\mathfrak{r} \in \mathsf{Adm}(M)} \mathbb{R}\mathsf{Spec}((M(\mathfrak{r})^{\times^\circ}) \simeq \coprod_{\mathfrak{r} \in \mathsf{Adm}(M)} \mathsf{Hom}(C_2 \otimes M(\mathfrak{r})^\times, C_2).$$

In particular,

$$\Pi_1(\mathbb{R}\mathsf{Spec}(M)) \simeq \coprod_{\mathfrak{r}\in\mathsf{Adm}(M)} \mathcal{D}^{d_\mathfrak{r}},$$

and

$$\pi_0(\mathbb{R}\mathsf{Spec}(M)) \simeq \biguplus_{\mathfrak{r}\in\mathsf{Adm}(M)} C_2^{d_\mathfrak{r}},$$

where $\mathcal{D}^{d_\mathfrak{r}}$ is the discrete groupoid with $2^{d_\mathfrak{r}}$ objects, $\uplus$ denotes the disjoint union and the coproduct is taken in the 2-category of groupoids (and is equivalent to their 2-coproduct). Moreover, $d_\mathfrak{r}$ is the dimension (over $\mathbb{F}_2$) of $C_2 \otimes M(\mathfrak{r})^\times$. Thus, any point $\mathfrak{x} \in \pi_0(\mathbb{R}\mathsf{Spec}(M))$ can be seen as

$$\mathfrak{x} = ((-1)^{\delta_1}, (-1)^{\delta_2} \ldots, (-1)^{\delta_{d_r}}), \quad \delta_i \in \{0,1\},$$

for some $r \in \mathsf{Adm}(M)$.

Proof. This is a direct result of Theorem 5.2.7, Proposition 5.3.1 and Corollary 5.3.2. The last assertion (the equivalence of the coproduct and the 2-coproduct) will be shown in Corollary 3.2.3. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Example 5.4.2. *Let us consider the binoid given by $M = \langle x, y, y^{-1}, z\rangle / x^2y = z^2$. It is easily seen that M has no non-trivial idempotents. As such,*

$$\mathbb{R}\mathsf{Spec}(M) \simeq \mathbb{R}\mathsf{Spec}(C_2^\circ).$$

5.4.1 Morphisms

Let $\alpha : M \to N$ be a binoid homomorphism. This induces a map

$$\alpha_* : \pi_0(\mathbb{R}\mathsf{Spec}(N)) \to \pi_0(\mathbb{R}\mathsf{Spec}(M)).$$

Due to Corollary 5.4.1, we may rewrite it as

$$\alpha_* : \biguplus_{\mathfrak{r}\in\mathsf{Adm}(N)} (C_2)^{d_\mathfrak{r}} \to \biguplus_{\mathfrak{t}\in\mathsf{Adm}(M)} (C_2)^{d_\mathfrak{t}}.$$

Let $\mathfrak{x}$ be a point of $\pi_0(\mathbb{R}\mathsf{Spec}(N)) \simeq \biguplus_{\mathfrak{r}\in\mathsf{Adm}(N)} (C_2^\circ)^{n_\mathfrak{r}}$. Since its an element of a disjoint union, it must lie inside a single one. Hence, we may simply write $\mathfrak{x} \in (C_2^\circ)^n$ and assume this copy corresponds to $\mathfrak{r} \in \mathsf{Adm}(N)$. We can look at $\mathfrak{r}$ as a monoid homomorphism $\mathsf{Idem}(N) \to \{0,1\}$. Thus, the image $\mathfrak{x}$ under α_* will lie in the subset labeled by the homomorphism $M \xrightarrow{\mathfrak{r}\alpha} \{0,1\}$, which we denote by $\alpha_*(\mathfrak{r})$. It follows that the induced map $C_2^m \to C_2^n$ is the same as the induces map between the grouplike binoids $N(\mathfrak{r})^{\times^\circ} \to M(\alpha_*(\mathfrak{r}))^{\times^\circ}$. We can thusly use the explicit description given in Lemma 5.3.4.

Corollary 5.4.3. *Let $\alpha : M \to N$ be a homomorphism of binoids with no non-trivial idempotents. The induced map*

$$\alpha_* : \pi_0(\mathbb{R}\mathsf{Spec}(N)) \to \pi_0(\mathbb{R}\mathsf{Spec}(M))$$

is a group homomorphism.

Remark 5.4.4. *This immediately implies that for binoids without non-trivial idempotents, the fibre of every* $\mathfrak{y} \in \pi_0(\mathbb{R}\mathrm{Spec}(M))$ *has the same number of elements. As such, in order to calculate the number of elements in the image of* α_**, it suffices to calculate the number of elements mapping to the unit* $(1, 1, \ldots, 1)$*, which can be handled by Lemma 5.3.4.*

5.5 Schemes

5.5.1 Realisation of binoid schemes as varieties

Recall that to a binoid scheme X and a ring K, we can associate a ring scheme $\mathsf{K}[X]$, see [18]. Locally, this is given as the binoid algebra $\mathsf{K}[M]$. These binoid algebras glue if the monoids glue (meaning, $M_f \simeq N_g$ implies $\mathsf{K}[M_f] \simeq \mathsf{K}[N_g]$), allowing us to define schemes $\mathsf{K}[X]$ in the classical sense.

Quite similarly, if M, N are binoids and $f \in M$, $g \in N$ elements such that $M_f \simeq N_g$ are isomorphic binoids, we have $\mathrm{KSpec}(M_f) \simeq \mathrm{KSpec}(N_g)$. This allows us to extend the construction $\mathsf{K}X$ from affine binoid schemes $X = \mathrm{Spec}(M)$ to arbitrary binoid schemes X, which is the set of K-valued points of the scheme $\mathsf{K}[X]$. If K is $\mathbb{R}$ or $\mathbb{C}$, $\mathsf{K}X$ becomes equipped with the classical topology. It follows immediately that an affine cover $\mathcal{U} = \{U_i\} = \{\mathrm{Spec}(M_i)\}$ gives rise to an open cover of $\mathsf{K}X$.

We aim to give a method of calculating all the homology groups, as well as the fundamental groupoid $\Pi_1(\mathbb{R}X)$ of the 'real' geometric object $\mathbb{R}X$, associated to X.

5.5.2 Homology of $\mathbb{R}X$

5.5.2.1 Preliminary discussion

Recall that a *simplicial complex* is a set K, together with a collection of finite non-empty subsets of K, called *faces* or *simplices*, such that any singleton is a face (or simplex) and every non-empty subset of a face is a face. A face σ is of dimension k, or simply k-face, if $|\sigma| = k+1$. The collection of k-faces is denoted by K_k. We assume that there is given a total order on K. For a k-face $\sigma = \{v_0 < v_1 < \cdots < v_k\}$, $v_i \in K$ and any integer $i \in \{0, 1, \cdots, k\}$, one denotes by $\partial_i(\sigma)$ the $(k-1)$-dimensional face

$$\{v_0 < \cdots < v_{i-1} < v_{i+1} < \cdots < v_k\}.$$

The natural ordering of the power set of K induces an ordering on the set of all faces. Hence, one can consider contravariant functors defined on the set of all faces. Any such functor A assigns an abelian group $A(\sigma)$ to a face σ. Moreover, if $\tau \subseteq \sigma$, there is a well-defined map $\alpha^{\sigma,\tau} : A(\sigma) \to A(\tau)$ such that $\alpha^{\sigma,\sigma} = \mathrm{Id}_{A(\sigma)}$, and for $\rho \subseteq \tau \subseteq \sigma$, one has $\alpha^{\sigma,\rho} = \alpha^{\tau,\rho} \circ \alpha^{\sigma,\tau}$.

One can define a chain complex $C_*(K, A)$ as follows: The group of n-chains is given by

$$C_n(K, A) = \bigoplus_{\sigma \in K_n} A(\sigma).$$

The inclusion map $A(\sigma) \to C_n(K, A)$ is denoted by j^σ. The boundary map

$$d_n : C_n(K, A) \to C_{n-1}(K, A)$$

is given by

$$d_n(j^\sigma(a)) = \sum_{i=0}^{n} (-1)^i j^{\partial_i \sigma}(\alpha^{\sigma, \partial_i \sigma}(a)).$$

Here, $\sigma \in K_n$ and $a \in A(\sigma)$. The homology of the chain complex $C_*(K, A)$ is denoted by $H_*(K, A)$. If A is the constant functor with values in $\mathbb{Z}$, these groups coincide with the classical groups $H_*(K, \mathbb{Z})$.

These groups appear in the classical spectral sequence of the homology of a covering (which is a homological version of [23, Section II. 2.5]). Let S be a topological space and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of S. We assume that I is a totally ordered set. For $\sigma = (i_0 < i_1 < \cdots < i_k)$, we set

$$U_\sigma = U_{i_0 \cdots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}.$$

Recall that the nerve $\mathcal{N}$ of $\mathcal{U} = \{U_i\}_{i \in I}$ is the simplicial complex, whose vertices are elements of I. Moreover, σ is a k-simplex of $\mathcal{N}$ if $U_\sigma \neq \varnothing$. We denote by $\mathcal{N}_k$ the set of all k-simplexes of $\mathcal{N}$.

If $\tau \subseteq \sigma$, then $U_\sigma \subseteq U_\tau$. Thus, by the functoriality of the singular homology, one obtains an induced homomorphism $H_q(U_\sigma) \to H_q(U_\tau)$, $q \geq 0$. This allows us to consider the homology groups $H_*(\mathcal{N}, H_q)$. The p-th component of these homology groups is given by

$$\bigoplus_{(i_0 < i_1 < \cdots < i_p) \in \mathcal{N}_p} H_q(U_{i_0 \cdots i_p}).$$

It is well-known that there exists a first quadrant homological spectral sequence for which

$$E^2_{pq} = H_p(\mathcal{N}, H_q) \Longrightarrow H_{p+q}(S).$$

5.5.2.2 Calculating the homology of $\mathbb{R}X$

Let $\{\mathrm{Spec}(M_i)\}_{i \in \{1, \ldots, k\}}$ be the open and affine covering of a quasi-separated binoid scheme X. For the corresponding topological space $\mathbb{R}X$, we have an open cover given by $U_i = \mathbb{R}\mathrm{Spec}(M_i)$. It follows from Corollary 3.1.2 and the well known identity $H_*(X \sqcup Y) \simeq H_*(X) \oplus H_*(Y)$, that $H_q(\mathbb{R}\mathrm{Spec}(M_i)) = 0$ for all $q \geq 1$. Here, $H_*(-)$ is used to denote $H_*(-, \mathbb{Z})$. As such, $E^2_{pq} = 0$ for $q \geq 1$. The spectral sequence degenerates to an isomorphism

$$H_*(\mathbb{R}X) = H_*(\mathcal{N}, H_0).$$

In other words, these groups are isomorphic to the homology of the following cochain complex

$$\bigoplus_i H_0(U_i) \longleftarrow \cdots \longleftarrow \bigoplus_{i_0 < \cdots < i_k} H_0(U_{i_0 \cdots i_k}) \longleftarrow \cdots.$$

Here, the sum is taken over all $i_0 < \cdots < i_k$, for which $U_{i_0 \cdots i_k} \neq \varnothing$.

5.5.3 The Fundamental Groupoid $\Pi_1(\mathbb{R}X)$:

We will give a brief description of how to calculate the fundamental groupoid of $\mathbb{R}X$. As one might expect, this will require several results of 2-category theory.

5.5.3.1 Preliminaries The fundamental groupoid

Recall from Paragraph 1.1.2.1, that the *fundamental groupoid* of a topological space X is defined as the groupoid $\Pi_1(X)$, whose objects are the points of X and for any two objects $x, y \in \Pi_1(X)$, $\mathrm{Hom}_{\Pi_1}(x, y)$ is defined to be the homotopy classes of paths between them.

The Seifert-van Kampen theorem for fundamental groupoids does not require a common point [12],[11] and works for every open covering, which is something we will heavily exploit in the upcoming section. It can also be calculated using the 2-colimit, rather than the colimit, as we mentioned in Theorem 1.3.15 and the subsequent discussion. We will use this, in conjunction with our results obtained in Section 3.2 to calculate the fundamental groupoid of $\mathbb{R}X$.

For a topological space X and a covering $\mathcal{U} = \{U_i\}_{i\in I}$, we use the notations $U_{ij} = U_i \cap U_j$, $U_{ij} = U_i \cap U_j \cap U_k$, and so on. Recall that for a set I, we used the notation $\mathfrak{B}(I)$ for the set of all proper subsets of I. For a subset $S \subseteq I$, let $S^{\mathsf{c}} = I \setminus X$ be the complement. Define a strict covariant functor $\Phi : \mathfrak{B}(I) \to \mathfrak{Grpd}$ from the poset $\mathfrak{B}(I)$ to the 2-category of groupoids by

$$\Phi(S) = \Pi_1(U_{S^c}) = \Pi_1(\bigcap_{i\notin S} U_i)$$

The following is a direct consequence of our version of Seifert-van Kampen theorem [50] and Corollary 3.2.15.

Theorem 5.5.1. *The fundamental groupoid of X is equivalent to the 2-colimit of the covariant strict functor $\Phi : \mathfrak{B}(I) \to \mathfrak{Grpd}$. It is also equivalent to the 2-colimit of the restriction of Φ on the subposet $\mathfrak{B}(\mathbf{n}, \mathbf{n-3})$.*

According to [50], $\Pi_1(X)$ is equivalent to the 2-colimit of $\Phi : \mathfrak{B}(\mathbf{n}, \mathbf{n-3}) \to \mathfrak{Grpd}$. We can then use Corollary 3.2.15.

5.5.3.2 Calculating the fundamental groupoid of $\mathbb{R}X$

Let X be a quasi-separated binoid scheme, with an affine covering $\mathsf{A} := \{\mathsf{Spec}(M_i)\}_{i\in\{1,\dots,n\}}$. Our aim is to calculate $\Pi_1(\mathbb{R}X)$. For this, we use an affine covering of X by $U_i = \mathsf{Spec}(M_i)$. As in previous case we consider intersections $U_{ij} = \mathsf{Spec}(M_{ij})$ etc, where M_{ij} usually are appropriate localisations of M_i and M_j.

We define a functor from the poset $\mathfrak{B}(\mathbf{n})$ to the category of topological spaces

$$\mathcal{R} : \mathfrak{B}(\mathbf{n}) \to \mathsf{Top},$$

given by

$$\mathcal{R}(S) = \coprod_{r\in\mathsf{Adm}(M_{S^c})} \mathbb{R}\mathsf{Spec}(M_{S^c}(\mathfrak{r})^{\times^\circ} \otimes C_2^\circ).$$

Since for all $I \subseteq J$ we have the localisation map $M_I \to M_J$, and hence $\mathbb{R}\mathsf{Spec}(M_J) \to \mathbb{R}\mathsf{Spec}(M_I)$, it is clear that $\mathcal{R}$ defines a functor. Replacing S with S^c, we see that $\mathcal{R}$ is indeed a functor, see Corollary 5.4.1.

Using this, we define a strict 2-functor

$$\mathfrak{R} : \mathfrak{B}(\mathbf{n}) \to \mathfrak{Grpd},$$

with values in the 2-category of groupoids, given by

$$\mathfrak{R}(S) = \Pi_1 \left(\coprod_{r \in \mathsf{Adm}(M_{S^c})} \mathbb{R}\mathsf{Spec}(M_{S^c}(\mathfrak{r})^{\times^\circ} \otimes C_2^\circ) \right).$$

Theorem 5.5.1, and the fact that we can replace equivalent groupoids with equivalent ones in a 2-colimit, says that the fundamental groupoid of $\mathbb{R}X$ is equivalent to the 2-colimit of $\mathfrak{R}$. Since calculating the colimit is significantly easier than calculating the 2-colimit, we wish to use Theorem 3.2.6. Our discussion regarding the stretching of functors on page 89 allows us to do just that.

5.6 Applications

We aim to give a practical demonstration on how to use the above results to calculate the homology groups, as well as the fundamental groupoid of $\mathbb{R}X$, where X is a binoid scheme. We will do so both for a general class (see the Stanley-Reisner case) as well as a specific example.

We point out that if we glue binoid schemes on affine subsets, the obtained binoid scheme will be topologically separated.

5.6.1 Punctured spectra of Stanley-Reisner rings

Let M be a binoid. Since the union of prime ideals is prime, it has a unique maximal ideal, denoted $\mathfrak{m}_M$ or simply $\mathfrak{m}$. It is prime, as one would expect and consists of the set of non-invertible elements. It is also the unique closed point of $\mathsf{Spec}(M)$. Its complement is denoted by

$$\mathsf{Spec}^\bullet(M) \coloneqq \mathsf{Spec}(M) \setminus \mathfrak{m}_M.$$

By restricting the structure sheaf $\mathcal{O}_{\mathsf{Spec}(M)}$ to $\mathsf{Spec}^\bullet(M)$, we get an open subscheme, called the *punctured spectrum* of M.

A simplicial complex Δ on $\mathbf{n} = \{1, \dots, n\}$ defines the (multiplicatively written) binoid

$$M_\Delta = \{X_1^{r_1} \cdots X_n^{r_n},\ \prod_{i \in I} X_i = 0 \text{ for nonfaces } I\}$$

and its binoid algebra is the *Stanley-Reisner Ring of* Δ. We want to compute the fundamental groupoid of the punctured spectrum of the Stanley-Reisner ring with our methods. In other

words, we want to calculate $\Pi_1(\mathbb{R}\mathrm{Spec}^\bullet(M_\Delta))$, or $\mathbb{R}^\bullet[\Delta]$ for short. It is clear that there exists an open covering of $\mathbb{R}^\bullet[\Delta]$, given by

$$\{D(\mathbb{R}[X_i])\}_{i\in\mathbf{n}} := \mathbb{R}\mathrm{Spec}((M_\Delta)_{X_i})_{i\in\mathbf{n}}.$$

Theorem 5.6.1. *Let Δ denote a simplicial complex on the vertex set $\mathbf{n} = \{1,\dots,n\}$. The fundamental groupoid of the real punctured spectrum of the Stanley-Reisner ring $\mathbb{R}^\bullet[\Delta]$ is given as the following groupoid: The objects are*

$$\{F \text{ facet of } \Delta,\ \sigma : F \to \{+,-\}\}.$$

The isomorphisms are generated by

$$\alpha_{i,[\sigma_F,\tau_G]} : (F,\sigma) \to (G,\tau),$$

whenever there exists $i \in F \cap G$ with $\sigma(i) = \tau(i)$. These are subject to the relations generated by:

R1) For a fixed $i \in \mathbf{n} = \{1,\dots,n\}$, $\alpha_{i,[\sigma_F,\rho_H]} = \alpha_{i,[\tau_G,\rho_H]} \circ \alpha_{i,[\sigma_F,\tau_G]}$. That is, the diagram

$$(F,\sigma) \xrightarrow{\alpha_{i,[\sigma_F,\tau_G]}} (G,\tau) \xrightarrow{\alpha_{i,[\tau_G,\rho_H]}} (H,\rho), \qquad (F,\sigma) \xrightarrow{\alpha_{i,[\sigma_F,\rho_H]}} (H,\rho)$$

commutes;

R2) For every $\{i,j\} \subseteq F \cap G$ with $\sigma(i) = \tau(i)$ and $\sigma(j) = \tau(j)$, we have

$$\beta_{i,[\sigma_F,\tau_G]} = \beta_{j,[\sigma_F,\tau_G]} : (F,\sigma) \to (G,\tau)$$

Here, β denotes a concatenation of some α-s.

Proof. For each subset $\varnothing \neq I \subseteq \{1,\dots,n\}$, define the groupoid $P(I)$, whose objects are

$$\{(F,\sigma),\ F \text{ a facet of } \Delta \text{ with } I \subseteq F\}$$

with isomorphisms

$$\alpha_{I,[\sigma_F,\tau_G]} : (F,\sigma) \to (G,\tau), \text{ whenever } \sigma \text{ and } \tau \text{ agree on } I,$$

subject to the relations $\alpha_{I,[\sigma_F,\rho_H]} = \alpha_{I,[\tau_G,\rho_H]} \circ \alpha_{I,[\sigma_F,\tau_G]}$. This is empty if I is a nonface. For $I \subseteq J$, there are natural injections $P(J) \to P(I)$ sending (F,σ) to itself and $\alpha_{J,[\sigma_F,\tau_G]}$ to $\alpha_{I,[\sigma_F,\tau_G]}$.

The groupoid $P(I)$ is equivalent to the discrete groupoid whose objects are $\{+,-\}^I$. To see this, first observe that the relation $\alpha_{I,[\sigma_F,\rho_H]} = \alpha_{I,[\tau_G,\rho_H]} \circ \alpha_{I,[\sigma_F,\tau_G]}$ implies that $P(I)$ is a groupoid, as there is at most one isomorphism between any two objects. To see that it has exactly 2^I isomorphism classes of objects, let $I \subseteq F,G$ be two extensions of $\rho : I \to \{+,-\}$ to

facets. There is a unique isomorphism relating them. The group of units of the binoid M_Δ, localised at the elements of I, is $\mathbb{Z}^I$ (which is 0 if I is a nonface) and tensoring with $\mathbb{Z}/(2)$ gives $\{+,-\}^I$. This is equivalent to the real fundamental groupoid $\Pi_1(D(\mathbb{R}[\prod_{i\in I} X_i]))$ by Theorem 5.2.7.

Hence, we may compute $\Pi_1(\bigcup_{i=1}^n D(\mathbb{R}[X_i])) \simeq \Pi_1(\mathbb{R}^\bullet[\Delta])$ as the 2-colimit of P. We claim that the conditions of Theorem 3.2.6 are fulfilled, thereby allowing us to compute it as the colimit of P. As mentioned above, the maps $P(J) \to P(I)$, for $I \subseteq J$, are injective on objects. Moreover, for each collection $I_1, \ldots, I_s$, we have that

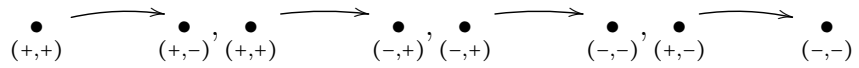
$$\mathrm{colim}_j\, P(I_j) = P(I_1) \cup \ldots \cup P(I_s) \to P(I_1 \cap \ldots \cap I_s)$$

is injective on objects.

For the computation of the colimit of P only the subsets I with 1 or 2 elements are relevant. Its objects consist of the set of all (F, σ), with F a facet of Δ. The isomorphisms are generated by the generators of the isomorphisms of $P(\{i\})$, modulo the relations coming from $P(\{i,j\})$. $\mathcal{Q.E.D}$

We give some easy examples of how to apply this method. For these examples, there are many other ways to compute the fundamental groupoid, they merely serve as an illustration.

Example 5.6.2. *Consider the full simplicial complex on $\{1,\ldots,n\}$. There is only the facet $F = \{1,\ldots,n\}$ and there are 2^n $\pm$-tuples. For $n = 1$, the objects are + and − and there is only one automorphism for each object. For $n = 2$, the generating isomorphisms are*



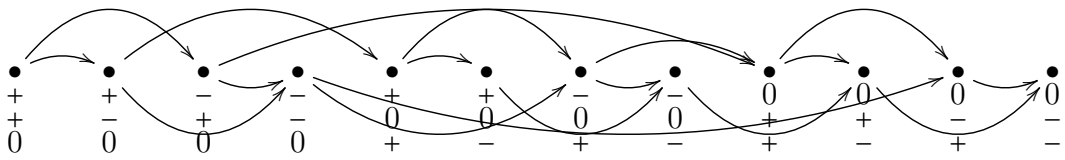
and there are no non-identity relations. As an example, the isomorphism $(+,+) \longrightarrow (+,-)$ would be $\alpha_{1,[\sigma_F,\tau_F]}$ in our notation, where $F = \{1,2\}$ with $\sigma(1) = +, \sigma(2) = +$ and $\tau(1) = +, \tau(2) = -$. This groupoid is connected with 4 objects and 4 isomorphisms. Hence, it is equivalent to the groupoid with one element, together with one generating automorphism without relations. So, the fundamental group is $\mathbb{Z}$.

For $n \geq 3$, we show that every generating isomorphism $\alpha_{i,[\sigma_F,\tau_F]} : (F,\sigma) \to (F,\tau)$ has a factorisation into isomorphisms, which are the restrictions of two subset isomorphisms. The maps σ and τ agree at least on i. If they agree also on another spot, the statement is clear. If they agree only on i, we pick two spots j, k where they differ. Then there exists an isomorphism φ which fixes i and j and another isomorphism ψ which fixes i and k such that $\alpha_{i,[\sigma_F,\tau_F]}$ can be written as

$$(F,\sigma) \overset{\varphi}{\to} (F,\rho) \overset{\psi}{\to} (F,\tau).$$

Hence, the fundamental group is trivial.

Example 5.6.3. *Consider the simplicial complex on $\{1,2,3\}$, given by the facets $\{1,2\}, \{1,3\}$ and $\{2,3\}$. There are twelve objects which are tuples of the form $(\pm,\pm,0)$ (and permutations), the generating isomorphisms are given whenever two such tuples coincide at a nonzero spot. As such, the groupoid can be visualised by the diagram*



Here, the upper arrows are induced from the sign functions taking + as their value, and the bottom arrows are induced by the − sign.

The relations coming from R1) are already taken into consideration in the above diagram, by not drawing in 'unnecessary' relations in the first place. An example of this is the following: By the construction, we have isomorphisms connecting $(+,+,0)$*,* $(+,-,0)$ *and* $(+,0,+)$ *in all pairwise combinations. However, one of them is simply the composition of the other two by R1), so we omit it in the diagram.*

The only relations coming from R2) are the identities. As such, we have 12 objects with 18 isomorphisms. This is equivalent to one object with $18-(12-1)=7$ *automorphisms. The fundamental group of the real punctured spectrum is the free group with* 7 *generators.*

5.6.2 Example 1:

Let $I=\{1,2,3\}$ and

$$\begin{aligned} M_1 &= \langle x_1,y_1,z_1\rangle/x_1^2y_1=z_1^2,\\ M_2 &= \langle x_2,y_2,z_2\rangle/x_2y_2=z_2^2,\\ M_3 &= \langle x_3,y_3\rangle. \end{aligned}$$

The gluing isomorphisms are given as

$$\begin{aligned} &\lambda_{12}:(M_1)_{x_1}\to(M_2)_{x_2} \quad | \quad x_1\mapsto x_2, \quad y_1\mapsto x_2^{-1}y_2, \quad z_1\mapsto z_2\\ &\lambda_{23}:(M_2)_{y_2}\to(M_3)_{x_3} \quad | \quad x_2\mapsto x_3y_3^2,\ y_2\mapsto x_3^{-1}, \quad z_2\mapsto y_3. \end{aligned}$$

We denote the binoid scheme obtained in this way by X.

5.6.2.1 The homology groups

We have $\mathbb{R}X=U_1\cup U_2\cup U_3$, where $U_i=\mathbb{R}\mathsf{Spec}(M_i)$. The homology of $\mathbb{R}X$ can be computed as the homology of the chain complex

$$\cdots\to 0\to H_0(U_{123})\xrightarrow{d_2}H_0(U_{12})\oplus H_0(U_{13})\oplus H_0(U_{23})\xrightarrow{d_1}H_0(U_1)\oplus H_0(U_2)\oplus H_0(U_3).$$

Since $U_{123}\simeq\mathbb{R}\mathsf{Spec}(\mathbb{Z}\times\mathbb{Z})\simeq U_{13}$, it follows that the map $H_0(U_{123})\to H_0(U_{13})$ is an isomorphism. Hence, the same homology can be computed by the chain complex

$$\cdots\to 0\to 0\to H_0(U_{12})\oplus H_0(U_{23})\xrightarrow{d}H_0(U_1)\oplus H_0(U_2)\oplus H_0(U_3).$$

Since $M_i^\times=\{1\}$, we have $H_0(U_i)=\mathbb{Z}$. We also have $U_{12}\simeq\mathbb{R}\mathsf{Spec}(\mathbb{N}\times\mathbb{Z})\simeq U_{23}$. Thus,

$$H_0(U_{12})\simeq H_0(U_{23})=\mathbb{Z}^2$$

and the map $d : \mathbb{Z}^4 \to \mathbb{Z}^3$ is given by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

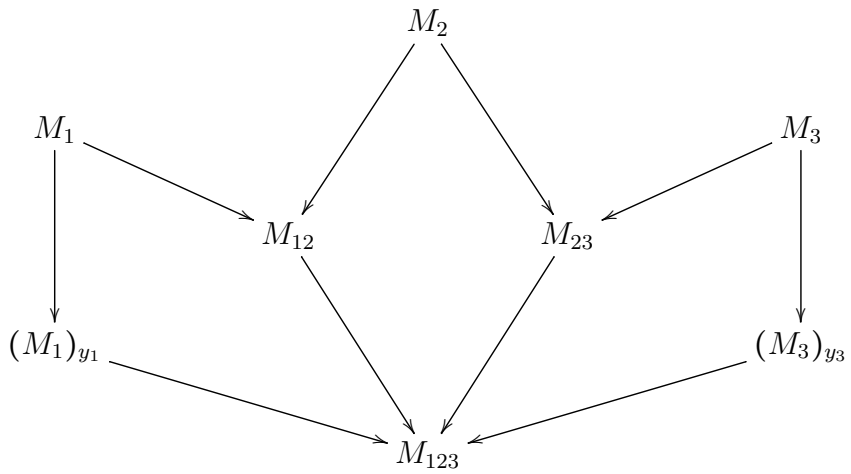
It follows that

$$H_i(\mathbb{R}X) = \begin{cases} 0, & \text{if } i \geq 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0. \end{cases}$$

5.6.2.2 The fundamental groupoid

We wish to calculate the fundamental groupoid. We will give a more detailed description since this involves calculating the 2-colimit. While calculating the 2-colimit of an arbitrary 2-functor can be quite difficult, the situation can be significantly simplified when the source 2-category is a finite poset and the target only involves finite discrete groupoids.

Step 1: We first draw our binoid scheme X, as follows:



Here, $M_{12} := (M_1)_{x_1} \simeq (M_2)_{x_2}$,$M_{23} := (M_2)_{y_2} \simeq (M_3)_{x_3}$ and M_{123} is the generic point. Examples of open sets are

$$\{M_{123}\}, \{(M_1)_{y_1}, M_{123}\}, \{M_2, M_{12}, M_{23}, M_{123}\} \text{ and } \{(M_1)_{y_1}, M_{12}, M_{23}, M_{123}\}.$$

In other words $U \subseteq X$ is open, if $M_\alpha \in U$ implies that every localisation of M is also in U.

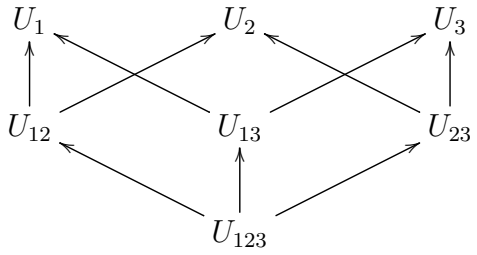
A binoid scheme has a canonical covering. In this case, it is

$$U_1 := \{M_1, (M_1)_{y_1}, M_{12}, M_{123}\}, \quad U_2 := \{M_2, M_{12}, M_{23}, M_{123}\}$$

and

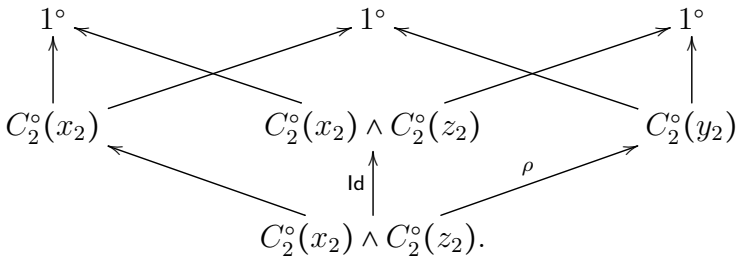
$$U_3 := \{M_3, M_{23}, (M_3)_{y_3}, M_{123}\}.$$

Thus we obtain the following diagram on the poset $\mathfrak{B}(\mathbf{3})$ X:



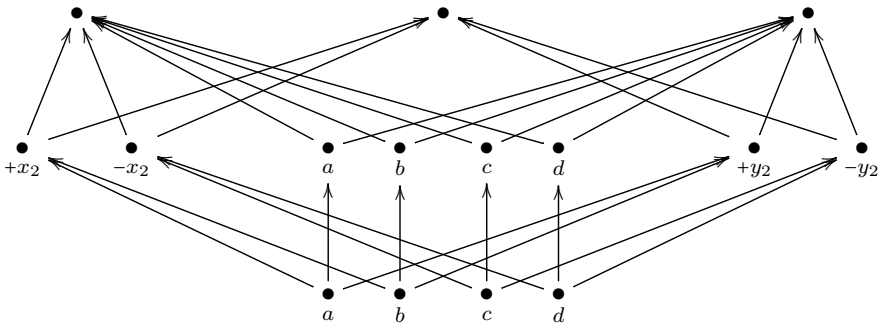
where, $U_{13} = U_{123} = \{M_{123}\}$.
Step 2: We calculate the fundamental groupoid of every U_i. We start by simplifying them using Thm. 5.2.7, by replacing every U_i with $\coprod_{r \in \mathsf{Adm}(M_i)} M_i(\mathfrak{r})^{\times^\circ} \otimes C_2^\circ$. Note that, as every U_i, U_{ij} and U_{ijk} has a maximal binoid, their global section is going to agree with said maximal binoid. That is, $\mathcal{O}_X(U_\alpha) = M_\alpha$. We require topological separatedness here, so that the intersections have a maximal binoid.

For this, we first determine the idempotents of M_i. It is easily seen that none of the binoids have any non-trivial idempotents. As such, we obtain:



Here, x_2 in $C_2^\circ(x_2)$ just indicates the generator. The only non-obvious morphism $\rho : C_2^\circ(y_2) \to C_2^\circ(x_2) \wedge C_2^\circ(z_2)$ is given by $y_2 \mapsto x_2^{-1} z_2^2 = x_2$.

Step 3: We compose these groupoids with $\mathbb{R}\mathsf{Spec}(-) = \mathsf{Hom}_{\mathsf{bn}}(-, \mathbb{R})$. In the case of $C_2^\circ(x_2)$, for example, this is $\{x_2 \mapsto 1, x_2 \mapsto -1\}$. These are all clearly determined by the sign. Since these are individual points, the objects of $\Pi_1(\mathbb{R}\mathsf{Spec}(C_2^\circ(x_2)))$ can be identified by $+x_2$ and $-x_2$. Similarly for all the other binoids. These groupoids are fully determined by this information as they have no non-trivial automorphisms. Hence, we can represent this as follows:



Here, $\{\underset{a}{\bullet}\ \underset{b}{\bullet}\ \underset{c}{\bullet}\ \underset{d}{\bullet}\}$ represents the groupoid $\Pi_1(C_2^\circ(x_2) \wedge C_2^\circ(z_2))$, with $a = (+x_2, +z_2)$, $b = (+x_2, -z_2)$, $c = (-x_2, +z_2)$ and $d = (-x_2, -z_2)$. Since $\rho(y_2) = x_2$, we only need look at the sign of x_2 to determine the functor. For more on the induced functors between the groupoids see our discussion in Subsection 5.4.1.

Step 4: We wish to use Theorem 3.2.6. Recall that this theorem allows us to compute the 2-colimit of a functor $\Phi : \mathfrak{B}(\mathbf{3}) \to \mathrm{Grpd}$ in terms of colimits, if Φ satisfies the condition $A(3)$. Spelled out, condition $A(3)$ is the collection of the following three conditions:

i) The natural functor $\Phi(\varnothing) \to \Phi(\{1\})$ is injective on object.

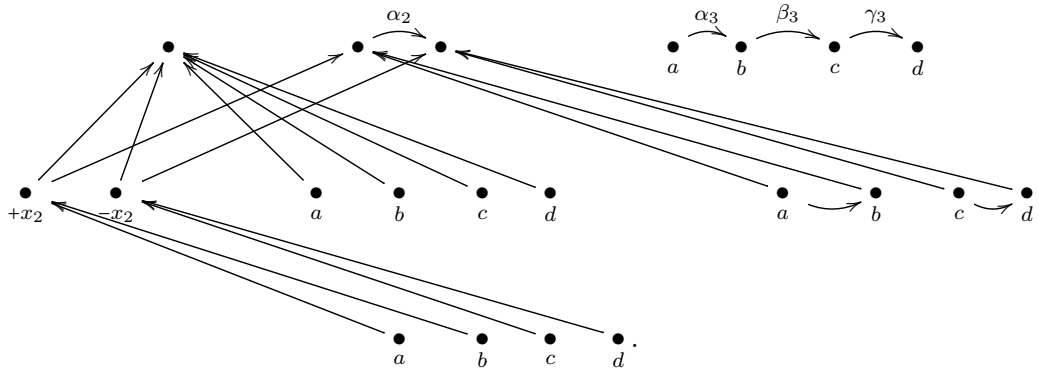
ii) The natural functor $\Phi(\{3\}) \to \Phi(\{1,3\})$ is injective on object.

iii) Denote the colimit of the diagram

$$\begin{array}{ccc} \Phi(\varnothing) & \longrightarrow & \Phi(\{1\}) \\ \downarrow & & \\ \Phi(\{2\}) & & \end{array}$$

by $\Phi'(\{1,2\})$. The natural functor $\Phi'(\{1,2\}) \to \Phi(\{1,2\})$ is injective on objects.

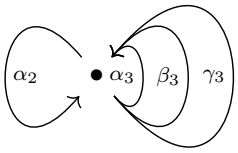
The diagram which we obtained above does not satisfy any of these conditions. However, we can replace it with the following equivalent diagram, which satisfies these conditions:



Here, $\bullet \overset{-}{\frown} \bullet$ denotes the connecting isomorphism between the two points. These morphisms must map to the connecting isomorphisms above them. This groupoid is clearly equivalent to the trivial groupoid $\bullet$. Further, the endofunctors between $\underset{a}{\bullet}\ \underset{b}{\bullet}\ \underset{c}{\bullet}\ \underset{d}{\bullet}$ have been omitted, as they are clear.

Step 5: The colimit and 2-colimit of the above diagram are equivalent by Proposition 3.2.6. The set of objects in the colimit of this diagram, is the colimit of the objects in this diagram.

Morphisms in the colimit are generated by the colimit of the morphisms, modulo the relations coming from the intersections. We get the following as the 'generator' of our groupoid:



The relations are the ones induced by the morphisms in the intersections. We have

$$\alpha_2 = \alpha_3 = \gamma_3.$$

As such, $\Pi_1(\mathbb{R}X)$ is equivalent to F_2, the free group with 2 generators $\langle \alpha_2, \beta_3 \rangle$, seen as a groupoid with one object.

Chapter 6

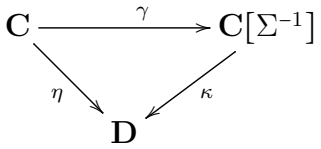
General theory of topoi

6.1 Preliminaries of topos theory

6.1.1 Localisation of categories

Let $\mathbf{C}$ be a category and Σ a class of morphisms which is closed under compositions and contains every identity morphisms. We can think of it as a (not necessarily full) subcategory of $\mathbf{C}$ by considering its objects to be exactly the objects of $\mathbf{C}$. Denote by $\mathbf{C}[\Sigma^{-1}]$ the category which is obtained by inverting all the morphisms in Σ, if it exists. To express it more formally, $\mathbf{C}[\Sigma^{-1}]$ satisfies the following universal property:

There exists a functor $\gamma : \mathbf{C} \to \mathbf{C}[\Sigma^{-1}]$ such that $\gamma(f)$ is an isomorphism for every $f \in \Sigma$. If, further, $\eta : \mathbf{C} \to \mathbf{D}$ is a functor such that $\eta(f)$ isomorphism for all $f \in \Sigma$, there exist a unique functor $\kappa : \mathbf{C}[\Sigma^{-1}] \to \mathbf{D}$ for which the diagram



commutes. Of course, if $\mathbf{C}[\Sigma^{-1}]$ exists, it is unique up to isomorphism. The existence of $\mathbf{C}[\Sigma^{-1}]$ is, in general, a big problem, because the collection of all arrows in $\mathbf{C}$ is a class, not a set.

There are several important cases, where the existence of $\mathbf{C}[\Sigma^{-1}]$ is well-known. One such case is the localisation of categories which, we now recall. The importance of this class comes from the fact that any Grothendieck topos (defined later) is a localisation of a category of presheaves.

Definition 6.1.1. *Let $\mathbf{D}$ and $\mathbf{C}$ be categories. A functor $j^\bullet : \mathbf{C} \to \mathbf{D}$ is called a* localisation of categories *if it commutes with finite limits. Furthermore, $j^\bullet$ needs to posses a right adjoint $j_\bullet : \mathbf{D} \to \mathbf{C}$, which is full and faithful.*

Recall that the existence of a right adjoint implies that $j^\bullet$ commutes with all colimits (and that $j_\bullet$ commutes with all limits).

As an example, we can take $\mathbf{C}$ to be the category of modules over a commutative ring R and $\mathbf{D}$ the category of modules over $S^{-1}R$, where S is a multiplicative subset of R and $S^{-1}R$ the localisation of R with respect to S. The functor $j^\bullet$ is given by $M \mapsto S^{-1}M$, while $j_\bullet$ sends any $S^{-1}M$-module N to N, considered now as an R-module via the canonical ring homomorphism $R \to S^{-1}R$.

Another important example comes from topology. Let U be an open subset of a topological space X. The inclusion $j : U \to X$ induces a localisation of categories $j^\bullet : \mathsf{Sh}(X) \to \mathsf{Sh}(U)$, given by restricting a sheaf on the open subset $U \subseteq X$.

We come back to the general situation. It is well-known that for the adjoint pair $j_\bullet : \mathbf{D} \to \mathbb{C}$, $j^\bullet : \mathbb{C} \to \mathbf{D}$, the left adjoint $j^\bullet$ is a localisation of categories if and only if $j^\bullet$ respects finite limits, and the counit of the adjunction $j^\bullet \circ j_\bullet \to \mathrm{Id}_{\mathbf{D}}$ is an isomorphism. Denote by Σ_j the collection of morphisms f of $\mathbb{C}$ for which $j^\bullet(f)$ is an isomorphism. It is also well-known, that $\mathbb{C}[\Sigma_j^{-1}]$ exists and the induced functor $\mathbb{C}[\Sigma_j^{-1}] \to \mathbf{D}$ is an equivalence of categories.

If $j^\bullet : \mathbb{C} \to \mathbf{D}$ is a localisation, we have seen that $j_\bullet : \mathbf{D} \to \mathbb{C}$ is a full and faithful embedding. Thus, we can and we will identify $\mathbf{D}$ with a full subcategory of $\mathbf{C}$. When we will talk about a localisation of a category $\mathbb{C}$, we will assume that it is a full subcategory $\mathbf{D}$ of $\mathbf{C}$ such that the following conditions hold:

i) If x belongs to $\mathbf{D}$ and $y \in \mathbf{C}$ is isomorphic to x, then y belongs to $\mathbf{D}$.

ii) The inclusion $j_\bullet : \mathbf{D} \to \mathbf{C}$ has a left adjoint $j^\bullet : \mathbf{C} \to \mathbf{D}$, called the *localisation functor*.

iii) The localisation functor $j^\bullet$ respects finite limits.

We denote by $\mathfrak{Loc}(\mathbf{C})$ the collection of all localisations of $\mathbf{C}$. The collection $\mathfrak{Loc}(\mathbf{C})$ has a natural poset structure induced by inclusion of subcategories.

6.1.2 Elementary topoi

The category of Sets has several important properties. One of them is the fact that for any sets A and B, there is a set A^B which is the set of all maps from B to A. In the language of category theory, this means that the functor $X \mapsto \mathsf{Hom}_{\mathsf{Sets}}(X \times B, A)$ is representable by A^B, meaning that there exists a functorial isomorphism

$$\mathsf{Hom}_{\mathsf{Sets}}(X \times B, A) \cong \mathsf{Hom}_{\mathsf{Sets}}(X, A^B).$$

Another important property is that for any set X, there exists a set $\mathfrak{P}(X)$, called the power set, which is the set of all subsets of X. The association $X \mapsto \mathfrak{P}(X)$ forms a contravariant functor. Moreover, the functor $\mathfrak{P}$ is representable by a two point set $\Omega = \{t, f\}$, meaning that there is a functorial isomorphism

$$\mathfrak{P}(X) \cong \mathsf{Hom}_{\mathsf{Sets}}(X, \Omega).$$

It sends a subset $A \subseteq X$ to the characteristic map $\chi_A : X \to \Omega$, defined by

$$\chi_A(a) = \begin{cases} t, & \text{if} \quad a \in A, \\ f, & \text{if} \quad a \notin A. \end{cases}$$

We are in a position to introduce the following definition (see for example [42]), originally due to William Lawvere.

Definition 6.1.2. *An* elementary topos *is a category $\mathcal{E}$ for which the following conditions hold:*

i) $\mathcal{E}$ has finite limits.

ii) For any two objects A and B, there exist an exponential object A^B. This means that one has a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(X \times B, A) \cong \mathrm{Hom}_{\mathcal{E}}(X, A^B). \tag{6.1}$$

iii) A subobject classifier *exits. This it to say, there exists an object Ω, together with a morphism from the terminal object $t : 1 \to \Omega$, such that for any object A and any subobject $i : B \subseteq A$, there is a unique morphism $\chi_B : A \to \Omega$, for which*

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & A \\ {\scriptstyle {}_B!}\downarrow & & \downarrow{\scriptstyle \chi_B} \\ 1 & \xrightarrow[\ t\ ]{} & \Omega \end{array}$$

is a pullback diagram in $\mathscr{T}$. Here and elsewhere, ${}_B!$ denotes the unique morphism from B to the terminal. The map $\chi_B : A \to \Omega$ is called the characteristic map *of a subobject B.*

Logical morphisms of topoi Let $\mathcal{E}$ and $\mathcal{D}$ be elementary topoi. A *logical morphism* $\mathcal{E} \to \mathcal{D}$ is a functor $F : \mathcal{E} \to \mathcal{D}$ which preserves finite limits, exponential objects and subobject classifiers.

The following fact shows that there are plenty of elementary topoi.

Proposition 6.1.3. *Let $\mathbf{X}$ be a small category. The category* $\mathrm{Psh}(X)$ *of presheaves is an elementary topos. More precisely, the subobject classifier Ω is the preasheaf which assigns to an object U the subfunctors of the representable functor* $\mathrm{Hom}_{\mathbf{X}}(-, U)$*, meaning the set of sieves. Moreover, if F and G are presheaves over $\mathbf{X}$, the exponential object F^G is the presheaf which assigns the set of natural transformations* $\mathrm{Hom}_{\mathbf{X}}(-, U) \times G \to F$ *to an object U of $\mathbf{X}$.*

Proof. This is a well-known fact, see for example [42]. Observe only that the descriptions of Ω and the exponential object easily follows from the Yoneda lemma. For example, if we put $X = \mathrm{Hom}_{\mathbf{X}}(-, U)$, $B = G$, $A = F$ and $\mathcal{E} = \mathrm{Psh}(\mathbf{X})$ in the isomorphism (6.1), we obtain the expected result for F^G. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

6.1.3 Grothendieck topoi, Grothendieck topologies and localizations of topoi

Recall that a *Grothendieck topos* is a category $\mathscr{T}$, equivalent to the category of set valued sheaves over a site [42]. We have the following classical fact [42]:

Theorem 6.1.4. *Any Grothendieck topos is an elementary topos.*

The converse is not true in general: The category of finite sets is an elementary topos, but not a Grothendieck topos. In what follows, all topoi will be Grothendieck topoi. Therefore, we will refer to Grothendieck topoi simply as topoi.

Let $\mathbf{X}$ be a small category. Denote the collection of all Grothendieck topologies on $\mathbf{X}$ by $\mathsf{Top}(\mathbf{X})$. There is a strong relation between the localisations $\mathfrak{Loc}(\mathsf{Psh}(\mathbf{X}))$ of $\mathsf{Psh}(\mathbf{X})$ and the Grothendieck topologies $\mathsf{Top}(\mathbf{X})$ on $\mathbf{X}$, as we will show in this subsection.

It is well-known that $\mathsf{Top}(\mathbf{X})$ is a complete lattice [32, Lemma 0.34] under inclusion. The minimal element of this lattice is called the discrete topology and the maximal element is called the indiscrete topology. Over the discrete topology, the sheaves are exactly the presheaves. Over the indiscrete topology, the category of sheaves is just the trivial category, having only one object and one morphism.

Let J be a Grothendieck topology on $\mathbf{X}$. We can form the category of sheaves $\mathsf{Sh}(\mathbf{X}, J)$. Recall that the inclusion $\mathsf{Sh}(\mathbf{X}, J) \hookrightarrow \mathsf{Psh}(\mathbf{X})$ has the left adjoint functor $P \mapsto \hat{P}$, called sheafification. It is well-known, that sheafification commutes with finite limits and thus, is a localisation $\mathsf{Psh}(\mathbf{X}) \to \mathsf{Sh}(\mathbf{X}, J)$.

An other way to express this is the following: There exists a class of morphisms of presheaves Σ such that

$$\mathsf{Sh}(\mathbf{X}, J) \cong \mathsf{Psh}(\mathbf{X})[\Sigma^{-1}],$$

meaning the morphisms in Σ become isomorphisms after shifification.

In other words, sheafification is determined by a class of morphisms, with which we localise (make invertible). Thus, we constructed a map $\mathsf{Top}(\mathbf{X}) \to \mathfrak{Loc}(\mathsf{Psh}(\mathbf{X}))$. Now we wish to construct a map $\mathfrak{Loc}(\mathsf{Psh}(\mathbf{X})) \to \mathsf{Top}(\mathbf{X})$.

Assume $j^\bullet : \mathsf{Psh}(\mathbf{X}) = \mathbf{C} \to \mathbf{D}$ is a localisation. For an object U of $\mathbf{X}$, we denote by JU the collection of all subobjects (i.e. sieves) $S \subseteq \mathsf{Hom}_{\mathbf{X}}(-, U)$ such that after applying the functor $j^\bullet$, we have an isomorphism

$$j^\bullet(S) \xrightarrow{\cong} j^\bullet(\mathsf{Hom}_{\mathbf{X}}(-, U)).$$

It is well-known that such sieves S form a Grothendieck topology and the category of sheaves over this topology is equivalent to $\mathbf{D}$. Thus, any localisation of a topos is a topos. Moreover, in this way, one obtains a bijective correspondence between localisations, $\mathfrak{Loc}(\mathsf{Psh}(\mathbf{X}))$ and Grothendieck topologies $\mathsf{Top}(\mathbf{X})$. These facts are well-known and the reader is referred to [6, Propositions 3.5.4 and 3.5.7].

For a site $(\mathbf{X}, J)$, we write $\mathsf{Top}_J(\mathbf{X})$ to denote the subclass of Grothendieck topologies containing J. This class is in one-to-one correspondence to two other classes, which can be described entirely in terms of the topos $\mathscr{T} = \mathsf{Sh}(\mathbf{X}, J)$. To describe them, we recall the definition of Lawvere-Tierney topologies:

Lawvere-Tierney topology Let $\mathscr{T}$ be a topos. Since it is also an elementary topos, it has a subobject classifier. Recall that a *Lawvere-Tierney topology* on $\mathscr{T}$ is given by a morphism $j : \Omega \to \Omega$ such that $j \circ j = j$ and the following diagrams commute:

$$\begin{array}{ccc} 1 & \xrightarrow{t} & \Omega \\ & \searrow^{t} & \downarrow{\scriptstyle j} \\ & & \Omega \end{array} \qquad \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ \downarrow{\scriptstyle j\times j} & & \downarrow{\scriptstyle j} \\ \Omega \times \Omega & \xrightarrow{\wedge} & \Omega. \end{array}$$

Here, $\wedge : \Omega \times \Omega \to \Omega$ is the characteristic map of the subobject $1 = 1 \times 1 \xrightarrow{(t,t)} \Omega \times \Omega$. The collection of all Lawvere-Tierney topologies on $\mathscr{T}$ is denoted by $\mathfrak{LT}(\mathscr{T})$.

Let j be a Lawvere-Tierney topology on $\mathscr{T}$. An subobject $B \hookrightarrow A$ is called j-*dense* if the following diagram commutes

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{\chi_B} & & \searrow^{\chi_A} & \\ \Omega & & \xrightarrow{\quad j \quad} & & \Omega. \end{array}$$

A object F of the topos $\mathscr{T}$ is called a j-*sheaf*, if for any object A and any dense subobject $B \subseteq A$, the induced map

$$\mathsf{Hom}_{\mathscr{T}}(A, F) \to \mathsf{Hom}_{\mathscr{T}}(B, F)$$

is a bijection. Denote by $\mathsf{Sh}_j(\mathscr{T})$ the full subcategory of $\mathscr{T}$ consisting of j-sheaves. It is well-known that $\mathsf{Sh}_j(\mathscr{T})$ is also a topos and the inclusion $\mathsf{Sh}_j(\mathscr{T})\mathfrak{t}\mathscr{T}$ has a left adjoint functor, which commutes with finite limits.

Denote by $\mathfrak{LT}(\mathscr{T})$ the collection of all Lawvere-Tierney topologies on $\mathscr{T}$.

Lemma 6.1.5. *Let $(\mathbf{X}, J)$ be a site and $\mathscr{T} = \mathsf{Sh}(\mathbf{X}, J)$ the corresponding topos.*

i) For $K \in \mathsf{Top}_J(\mathbf{X})$, the category $\mathsf{Sh}(\mathbf{X}, K)$ is a localisation of $\mathscr{T}$ and the localisation functor is the composite

$$\mathscr{T} = \mathsf{Sh}(\mathbf{X}, J) \subseteq \mathsf{Psh}(\mathbf{X}) \to \mathsf{Sh}(\mathbf{X}, K),$$

where the last functor is the sheafification functor.

ii) If $\mathscr{D}$ is a localisation of $\mathscr{T}$ with localisation functor $\rho^\bullet : \mathscr{T} \to \mathscr{D}$, we denote by K the collection of all sieves $R \subseteq \mathsf{Hom}_{\mathbf{X}}(-, U)$ for which the induced morphism

$$\rho(R) \to \rho(\mathsf{Hom}_{\mathbf{X}}(-, U))$$

is an isomorphism. Then $K \in \mathsf{Top}_J(\mathbf{X})$.

iii) The correspondences defined in i) and ii) induces an order reversing bijection between $\mathfrak{Loc}(\mathscr{T})$ and $\mathsf{Top}_J(\mathbf{X})$.

iv) The result remains true if one uses the Lawvere-Tierney topologies on $\mathscr{T}$. That is, if j is a Lawvere-Tierney topology on $\mathscr{T}$, then $\mathsf{Sh}_j(\mathscr{T})$ is a localisation of $\mathscr{T}$. The localisation

functor in this case is given by sheafification. In this way, one obtains an order reversing bijection

$$\mathfrak{LT}(\mathscr{T}) \cong \mathfrak{Loc}(\mathscr{T}).$$

This fact is well-known, see for example [6, Propositions 3.5.4 and 3.5.7] for the bijection in iii) and [32, Exercise 2. Ch3] for the bijection in iv).

6.1.4 Giraud's theorem

Recall that a *Grothendieck topos*, or simply a topos, is a category $\mathscr{T}$, equivalent to the category of set valued sheaves over a site [42].

The category $\mathrm{Sh}(X)$ of all set valued sheaves over a topological space X is an example of a topos. Moreover, for any small category $\mathbf{X}$ the category of presheaves $\mathrm{Psh}(\mathbf{X})$ is a topos.

A famous theorem of Giraud [42] asserts that a category $\mathscr{T}$ is a topos if and only if the following axioms G1-G5 hold:

(G1) The category $\mathscr{T}$ has finite limits.

(G2) The category $\mathscr{T}$ has colimits and they commute with pullbacks.

(G3) Let A_i, $i \in I$ be a family of objects in $\mathscr{T}$. The following

$$\begin{array}{ccc} \varnothing & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ A_j & \longrightarrow & \coprod A_i \end{array}$$

is a pullback diagram for any $i, j \in I$ such that $i \neq j$. Throughout this dissertation $\varnothing$ denotes the initial object of $\mathscr{T}$.

(G4) Let $f : B \to A$ be an epimorphism and

$$\begin{array}{ccc} C & \xrightarrow{p_1} & B \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[f]{} & A \end{array}$$

a pullback diagram. The following

$$C \rightrightarrows B \xrightarrow{f} A$$

is a coequaliser.

(G5) There exist a set I and a family of objects G_i, $i \in I$, called *generators*, such that for any two parallel arrows $u, v : A \to B$ with $u \neq v$, there exist an $i \in I$ and an arrow $a : G_i \to A$ for which $ua \neq va$.

Lemma 6.1.6. *Let A be an object of a topos $\mathscr{T}$ and $B \subseteq A$. If every morphism $f : G_i \to A$ factors through B, then $B = A$.*

Proof. Let α be the inclusion $B \to A$. We need to show that α is an isomorphism. Let us recall that a morphism in a topos is an isomorphism, if and only if it is both a monomorphism and an epimorphism. To see that α is an epimorphism, let

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & A \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ A & \xrightarrow[\beta_2]{} & C \end{array}$$

be a pushout diagram. It suffices to show that $\beta_1 = \beta_2$. Take any morphism $\gamma : G_i \to A$. By assumption, $\gamma = \alpha\delta$ for some morphism $\delta : G_i \to B$. Hence

$$\beta_1\gamma = \beta_1\alpha\delta = \beta_2\alpha\delta = \beta_2\gamma,$$

and we obtain $\beta_1 = \beta_2$ since the objects G_i, $i \in I$ are generators. This finishes the proof. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

6.1.5 Geometric morphisms

A *geometric morphism*, or simply morphism, between topoi $f : \mathscr{T} \to \mathscr{T}_1$ consists of a pair of functors $f^\bullet : \mathscr{T}_1 \to \mathscr{T}$ and $f_\bullet : \mathscr{T} \to \mathscr{T}_1$, called the *inverse* and *direct image* respectively. These functors must satisfy the following two properties:

(i) $f^\bullet \dashv f_\bullet$, that is $f^\bullet$ is the left adjoint of $f_\bullet$, (in particular $f^\bullet$ commutes with all colimits, while $f_\bullet$ commutes with all limits).

(ii) $f^\bullet$ commutes with finite limits.

The adjoint $f_\bullet$ (resp. $f^\bullet$) of a given functor $f^\bullet$(resp. $f_\bullet$), if it exists, is determined uniquely up to an isomorphism. As such, we can also define a geometric morphism $f : \mathscr{T} \to \mathscr{T}_1$ to simply be a functor $f^\bullet : \mathscr{T}_1 \to \mathscr{T}$ that commutes with all colimits and with finite limits, and that has a left adjoint. There are many technical advantages in looking at geometric morphisms in this way, and we shall do so many times throughout this thesis.

We mention, as an example, that any continuous map $f : X \to Y$ of topological spaces yields a pair of adjoint functors $(f^\bullet, f_\bullet) : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$, defining a geometric morphism of topoi $\mathsf{Sh}(X) \to \mathsf{Sh}(Y)$, which is still denoted by f. Let $\mathcal{F}$ be a sheaf over X and $\mathcal{G}$ a sheaf over Y. Recall that $f_\bullet(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ and $f^\bullet(\mathcal{G})_x = \mathcal{G}_{f(x)}$ for any open subset $V \subseteq Y$ and any $x \in X$.

Let $\mathbf{X}$ and $\mathbf{Y}$ be small categories and assume there is given a functor $f : \mathbf{X} \to \mathbf{Y}$. Then precomposition with f induces a functor

$$f^\bullet : \mathsf{Psh}(\mathbf{Y}) \to \mathsf{Psh}(\mathbf{X}).$$

Thus, for any presheaf $F : \mathbf{Y}^{op} \to \mathrm{Sets}$ we have

$$f^\bullet(F) = F \circ f.$$

It is well-known that $f^\bullet$ has both left and right adjoint functors. Denote them by $f_!$ and $f_\bullet$ respectively. Meaning

$$f_! \dashv f^\bullet \dashv f_\bullet.$$

They are known as left and right Kan extensions. It follows that $f^\bullet$ respects all limits and colimits. In particular, the pair $(f_\bullet, f^\bullet)$ is a geometric morphism

$$\mathrm{Psh}(\mathbf{X}) \to \mathrm{Psh}(\mathbf{Y}).$$

Having this example in mind, we call a geometric morphism $f = (f_\bullet, f^\bullet) : \mathscr{T} \to \mathscr{T}_1$ *essential* if the inverse image functor $f^\bullet$ has a left adjoint, usually denoted by $f_! : \mathscr{T} \to \mathscr{T}_1$.

By comparing the definition, we see that any localisation $f^\bullet : \mathscr{T} \to \mathscr{T}'$ is the inverse image part of a geometric morphism $\mathscr{T}' \to \mathscr{T}$. In this case, additionally $f^\bullet \circ f_\bullet \cong \mathrm{Id}$ holds. Geometric morphisms with the last property are known as *embeddings* [42]. We shall, however, stick with the notation of [20], and refer to them as localisations of topoi. Recall also the following basic fact:

Lemma 6.1.7. *Let U be an open subset of a topological space X. The inclusion $j : U \to X$ induces a localisation of categories $j^\bullet : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$.*

Topoi and morphisms of topoi form a 2-category Top. The 2-morphisms (or 2-cells) between morphism are natural transformations between the inverse image functors. The following is a classical result:

Lemma 6.1.8. *The category* Sets *is the 2-terminal object in this 2-category.*

Proof. Assume we have a geometric morphism $f = (f_\bullet, f^\bullet) : \mathscr{T} \to \mathrm{Sets}$. The functor $f^\bullet : \mathrm{Sets} \to \mathscr{T}$ respects all coproducts and the terminal object since $f^\bullet$ preserves finite limits. It follows that for any set X, we have

$$f^\bullet(X) = \coprod_{x \in X} f^\bullet(\{x\}) \cong \coprod_{x \in X} 1$$

Thus, $f^\bullet$ is unique up to isomorphism and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Points of topoi A *point* of a topos $\mathscr{T}$ is a geometric morphism $p = (p^\bullet, p_\bullet) : \mathrm{Sets} \to \mathscr{T}$. The points of a topos $\mathscr{T}$ form a category, which we denote by $\mathbf{Pts}(\mathscr{T})$. The isomorphism classes of $\mathbf{Pts}(\mathscr{T})$ will be denoted by $\mathsf{F}_{\mathscr{T}}$. Any morphism of topoi $f = (f^\bullet, f_\bullet) : \mathscr{T} \to \mathscr{T}_1$ yields the functor $\mathbf{Pts}(f) : \mathbf{Pts}(\mathscr{T}) \to \mathbf{Pts}(\mathscr{T}_1)$, which in turn induces the map $\mathsf{F}_\mathsf{f} : \mathsf{F}_{\mathscr{T}} \to \mathsf{F}'_{\mathscr{T}}$.

Lemma 6.1.9. *Let $j = (j^\bullet, j_\bullet) : \mathscr{T} \to \mathscr{T}_1$ be a geometric morphism of topoi, for which $j^\bullet$ is a localisation of categories. The induced map $\mathsf{F}_\mathsf{j} : \mathsf{F}_{\mathscr{T}} \to \mathsf{F}_{\mathscr{T}_1}$ is injective.*

Proof. Take two points $p = (p^\bullet, p_\bullet)$ and $q = (q^\bullet, q_\bullet)$ of the topos $\mathscr{T}$. Assume there exist an isomorphism of functors $\xi : p^\bullet j^\bullet \to q^\bullet j^\bullet$. It induces an isomorphism $p^\bullet j^\bullet j_\bullet \simeq q^\bullet j^\bullet j_\bullet$. The result follows, since $j^\bullet j_\bullet \simeq \mathrm{Id}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Diaconescu's theorem Let $\mathbf{X}$ be a small category. A covariant functor $\mathbf{X} \xrightarrow{F}$ Sets is called *filtered*, if the following properties hold

- There is an object X of $\mathbf{X}$ such that $F(X) \neq \varnothing$.
- Given elements $x \in F(X)$ and $y \in F(Y)$, there exist an object Z, morphisms $u : Z \to X$, $v : Z \to Y$, and an element $z \in F(Z)$ such that $F(u)(z) = x$ and $F(v)(z) = y$.
- Given two parallel arrows $u, v : X \to Y$ in $\mathbf{X}$ and an $x \in F(X)$, such that $F(u)(x) = F(v)(x)$, there is a morphism $w : Z \to X$ and an element $z \in F(Z)$ such that $uw = vw$ and $F(w)(z) = x$.

Now we are in a position to state the following theorem of Diaconescu [32].

Theorem 6.1.10. *For any small category* $\mathbf{X}$*, the category of filtered covariant functors* $\mathbf{X} \to$ Sets *is equivalent to the category* $\mathbf{Pts}(\mathrm{Psh}(\mathbf{X}))$.

6.2 Locally s-Noetherian topoi

We aim to introduce a topos theoretical analogue of the classical locally Noetherian abelian categories, see [20], called locally s-Noetherian topoi. We will prove that gluing of such topoi is again a locally s-Noetherian topos. Most of the material discussed in this section is a topos theoretical analogue of the technique developed by Gabriel in [20, Chapter 4].

Recall that an object in an abelian category is Noetherian, if it satisfies the ascending chain condition for its subobjects. A basic, but fundamental, property of abelian categories is the one to one correspondence between subobjects and quotients. Hence, Noetherianness can be equivalently defined in terms of quotient objects.

This is no longer true for topoi as there are congruences that are not induced by subobjects. As such, there are two non-equivalent analogues of a Noetherian object in topoi. One is based on subobjects and the other on quotients. We shall only study the first concept in this dissertation. We will refer to them as s-Noetherian objects to make this distinction clear.

An object A of a topos $\mathscr{T}$ is called *s-Noetherian* if for any ascending sequence of subobjects

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A$$

of A, there is an integer n such that $A_k = A_{k+1}$ for all $k \geq n$.

Lemma 6.2.1. *i) An object A of a topos $\mathscr{T}$ is s-Noetherian if and only if any family of subobjects $(B_\lambda \subseteq A)_{\lambda \in \Lambda}$ has a maximal element.*

ii) Let $f : A \to B$ be a monomorphism in $\mathscr{T}$. If B is s-Noetherian, then A is s-Noetherian.

iii) Let $f : A \to B$ be an epimorphism in $\mathscr{T}$. If A is s-Noetherian, then B is s-Noetherian.

iv) If A and B are s-Noetherian, then $A \coprod B$ is s-Noetherian.

v) If A and B are s-Noetherian subobjects of C, then $A \cup B$ is an s-Noetherian subobject of C.

Proof. i) Assume A is s-Noetherian and the family (B_λ) does not contain a maximal element. Choose $\lambda_1 \in \Lambda$ and consider the corresponding subobject B_{λ_1}. It is not maximal by our assumption. Thus, there exists $\lambda_2 \in \Lambda$ with $B_{\lambda_1} \subsetneq B_{\lambda_2}$. We obtain a non-stabilising, ascending sequence of subobjects of A. This contradicts our assumption.

Conversely, assume the maximality condition holds and let $A \in \mathscr{T}$ be an object. Consider any ascending sequence of subobjects

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A.$$

This family has a maximal element by assumption. Call it A_n. We have $A_k \subseteq A_n$ for all k. Thus, A is s-Noetherian since $A_n = A_{n+1} = \cdots$.

ii) This is trivial since any subobject of A is also a subobject of B.

iii) Take any ascending sequence of subobjects

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B.$$

By assumption, there exists a natural number n such that $f^{-1}(B_n) = f^{-1}(B_{n+1}) = \cdots$. We have $ff^{-1}(B_k) = B_k$ since f is an epimorphism. The result follows.

iv) Let

$$C_1 \subseteq C_2 \subseteq \cdots \subseteq C = A \coprod B$$

be any ascending sequence of subobjects of $A \coprod B$. Set $A_i = C_i \cap A$ and $B_i = C_i \cap B$. Both (A_i) and (B_i) form ascending sequences of subobjects of A and B, respectively. As such, they stabilise. It follows that $A_n = A_{n+1} = \cdots$ and $B_n = B_{n+1} = \cdots$ for some n. Since $C_k = A_k \coprod B_k$, one obtains $C_n = C_{n+1} = \cdots$, and the result follows.

v) The last statement follows from iv) and iii). $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Definition 6.2.2. *A topos $\mathscr{T}$ is said to be* locally s-Noetherian*, provided it posses a family of s-Noetherian generators.*

Remark 6.2.3. *The product of two s-Noetherian objects in a locally s-Noetherian topos is not s-Noetherian in general, see Example 7.2.8 below.*

Lemma 6.2.4. *Let $\mathscr{T}$ be a locally s-Noetherian topos. One has $A = \bigcup_N N$ for any object A of $\mathscr{T}$. Here, N runs through all the s-Noetherian subobjects of A.*

Proof. We set $B = \bigcup_N N$, where N runs through all the s-Noetherian subobjects of A. It is clear that $B \subseteq A$. Take any Noetherian generator G_i and a morphism $\alpha : G_i \to A$. We know from Lemma 6.2.1, part iii) that $\mathrm{Im}(\alpha)$ is an s-Noetherian subobject of A. Hence, $\mathrm{Im}(\alpha) \subseteq B$ and we obtain $B = A$, by Lemma 6.1.6. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 6.2.5. *Let $\mathscr{T}$ be a locally s-Noetherian topos and $(G_s)_{s \in S}$ a family of s-Noetherian generators. Any s-Noetherian object is a quotient of a finite coproduct of G_s, $s \in S$.*

Proof. Let N be an s-Noetherian object in $\mathscr{T}$. Consider the collection of all subobjects of N, which are quotients of finite coproducts of the G_s-s. This collection contains a maximal subobject $M \subseteq N$, thanks to Lemma 6.2.1, as N is s-Noetherian. Note that it does not need to be unique. Hence, there is a finite subset $R \subseteq S$ with an epimorphism $\alpha : \coprod_{t\in R} G_t \to M$. We claim that $M = N$.

If this is not the case, there exists a morphism $\beta : G_i \to N$ which does not factor trough M. This follows from Lemma 6.1.6. We obtain a map $f = (\alpha, \beta) : (\coprod_{t\in R} G_t) \coprod G_i \to N$ such that $M \subsetneq \mathrm{Im}(f)$ is a proper inclusion. This contradicts the maximality of M, proving our result. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 6.2.6. *The isomorphism classes of s-Noetherian objects in a locally s-Noetherian topos form a set.*

Lemma 6.2.7. *Let $\mathscr{T}_1$ be a locally s-Noetherian topos and $j^\bullet : \mathscr{T}_1 \to \mathscr{T}_0$ a localisation of categories.*

i) The topos $\mathscr{T}_0$ is locally s-Noetherian.

ii) Let B be any object of $\mathscr{T}_1$ and $A \subseteq j^\bullet(B)$ any s-Noetherian subobject. There is an s-Noetherian subobject $N \subseteq B$ such that $j^\bullet(N) = A$.

Proof. i) Denote by $\xi : Id_{\mathscr{T}_1} \to j_\bullet j^\bullet$ the unit of the adjoint pair $(j^\bullet, j_\bullet)$. Let X be an object of $\mathscr{T}_1$. It follows from the general properties of adjoint functors that we have a morphism $\xi_X : X \to j_\bullet j^\bullet X$ in $\mathscr{T}_1$ satisfying the following: For any object Y of $\mathscr{T}_0$ and any morphism $\alpha : X \to j_\bullet Y$, there is a unique morphism $\eta : j^\bullet X \to Y$ in $\mathscr{T}_0$, such that $\alpha = j_\bullet(\eta)\xi_X$. We refer to this property as the universality of ξ.

The counit $j^\bullet j_\bullet \to Id_{\mathscr{T}_0}$ is an isomorphism since $j_\bullet$ is full and faithful. This allows us to identify $j^\bullet j_\bullet Y$ with Y for any object Y of $\mathscr{T}_0$.

Take an object $N \in \mathscr{T}_1$, any subobject $Y \subseteq j^\bullet(N)$ and consider the following pullback diagram in $\mathscr{T}_1$:

$$\begin{array}{ccc} \xi_N^{-1}(j_\bullet(Y)) = X & \xrightarrow{i} & N \\ \downarrow & & \downarrow{\scriptstyle \xi_N} \\ j_\bullet(Y) & \longrightarrow & j_\bullet j^\bullet(N). \end{array}$$

Our first claim is that one has $j^\bullet(\xi_N^{-1}(j_\bullet(Y))) = Y$. We show this by applying $j^\bullet$ to the above diagram to obtain the following commutative diagram:

$$\begin{array}{ccc} j^\bullet(X) & \xrightarrow{j^\bullet(i)} & j^\bullet(N) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ j^\bullet(j_\bullet(Y)) & \longrightarrow & j^\bullet(j_\bullet j^\bullet(N)) = j^\bullet(N). \end{array}$$

Since $j^\bullet$ respects all finite limits, this is in fact a pull-back diagram. We observe that the right vertical arrow is an isomorphism. It follows that the left vertical arrow $j^\bullet(X) \to j^\bullet j_\bullet Y = Y$ is an isomorphism as well since this is a pullback diagram. This proves the first claim.

Let $(G_s)_{s\in S}$ be a family of generators of $\mathscr{T}_1$. Our second claim is that $(j^\bullet(G_s))_{s\in S}$ are generators of $\mathscr{T}_0$. Take two morphisms $\alpha, \beta : A \to B$ in $\mathscr{T}_0$ and assume $\alpha \neq \beta$. Since $j_\bullet$ is full and faithful, $j_\bullet(\alpha) \neq j_\bullet(\beta)$. The collection $\{G_s\}_{s\in S}$ is a family of generators, which means that there exists a morphism $\gamma : G_s \to j_\bullet A$ such that $j_\bullet(\alpha)\gamma \neq j_\bullet(\beta)\gamma$. We have $\gamma = j_\bullet(\delta)\xi_{G_s}$ for some $\delta : j^\bullet(G_s) \to A$, by the universality property of ξ. Thus, $j_\bullet(\alpha\delta)\xi_{G_s} \neq j_\bullet(\beta\delta)\xi_{G_s}$ and hence, $\alpha\delta \neq \beta\delta$. This proves the second claim.

Our third claim is that the image $j^\bullet(N)$ of an s-Noetherian object $N \in \mathscr{T}_1$ is an s-Noetherian object in $\mathscr{T}_0$. We consider the sequence of subobjects

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq j^\bullet(N)$$

in $\mathscr{T}_0$. Consider the corresponding sequence of subobjects of N in $\mathscr{T}_1$

$$\xi_N^{-1}(j_\bullet(A_1)) \subseteq \xi_N^{-1}(j_\bullet(A_2)) \subseteq \cdots \subseteq N.$$

Since N is s-Noetherian, there exist $n \geq 1$ such that $\xi_N^{-1}(j_\bullet(A_i)) = \xi_N^{-1}(j_\bullet(A_{i+1}))$, for all $i \geq n$. It follows from the first claim that $A_i = A_{i+1}$ as well. Thus, $j^\bullet(N)$ is s-Noetherian. This, in conjunction with the second claim, implies the result.

ii) Take any object B of $\mathscr{T}_1$ and any s-Noetherian subobject $A \subseteq j^\bullet(B)$. Consider the family of subobjects $Y \subseteq A$ for which there exists an s-Noetherian subobject $X \subseteq \xi_B^{-1}(j_\bullet A)$ such that $j^\bullet(X) = Y$. This family has a maximal object Y' since A is s-Noetherian by i) Lemma 6.2.1. So, $Y' = j^\bullet(X')$ for an s-Noetherian subobject $X' \subseteq \xi_B^{-1}(j_\bullet A)$. We have to prove that $Y' = A$.

Since $j_\bullet$ is full and faithful, it suffice to show that $j_\bullet Y' = j_\bullet A$. Assume it is not. Then $\xi_B^{-1} j_\bullet(Y') \neq \xi_B^{-1}(j_\bullet(A))$ and by Lemma 6.2.4, there exists an s-Noetherian subobject $Z \subseteq \xi^{-1}(j_\bullet(A))$ such that Z does not lie in $\xi^{-1}(j_\bullet(Y'))$. We set $W = Z \cup X' \subseteq \xi^{-1}(j_\bullet(A))$, which is s-Noetherian by v) Lemma 6.2.1. It follows that $j^\bullet(W)$ is also a member of the above collection, which contradicts the maximality of Y'. This shows that $Y' = A$ and the lemma follows. $\mathcal{Q.E.D}$

6.2.1 Gluing of topoi

Let $\mathscr{T}_i$, $i = 0, 1, 2$ be topoi and $j_i^\bullet : \mathscr{T}_i \to \mathscr{T}_0$, $i = 1, 2$ localisations of categories. The *gluing* (compare with [20, p. 439]) of $\mathscr{T}_1$ and $\mathscr{T}_2$ along $\mathscr{T}_0$ is defined to be the 2-pullback, which is just a special case of the 2-limit defined in Section 1.2.2.

As a reminder, this category is defined as follows: Its objects are triples (A_1, A_2, α), where A_i is an object of $\mathscr{T}_i$, $i = 1, 2$ and $\alpha : j_1^\bullet(A_1) \to j_2^\bullet(A_2)$ is an isomorphism in $\mathscr{T}_0$. A morphism $(A_1, A_2, \alpha) \to (B_1, B_2, \beta)$ in $\mathscr{T}$ is a pair (ξ_1, ξ_2), where $\xi_i : A_i \to B_i$ is a morphism in $\mathscr{T}_i$, $i = 1, 2$ such that $j^\bullet(\xi_2)\alpha = \beta j^\bullet(\xi_1)$. We have the following diagram

$$\begin{array}{ccc} \mathscr{T} & \xrightarrow{\pi_1^\bullet} & \mathscr{T}_1 \\ {\scriptstyle \pi_2^\bullet}\downarrow & & \downarrow{\scriptstyle j_1^\bullet} \\ \mathscr{T}_2 & \xrightarrow[j_2^\bullet]{} & \mathscr{T}_0, \end{array}$$

where the projection $\pi_i^\bullet : \mathscr{T} \to \mathscr{T}_i$ is given by $\pi_i^\bullet(A_1, A_2, \alpha) = A_i$, $i = 1, 2$.

6.2.2 Gluing of locally s-Noetherian topoi

Theorem 6.2.8. *Assume $\mathscr{T}_1$ and $\mathscr{T}_2$ are locally s-Noetherian topoi and the category $\mathscr{T}$ is the gluing of $\mathscr{T}_1$ and $\mathscr{T}_2$ along the topos $\mathscr{T}_0$:*

$$\begin{array}{ccc} \mathscr{T} & \xrightarrow{\pi_1^\bullet} & \mathscr{T}_1 \\ {\scriptstyle \pi_2^\bullet}\downarrow & & \downarrow{\scriptstyle j_1^\bullet} \\ \mathscr{T}_2 & \xrightarrow[j_2^\bullet]{} & \mathscr{T}_0. \end{array}$$

Here, $j_1^\bullet$ and $j_2^\bullet$ are localisations of categories. Then $\mathscr{T}$ and $\mathscr{T}_0$ are locally s-Noetherian topoi and the functors $\pi_i^\bullet$, $i = 1, 2$ are localisations.

Proof. We showed in part i) of Lemma 6.2.7 that $\mathscr{T}_0$ is a locally s-Noetherian topos. One easily sees that $\mathscr{T}$ posses all colimits and finite limits and that the functors $\pi_i^\bullet$, $i = 1, 2$ preserve colimits and finite limits. Based on this, one easily checks that the Giraud Axioms G1-G4 hold. To check Axiom G5 we proceed as follows:

The functors $j_i^\bullet : \mathscr{T}_i \to \mathscr{T}_0$, $i = 1, 2$ have, by assumption, right adjoint functors $j_{\bullet i} : \mathscr{T}_0 \to \mathscr{T}_i$, $i = 1, 2$. We may assume, without loss of generality, that $j_i^\bullet j_{\bullet i}(A_0) = A_0$ for any object A_0 of $\mathscr{T}_0$. Let $\xi_i : \mathsf{Id}_{\mathscr{T}_i} \to j_{\bullet i} j_i^\bullet$ be the unit of the adjoint pair $(j_i^\bullet, j_{\bullet i})$, $i = 1, 2$. Define the functors

$$\pi_{\bullet i} : \mathscr{T}_i \to \mathscr{T}, \quad i = 1, 2$$

by

$$\pi_{\bullet 1}(A_1) = (A_1, j_{\bullet 2}(j_1^\bullet(A_1)), \mathsf{Id}_{j_1^\bullet(A_1)})$$

and

$$\pi_{\bullet 2}(A_2) = (j_{\bullet 1}(j_2^\bullet(A_2)), A_2, \mathsf{Id}_{j_2^\bullet(A_2)}).$$

This makes $\pi_{\bullet i}$ the right adjoint of $\pi_i^\bullet$, for $i = 1, 2$. The latter are localisations since $\pi_i^\bullet \pi_{\bullet i} = \mathsf{Id}_{\mathscr{T}_i}$, $i = 1, 2$.

It is obvious that (A_1, A_2, α) is an s-Noetherian object in $\mathscr{T}$, provided both A_1 and A_2 are s-Noetherian in $\mathscr{T}_i$. We claim that the converse is also true. We will need the following construction for the proof:

Let (A_1, A_2, α) be an object of $\mathscr{T}$ and let $X \subseteq A_1$ be a subobject. Then (X, Y, γ) is a subobject of (A_1, A_2, α) in $\mathscr{T}$. Here, $Y = \xi_2^{-1}(\alpha(j_1^\bullet(X)))$ and γ is the restriction of α on $j_1^\bullet(X)$. It is clear from this construction that any ascending chain of subobjects of A_1 can be lifted as an ascending chain of subobjects of (A_1, A_2, α). Thus, A_1 is s-Noetherian in $\mathscr{T}_1$ if (A_1, A_2, α) is s-Noetherian in $\mathscr{T}$. Similarly for A_2. We have proven that an object (A_1, A_2, α) of the category $\mathscr{T}$ is s-Noetherian if and only if A_1 and A_2 are s-Noetherian in $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. It follows that the isomorphism classes of s-Noetherian objects of $\mathscr{T}$ form a set.

Hence, it suffices to show that s-Noetherian objects generate $\mathscr{T}$. Take any object (A_1, A_2, α) in $\mathscr{T}$ and a strict subobject (B_1, B_2, β) of (A_1, A_2, α). Either $B_1 \neq A_1$ or $B_2 \neq A_2$ and we

may assume, without loss of generality, that we are in the first situation. In other words, that there exist an s-Noetherian subobject $N_1 \subseteq A_1$, which does not lie in B_1. There exist an s-Noetherian subobject $N_2 \subseteq A_2$ such that $j_2^\bullet(N_2) = \alpha(j_1^\bullet(N_1))$ by part ii) of Lemma 6.2.7. We have constructed an s-Noetherian subobject $(N_1, N_2, \alpha_{|j_1^\bullet(N_1)})$ of (A_1, A_2, α) which does not factors through (B_1, B_2, β). This shows that s-Noetherian objects generate the category $\mathscr{T}$, which finishes the proof. $\mathcal{Q.E.D}$

6.2.3 Centre of a category and gluing

Recall that the *centre* $\mathcal{Z}(\mathsf{A})$ of a category A is the monoid of all natural transformation $\mathrm{Id}_\mathsf{A} \to \mathrm{Id}_\mathsf{A}$ of the identity functor to itself. It is well-known that the monoid $\mathcal{Z}(\mathsf{A})$ is commutative.

Thus, an element θ of the centre $\mathcal{Z}(\mathsf{A})$ of A is a collection of morphisms $\theta_A : A \to A$, where A is running through all the objects of A such that for any morphism $f : a \to b$ of the category A, the diagram

$$\begin{array}{ccc} a & \xrightarrow{\theta_a} & a \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ b & \xrightarrow{\theta_b} & b \end{array}$$

commutes.

Let $\mathscr{T}$ be a topos. If $\mathscr{D} \subseteq \mathscr{D}'$ are localisations of $\mathscr{T}$ and θ is an element of $\mathcal{Z}(\mathscr{D}')$, then the collection $(\theta_x)_{x\in\mathscr{D}}$ belongs to $\mathcal{Z}(\mathscr{D})$. Hence, one obtains the monoid homomorphism $\mathcal{Z}(\mathscr{D}') \to \mathcal{Z}(\mathscr{D})$, called *restriction*. This allows us to define a functor

$$\mathcal{Z} : \mathfrak{Loc}(\mathscr{T})^{op} \to \mathsf{ComMn},$$

where ComMn denotes the category of commutative monoids.

Proposition 6.2.9. *Let $\mathscr{T}_i$, $i = 0, 1, 2$ be locally s-Noetherian topoi, $\rho_i^* : \mathscr{T}_i \to \mathscr{T}_0$, $i = 1, 2$ localisations and $\mathscr{T}$ be the corresponding gluing topos. Thus we have the following diagram*

$$\begin{array}{ccc} \mathscr{T} & \xrightarrow{pr_1} & \mathscr{T}_1 \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \rho_1^*} \\ \mathscr{T}_2 & \xrightarrow[\rho_2^*]{} & \mathscr{T}_0, \end{array}$$

where $pr_i : \mathscr{T} \to \mathscr{T}_i$ is given by $pr_i(A_1, A_2, \alpha) = A_i$, $i = 1, 2$. Then the following is a pull-back diagram

$$\begin{array}{ccc} \mathcal{Z}(\mathscr{T}) & \longrightarrow & \mathcal{Z}(\mathscr{T}_1) \\ \downarrow & & \downarrow \\ \mathcal{Z}(\mathscr{T}_2) & \longrightarrow & \mathcal{Z}(\mathscr{T}_0). \end{array}$$

Proof. The functors $\rho_i^* : \mathscr{T}_i \to \mathscr{T}_0$ have, by assumption, right adjoint functors $\rho_{i*} : \mathscr{T}_0 \to \mathscr{T}_i$, $i = 1, 2$. Denote by $\kappa_i : \mathrm{Id}_{\mathscr{T}_i} \to \rho_{i*}\rho^*$, $i = 1, 2$ the unite of the adjoints. We may assume,

without loss of generality, that $\rho_i^*\rho_{i*}(A_0) = A_0$ for any object A_0 of $\mathscr{T}_0$. Hence, the categories $\mathscr{T}_0$, $\mathscr{T}_1$ and $\mathscr{T}_2$ can be identified respectively with the following full subcategories of $\mathscr{T}$:

$$\{(\rho_{1*}(A_0), \rho_{2*}(A_0), \mathsf{Id}_{A_0}) \mid A_0 \in \mathscr{T}_0\},$$

$$\{(A_1, \rho_{2*}\rho_1^*(A_1), \mathsf{Id}_{\rho_{1*}A_1}) \mid A_1 \in \mathscr{T}_1\},$$

$$\{(\rho_{1*}\rho_2^*(A_2), A_2, \mathsf{Id}_{\rho_{2*}A_2}) \mid A_2 \in \mathscr{T}_2\}.$$

We have to show that for any natural transformations $\xi_i : \mathsf{Id}_{\mathscr{T}_i} \to \mathsf{Id}_{\mathscr{T}_i}$, $i = 1, 2$ such that

$$\xi_1(\rho_{1*}(A_0)) = \xi_2(\rho_{2*}(A_0))$$

for every object A_0 of $\mathscr{T}_0$, there exists a unique natural transformation $\theta : \mathsf{Id}_{\mathscr{T}} \to \mathsf{Id}_{\mathscr{T}}$ for which

$$\theta(A_1, \rho_{2*}\rho_1^*(A_1), \mathsf{Id}_{\rho_{1*}A_1}) = (\xi_1(A_1), \rho_{2*}\rho_1^*\xi_1(A_1))$$

and

$$\theta(\rho_{1*}\rho_2^*(A_2), A_2, \mathsf{Id}_{\rho_{2*}A_2}) = (\rho_{1*}\rho_2^*\xi_2(A_2), \xi_2(A_2))$$

holds for every $A_i \in \mathscr{T}_i$, $i = 1, 2$.

To this end, take any object (A_1, A_2, α) of $\mathscr{T}$. Since $\theta(A_1, A_2, \alpha)$ must be a morphism in $\mathscr{T}$, we have

$$\theta(A_1, A_2, \alpha) = (\theta_1(A_1, A_2, \alpha), \theta_2(A_1, A_2, \alpha)),$$

where

$$\theta_1(A_1, A_2, \alpha) : A_1 \to A_1 \quad \text{and} \quad \theta_2(A_1, A_2, \alpha) : A_2 \to A_2$$

are morphisms in $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. We have the following morphism in $\mathscr{T}$:

$$(\mathsf{Id}_{A_1}, \rho_2^*(\alpha^{-1}) \circ \kappa_2(A_2)) : (A_1, A_2, \alpha) \to (A_1, \rho_{2*}\rho_1^*(A_1), \mathsf{Id}_{\rho_{1*}A_1}).$$

Applying the naturality of θ for this morphism, we obtain

$$\theta_1(A_1, A_2, \alpha) = \xi_1(A_1).$$

Similarly, we obtain $\theta_2(A_1, A_2, \alpha) = \xi_2(A_2)$. This, not only proves the uniqueness of θ, but also yields the method to construct it and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Chapter 7

Monoids and topoi

7.1 The topos of M-sets

This section deals with monoids and sets endowed with a monoid action. We will collect some well-known properties about these objects, which will be used in the forthcoming sections.

7.1.1 Monoids and the topos of M-sets

7.1.1.1 Notations

Having only discussed commutative monoids until now, we will slowly start moving towards the non-commutative setting as well. As such, even though we will for a while still prioritise the commutative case, we wish to introduce the following definitions in the general case.

As always, monoids will be written in the multiplicative way. Likewise, the subgroup of invertible elements of M will be denoted by $M^\times$. A subset $I \subseteq M$ is called a *right ideal* provided $IM \subseteq I$. *Left ideals* are defined in the expected way. An *ideal* of M is a subset I which is simultaneously a left and a right ideal.

A *right* M-set is a set A, together with a map $A \times M \to A$, $(a, m) \mapsto am$ such that

$$a1 = 1, \quad \text{and} \quad a(mn) = (am)n, \quad m, n \in M, a \in A.$$

A *left* M*-set* has a similar meaning. It is clear that right (resp. left) ideals of M are right (resp. left) M-sets. Denote the category of left and right M-sets by ${}_M\mathscr{S}$ and $\mathscr{S}_M$ respectively. Observe that ${}_M\mathscr{S} \simeq \mathscr{S}_{M^{op}}$, where M^{op} is the opposite monoid. Likewise, ${}_M\mathscr{S} \simeq \mathscr{S}_M^{op}$, where the latter denotes the dual category.

The category $\mathscr{S}_M$ (resp. ${}_M\mathscr{S}$) posses all limits and colimits. It is well-known that the forgetful functor from $\mathscr{S}_M$ (resp. ${}_M\mathscr{S}$) to $\mathscr{S}$ the category of sets preserves all limits and colimits. In particular, the coproduct is given by the disjoint union.

Recall also the construction of the tensor product of M-sets. Let A be a right M-set and B a left M-set. We denote by $A \otimes_M B$ the set $(A \times B)/\sim$, where $\sim$ is the equivalence relation generated by $(am, b) \sim (a, mb)$. Here, $a \in A, b \in B$ and $m \in M$. The class of (a, b) in $A \otimes_M B$ is denoted by $a \otimes b$. We have $am \otimes b = a \otimes mb$.

We also have the bijection

$$\mathsf{Hom}_{\mathscr{S}}(A \otimes_M B, C) \simeq \mathsf{Hom}_{\mathscr{S}_M}(A, \mathsf{Hom}_{\mathscr{S}}(B, C))$$

for any set C. We note that the M-set structure of $\mathsf{Hom}_{\mathscr{S}}(B, C)$ is given by

$$(f \cdot m)(b) = f(mb), \quad m \in M, b \in B, f \in \mathsf{Hom}_{\mathscr{S}}(B, C).$$

It follows that the functor

$$(-) \otimes_M B : \mathscr{S}_M \to \mathscr{S}$$

has a right adjoint functor $\mathsf{Hom}_{\mathscr{S}}(B, -)$ for any left M-set B. Hence, it commutes with arbitrary colimits. One can easily check that conversely, any covariant functor $F : \mathscr{S}_M \to \mathscr{S}$ commuting with all colimits is isomorphic to a functor of the type $(-) \otimes_M B$. Here, $B = F(M)$ as a set, while a left M-set structure on B is induced by $mb := F(l_m)(b)$, where $b \in F(M), m \in M$ and $l_m : B \to B$ is the homomorphism given by $l_m(b) = mb$.

For a monoid M we can consider a one object category, whose endomorphisms are M. We denote this category by $*_M$. It is readily observed that a presheaf on this category is the same thing as a right M-set. Thus, $\mathscr{S}_M = \mathsf{Psh}(*_M)$. This allows to apply the general theory of presheaves to M-sets. For example, we see that $\mathscr{S}_M$ is a topos [42].

7.1.1.2 Monoid homomorphisms and morphisms of topoi

Let $f : M \to N$ be a monoid homomorphism. We can consider a right N-set A as a right M-set via this homomorphism. We denote it by $f^\bullet(A)$. Similarly for left M-sets. In particular, N itself can be seen as both a right and a left M-set. Thus, we can form

$$f_\bullet(A) = \mathsf{Hom}_{\mathscr{S}_M}(N, A) \quad \text{and} \quad f_!(A) = A \otimes_M N$$

for any right M-set A. Both of these sets have right N-set structures given by

$$(g \cdot n)(x) = g(nx) \quad \text{and} \quad (a \otimes n)x := a \otimes (nx)$$

respectively. Here, $a \in A, n, x \in N$ and $g \in \mathsf{Hom}_{\mathscr{S}_M}(N, A)$. These N-sets define left and right adjoint functors of the functor $f^\bullet$, that is $f_! \vdash f^\bullet \vdash f_\bullet$. In particular, the pair $(f^\bullet, f_\bullet)$ defines a morphism of topoi $\mathscr{S}_M \to \mathscr{S}_N$, which will be denote by $\mathscr{S}_f$. Actually, it is an essential geometric morphism. We obtain a (pseudo)functor or 2-functor

$$\mathsf{Monoids} \to \mathsf{Top},$$

which assigns the topos $\mathscr{S}_M$ to a monoid M.

7.1.1.3 Subobject classifier for the topos of M-sets

In this section, we will recall the construction of the subobject classifier of the topos of right M-sets $\mathscr{S}_M$, where M is a multiplicatively written monoid. This is just specialisation of

Proposition 6.1.3, see also [42].

Elements of the set Ω (also denoted as Ω^M, to make explicit the role of the monoid M) are right ideals of M. In order to describe the action of M on Ω, for a right ideal $\mathfrak{m}$ and an element $a \in M$, we set

$$(\mathfrak{m} : a) = \{x \in M | ax \in \mathfrak{m}\}.$$

Lemma 7.1.1. *In the notation above, we have that*

i) $(\mathfrak{m} : a)$ *is a right ideal;*

ii) $(\mathfrak{m} : 1) = \mathfrak{m}$*;*

iii) $((\mathfrak{m} : a) : b) = (\mathfrak{m} : ab)$*;*

iv) $(M : a) = M$*.*

Proof. i) Take any $m \in M$ and $x \in (\mathfrak{m} : a)$. Then $ax \in \mathfrak{m}$. Since $\mathfrak{m}$ is a right ideal, $(ax)m \in \mathfrak{m}$, implying $xm \in (\mathfrak{m} : a)$.

ii) and iv) are obvious.

iii) For an element $x \in M$, one has $x \in ((\mathfrak{m} : a) : b)$ if and only if $bx \in (\mathfrak{m} : a)$. The last condition is equivalent to $abx \in \mathfrak{m}$. In other words, $x \in ((\mathfrak{m} : a) : b)$ if and only if $x \in (\mathfrak{m} : ab)$. $\mathcal{Q.E.D}$

Corollary 7.1.2. *The rule* $(\mathfrak{m}, a) \mapsto (\mathfrak{m} : a)$ *defines a right* M*-set structure on* Ω*.*

We can define the "truth" map $t : 1 \to \Omega^M$ by $t(1) = M \in \Omega^M$. We claim that Ω^M is subobject classifier.

By iv) of Lemma 7.1.1, the map $t : 1 \to \Omega$ is a morphism of M-sets. For any right M-set A and any M-subset $i : B \subseteq A$, define the map $\chi_B : A \to \Omega$ by

$$\chi_B(a) = \{m \in M | am \in B\}.$$

One easily sees that χ_B is a morphism of M-sets and the diagram

$$\begin{array}{ccc} B & \xrightarrow{i} & A \\ {\scriptstyle B!}\downarrow & & \downarrow{\scriptstyle \chi_B} \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

is a pullback diagram. Hence, Ω is the subobject classifier in $\mathscr{S}_M$ [42].

7.1.1.4 Grothendieck topologies on a monoid

According to Lemma 6.1.5, there is an order reversing bijection between the set $\mathfrak{Loc}(\mathscr{S}_M)$ and all Grothendieck topologies defined on the one object category corresponding to M. Let us recall the last notion in this particular case.

For a monoid M, a *Grothendieck topology*, or simply a *topology*, $\mathfrak{F}$ on M is a collection of right ideals such that

(T1) $M \in \mathfrak{F}$;

(T2) if $\mathfrak{a} \in \mathfrak{F}$ and $m \in M$, then $(\mathfrak{a} : m) \in \mathfrak{F}$;

(T3) if $\mathfrak{b} \in \mathfrak{F}$ and $\mathfrak{a}$ is a right ideal of M for which $(\mathfrak{a} : m) \in \mathfrak{F}$ for any $m \in \mathfrak{b}$, then $\mathfrak{a} \in \mathfrak{F}$.

Thus, $\mathfrak{F}$ is a subobject of the M-set Ω_M. Since Ω_M is a subobject classifier, it defines a map $\chi_{\mathfrak{F}} : \Omega_M \to \Omega_M$, which is a Lawvere-Tierney topology on $\mathscr{S}_M$ and any Lawvere-Tierney topology on $\mathscr{S}_M$ is of this form. Explicitly, the map $\chi_{\mathfrak{F}}$ is given by

$$\chi_{\mathfrak{F}}(\mathfrak{a}) = \{m \in M \mid (\mathfrak{a} : m) \in \mathfrak{F}\}.$$

This result draws a clear analogy to Gabriel's topologies in ring theory [69]. We will see that many of the results originally proven for rings by Gabriel [20] are still valid for monoids.

Lemma 7.1.3. *Let $\mathfrak{F}$ be a topology on M. The following hold:*

(i) If $\mathfrak{a} \subseteq \mathfrak{b}$ are right ideals and $\mathfrak{a} \in \mathfrak{F}$, then $\mathfrak{b} \in \mathfrak{F}$.

(ii) If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathfrak{F}$.

(iii) If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$, then $\mathfrak{ab} \in \mathfrak{F}$.

Proof. i) Take any element $m \in \mathfrak{a}$. Then $m \in \mathfrak{b}$ and $(\mathfrak{b} : m) = R \in \mathfrak{F}$. Hence, by (T3), we have $\mathfrak{b} \in \mathfrak{F}$.

ii) If $m \in \mathfrak{a}$, then $(\mathfrak{a} \cap \mathfrak{b} : m) = (\mathfrak{b} : m) \in \mathfrak{F}$ by (T2). Hence, $\mathfrak{a} \cap \mathfrak{b} \in \mathfrak{F}$, by (T3).

iii) For any $m \in \mathfrak{a}$, we have $\mathfrak{b} \subseteq (\mathfrak{ab} : m)$. Hence, by i), we have $(\mathfrak{ab} : m) \in \mathfrak{F}$ and (T3) implies that $\mathfrak{ab} \in \mathfrak{F}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

A right M-set A is called an *$\mathcal{F}$-sheaf* if the restriction map

$$A \to \mathsf{Hom}_M(\mathfrak{F}, A), \quad a \mapsto f_a$$

is a bijection for every $\mathfrak{F} \in \mathcal{F}$. Here, $a \in A$ and $f_a \in \mathsf{Hom}_M(\mathfrak{F}, A)$, where $f_a(x) = xa$, $x \in \mathfrak{F}$. We let $\mathsf{Sh}(M, \mathcal{F})$ denote the topos of $\mathcal{F}$-sheaves.

For a monoid M, we let $\mathsf{Top}(M)$ denote the set of all topologies on M. As was already mentioned in Section 6.1.3, the set $\mathsf{Top}(M)$ is closed under arbitrary intersections. It is moreover a complete lattice, where the ordering is given by inclusion.

The following obvious facts are well-known: For any monoid M, there is a minimal $\mathcal{F}_{\mathsf{min}}$ and a maximal $\mathcal{F}_{\mathsf{max}}$ topology, among all Grothendieck topologies. These are given by

$$\mathcal{F}_{\mathsf{min}} = \{M\}, \quad \mathcal{F}_{\mathsf{max}} = \Omega^M.$$

We also have $\mathsf{Sh}(M, \mathcal{F}_{\mathsf{min}}) = \mathscr{S}_M$ and $\mathsf{Sh}(M, \mathcal{F}_{\mathsf{max}}) \cong \{\mathbf{1}\}$, where $\{\mathbf{1}\}$ is the trivial category with one object and one morphism.

As we already stated in Lemma 6.1.5, there is an order reversing bijection between the set of localisations of $\mathfrak{Loc}(\mathscr{S}_M)$, and the set of all Grothendieck topologies defined on the one object category associated to M. Under this bijection, the localisation corresponding to a topology $\mathcal{F}$ is the category $\mathsf{Sh}(M, \mathcal{F})$ of sheaves on $\mathcal{F}$.

7.1.1.5 Monoid homomorphisms and subobject classifier

Let $f : M \to N$ be a monoid homomorphism. One has the following maps between the subobject classifiers of M and N:

$$f^* : \Omega^N \to \Omega^M, \text{ and } f_* : \Omega^M \to \Omega^N$$

defined by

$$f^*(\mathfrak{n}) = f^{-1}(\mathfrak{n}), \text{ and } f_*(\mathfrak{m}) = \bigcup_{x \in \mathfrak{m}} f(x)N.$$

We point out that $\bigcup_{x \in \mathfrak{m}} f(x)N$ is the right ideal generated by $f(\mathfrak{m})$.

Lemma 7.1.4. *For any $\mathfrak{m} \in \Omega^M$, $\mathfrak{n} \in \Omega^N$ and $a \in M$, the following hold:*

i) $\mathfrak{m} \subseteq f^* f_*(\mathfrak{m}), \quad f_* f^*(\mathfrak{n}) \subseteq \mathfrak{n}$,

ii) $f_* f^* f_*(\mathfrak{m}) = f_*(\mathfrak{m}), \quad f^* f_* f^*(\mathfrak{n}) = f^*(\mathfrak{n})$,

iii) $f_*(\mathfrak{m} : a) \subseteq (f_*(\mathfrak{m}) : f(a))$ *and*

iv) $(f^*(\mathfrak{n}) : a) = f^*(\mathfrak{n} : f(a))$.

Moreover, if $g : N \to K$ is a monoid homomorphism, then

v) $(gf)_*(\mathfrak{m}) = g_*(f_*(\mathfrak{m}))$.

Proof. i) is clear and ii) is a formal consequence of i).

To prove iii), observe that $f_*(\mathfrak{m} : a)$ is a right ideal of N, generated by elements of the form $f(x)$, where $ax \in \mathfrak{m}$. It follows that $f(ax) \in f_*(\mathfrak{m})$. Hence, $f(x) \in (f_*(\mathfrak{m}) : f(a))$, showing iii).

For iv), observe that

$$\begin{aligned}(f^*(\mathfrak{n}) : a) &= \{x \in M \mid ax \in f^*(\mathfrak{n})\} \\ &= \{x \in M \mid f(ax) \in \mathfrak{n}\} \\ &= \{x \in M \mid f(x) \in (\mathfrak{n} : f(a))\} \\ &= f^*(\mathfrak{n} : f(a)).\end{aligned}$$

v) By definition, $(gf)_*(\mathfrak{m})$ is an ideal of K, generated by $gf(\mathfrak{m})$. Since $g_*(f_*(\mathfrak{m}))$ is also an ideal containing $gf(\mathfrak{m})$, we have $(gf)_*(\mathfrak{m}) \subseteq g_*(f_*(\mathfrak{m}))$. To prove the converse, take an element $x \in g_*(f_*(\mathfrak{m}))$. We have $x = kg(y)$, where $y \in f_*(\mathfrak{m})$ and $k \in K$. Thus, $y = nf(m)$, where $m \in \mathfrak{m}$ and $n \in N$. It follows that $x = kg(nf(m)) = (kg(n))gf(m) \in (gf)_*(\mathfrak{m})$ and vi) follows. $\mathcal{Q.E.D}$

Corollary 7.1.5. *Let $f : M \to N$ be a monoid homomorphism. The map $f^* : \Omega^N \to \Omega^M$, given by $f^*(\mathfrak{n}) = f^{-1}(\mathfrak{n})$, is a morphism of M-sets. Here, the monoid M acts on Ω^N via f.*

7.1.1.6 Localisation of monoids

Let M be a commutative monoid. We recall that if $S \subseteq M$ is a submonoid, we can form the localisation of M by S, denoted $S^{-1}M$.

This construction can be extended to M-sets in an obvious way. It is also obvious that $S^{-1}A$ is a natural $S^{-1}M$-set, if A is an M-set. We have a monoid homomorphism

$$f : M \to S^{-1}M, \quad f(m) = \frac{m}{1},$$

called the localisation map.

One also has

$$S^{-1}A = A \otimes_M S^{-1}M = f_!(A)$$

for any M-set A. We discussed in the previous section that f induces a geometric morphism of topoi

$$\mathscr{S}_f = (f^\bullet, f_\bullet) : \mathscr{S}_M \to \mathscr{S}_{S^{-1}M}.$$

One easily checks that the localisation $f_!$ preserves finite limits. Hence, the pair $(f_!, f^\bullet)$ defines a geometric morphism

$$\mathscr{S}^f : \mathscr{S}_{S^{-1}M} \to \mathscr{S}_M.$$

For any $S^{-1}M$-set A, the M-set $f^\bullet(A)$ is isomorphic to A, considered as an M-set via the monoid homomorphism $f : M \to S^{-1}M$. It follows that $S^{-1}A \simeq A$, which implies that $f_! f^\bullet \simeq \mathrm{Id}$. Hence, $\mathscr{S}^f : \mathscr{S}_{S^{-1}M} \to \mathscr{S}_M$ is a localisation of topoi, i.e. $f_! : \mathscr{S}_M \to \mathscr{S}_{S^{-1}M}$ is a localisation of categories.

Lemma 7.1.6. *Let S be a submonoid of a commutative monoid M and let $f : M \to S^{-1}M$ be the corresponding localisation.*

i) If $\mathfrak{n}$ is an ideal of $S^{-1}M$, then $f_* f^*(\mathfrak{n}) = \mathfrak{n}$.

ii) $f_*(\mathfrak{m} : a) = (f_*\mathfrak{m} : f(a))$.

iii) $f^* f_*(\mathfrak{m}) = \bigcup_{s \in S}(\mathfrak{m} : s)$.

iv) For any $t \in S$, one has

$$(f^* f_*(\mathfrak{m}) : t) = f^* f_*(\mathfrak{m}).$$

Proof. i) By part i) of Lemma 7.1.4, it suffice to show that $\mathfrak{n} \subseteq f_* f^*(\mathfrak{n})$. Take an element $x = \frac{n}{s} \in \mathfrak{n}$. Since $\mathfrak{n}$ is an ideal, we see that $sx \in \mathfrak{n}$. Hence, $n \in f^*(\mathfrak{n})$ and $sx = f(n) \in f_* f^*(\mathfrak{n})$. Since $f_* f^*(\mathfrak{n})$ is an ideal of $S^{-1}M$, we see that $x = f(n) \cdot \frac{1}{s} \in f_* f^*(\mathfrak{n})$, and i) follows.

ii) By part iii) of Lemma 7.1.4, it suffices to show that $(f_*(\mathfrak{m}) : f(a)) \subseteq f_*(\mathfrak{m} : a)$. To this end, take an element $z \in (f_*\mathfrak{m} : f(a))$. Since $z \in S^{-1}M$, we can write $z = \frac{m}{s}$, where $m \in M$ and $s \in S$. By assumption, $zf(a) \in f_*(\mathfrak{m})$. So, $\frac{ma}{s} \in f_*(\mathfrak{m})$. Thus, there exist an element $u \in \mathfrak{m}$ and $t \in S$ such that

$$\frac{ma}{s} = \frac{u}{t}.$$

This implies that $matt' = ust'$, for an element $t' \in S$. We have $z = \frac{m'}{s'}$, where $m' = mtt'$ and $s' = stt'$. Since $am' = amtt' = ust' \in \mathfrak{m}$, we have $z \in (f_*(\mathfrak{m}) : a)$. This proves the result.

iii) We have $x \in f^*f_*(\mathfrak{m})$ if and only if $\frac{x}{1} \in f_*(\mathfrak{m})$. This is equivalent to saying that $\frac{x}{1} = \frac{m}{s}$, for some $m \in \mathfrak{m}$, $s \in S$. The last condition holds if and only if $xt \in \mathfrak{m}$ for some $t \in S$, i.e. when $x \in (\mathfrak{m} : t)$ and the result follows.

iv) Since $\mathfrak{n} \subseteq (\mathfrak{n} : a)$ for any ideal $\mathfrak{n}$ in a commutative monoid M, we only need to show $(f^*f_*(\mathfrak{m}) : t) \subseteq f^*f_*(\mathfrak{m})$. Take $z \in (f^*f_*(\mathfrak{m}) : t)$, then

$$zt \in f^*f_*(\mathfrak{m}) = \bigcup_{s \in S} (\mathfrak{m} : s).$$

Thus, there exists an $s \in S$ such that $zts \in \mathfrak{m}$. Hence,

$$z \in (\mathfrak{m} : ts) \subseteq \bigcup_{r \in S} (\mathfrak{m} : r) = f^*f_*(\mathfrak{m})$$

and iv) follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 7.1.7. *The map $f_* : \Omega^M \to \Omega^{S^{-1}M}$ is compatible with the monoid actions and as such, induces the map of $S^{-1}M$-sets*

$$f_{*S} : S^{-1}\Omega^M \to \Omega^{S^{-1}M},$$

which is surjective.

Proof. The first assertion follows from the part ii) of Lemma 7.1.6, while surjectivity property from part i) of Lemma 7.1.6. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Thus, by construction, we have a commutative diagram

$$\begin{array}{ccc} \Omega^M & \longrightarrow & S^{-1}\Omega^M \\ & {\scriptstyle f_*} \searrow \quad \swarrow {\scriptstyle f_{*S}} & \\ & \Omega^{S^{-1}M}, & \end{array}$$

where the horizontal map is the localisation: $\mathfrak{m} \mapsto \frac{\mathfrak{m}}{1}$.

7.2 s-Noetherian monoids

We have defined the notion of locally s-Noetherian topoi in Subsection 6.2, which we studied further in Subsection 6.2.2.

Our aim now is to show that, under some finiteness assumptions on M, the topos of M-sets is s-Noetherian. These types of monoids are going to be referred to as s-Noetherian monoids. We will then continue to study them in a bit more detail. The result of this section allows us to glue topoi of the form M-sets to obtain new topoi. Of course particularly important are the categories of coherent sheaves over locally Noetherian monoid schemes, which we will study in the last chapter.

Let X be a set and M a monoid. Recall that one can define a right M-set structure on $X \times M$ by $(x, n)m = (x, nm)$, $n, m \in M$, $x \in X$. A right M-set A is called *free* if there exists

a set X and an isomorphism of right M-sets $A \simeq X \times M$. It is well-known that any right M-set is a quotient of a free right M-set. We can consider M itself as a free right M-set by taking X to be a one-element set. We have $\mathrm{Hom}_{\mathscr{S}_M}(M, A) \simeq A$ for any M-set A. It follows that M generates the topos $\mathscr{S}_M$.

We call a right M-set A *s-Noetherian* if it satisfies the ascending chain condition on subobjects. According to part i) of Lemma 6.2.1, this happens if and only if any family of right M-subsets of A contains a maximal member. A monoid M is called *right s-Noetherian* if it is s-Noetherian considered as a right M-set. I.e., if M satisfies the ascending chain condition on right ideals. This is equivalent to saying that any family of right ideals of M contains a maximal one.

Remark 7.2.1. *We use the term s-Noetherian in order to distinguish it from* Noetherian monoids *in the sense of [22], which satisfy the ascending chain condition on congruences.*

Lemma 7.2.2. *Let M be a monoid. The following are equivalent:*

i) M is right s-Noetherian.

ii) The topos $\mathscr{S}_M$ is s-Noetherian.

Proof. It is clear that $i) \Longrightarrow ii)$ since M generates the topos $\mathscr{S}_M$.

Conversely, assume $\mathscr{S}_M$ is s-Noetherian and let $(G_i)_{i\in I}$ be a set of Noetherian generators of $\mathscr{S}_M$. Any object of $\mathscr{S}_M$ is a quotient of a coproduct of objects from the family of generators. In particular, there is an epimorphism $f : \coprod X_j \to M$, where each X_j is isomorphic to one of the G_i-s. Since epimorphisms in $\mathscr{S}_M$ are exactly surjective homomorphisms of right M-sets, there is an element $x \in \coprod X_j$, such that $f(x) = 1 \in M$.

Recall that the coproduct in $\mathscr{S}_M$ is the disjoint union. Hence, $x \in X_j$ for some j. It follows that the restriction $f_j : X_j \to M$ of f on X_j is an epimorphism. Since X_j is s-Noetherian (it is isomorphic to one of the s-Noetherian generator), it follows by part iii) of Lemma 6.2.1 that M is also s-Noetherian. Thus, $(ii) \Longrightarrow i)$. $\mathcal{Q.E.D}$

Corollary 7.2.3. *Let M be an s-Noetherian monoid and $S \subseteq M$ a multiplicative subset. The localisation $S^{-1}M$ is s-Noetherian as well.*

Proof. This follows directly from Lemmas 6.2.7 and 7.2.2, and the fact that as mentioned in Paragraph 7.1.1.6, the geometric morphism $\mathscr{S}^f : \mathscr{S}_{S^{-1}M} \to \mathscr{S}_M$ is a localisation of topoi. $\mathcal{Q.E.D}$

Recall that a right M-set A is called *finitely generated*, if there are elements $a_1, \cdots, a_k \in A$ such that any element $a \in A$ can be written as $a = a_i m$, for some $i \in \{1, \cdots, k\}$ and $m \in M$. A right ideal of M is called finitely generated if it is so as a right M-set.

Lemma 7.2.4. *A monoid M is s-Noetherian if and only if any right ideal of M is finitely generated.*

Proof. Assume M is s-Noetherian and there exists a right ideal $\mathfrak{m} \subseteq M$ which is not finitely generated. In particular, $\mathfrak{m}$ is non-empty. Take an element $m_1 \in \mathfrak{m}$ and consider the right

ideal $\mathfrak{m}_1$ generated by m_1. Clearly, $\mathfrak{m}_1 \subsetneq \mathfrak{m}$ and so, there exists an element $m_2 \in \mathfrak{m}$, such that $m_2 \notin \mathfrak{m}_1$. Let $\mathfrak{m}_2$ be the right ideal generated by m_1 and m_2. We see that $\mathfrak{m}_1 \subsetneq \mathfrak{m}_2$. We obtain an infinite sequence of right ideals $\mathfrak{m}_1 \subsetneq \mathfrak{m}_2 \subsetneq \cdots$. This contradicts our assumptions. Hence, any right ideal is finitely generated.

Conversely, assume any right ideal is finitely generated. Let $\mathfrak{m}_1 \subsetneq \mathfrak{m}_2 \subsetneq \cdots$ be an ascending sequence of ideals and set $\mathfrak{m} := \bigcup \mathfrak{m}_i$. By assumption, $\mathfrak{m}$ is finitely generated and as such, there exists an $n \in \mathbb{N}$ such that $\mathfrak{m}_n$ contains all the generators. Clearly, $\mathfrak{m}_n = \mathfrak{m}_{n+1} = \cdots$ and we are done. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Example 7.2.5. *It is clear that any finite monoid is s-Noetherian. Moreover, any group is s-Noetherian, because any non-empty right ideal of a group G is G. The additive monoid of natural numbers $\mathbb{N}$ is also s-Noetherian, because any ideal of $\mathbb{N}$ is principal. We refer to Lemma 7.2.7 for other examples.*

Lemma 7.2.6. *Let M be an s-Noetherian monoid and let A be a right M-set.*

i) If A is a finitely generated M-set and $B \subseteq A$ is an M-subset, then B is also finitely generated.

ii) The M-set A is s-Noetherian if and only if A is finitely generated.

iii) Let A be a finitely generated M-set. Any family of M-subsets of A contains a maximal member.

Proof. i) The statement is true if $A \cong M$ is free with one generator, because M-subsets of M are simply ideals. Lemma 7.2.4 now implies the result.

The union of finitely generated M-subsets is finitely generated and the coproduct in the category $\mathscr{S}_M$ is given by the disjoint union. It follows that the statement is also true when A is finitely generated and free as an M-set. In fact, we can think of A as $A = \coprod_{i=1}^{k} A_i$, where each A_i is free with one generator. Denote $B_i := B \cap A_i$. These are M-subsets of the A_i-s, and therefore finitely generated as M-sets. We have $B = \coprod_{i=1}^{k} B_i$ and thus, B is also a finitely generated M-set.

Let A and F be finitely generated M-sets, with F free and $\xi : F \to A$ a surjective homomorphism. Consider $\xi^{-1}(B)$. It is an M-subset of F and as shown, finitely generated. It follows that B is also finitely generated since $\xi^{-1}(B) \to B$ is surjective.

ii) Assume A is finitely generated and let

$$A_1 \subseteq A_2 \subseteq \cdots\cdots \subseteq A$$

be a sequence of M-subsets of A. Consider the M-subset $B = \bigcup_{n=1}^{\infty} A_n$. It is finitely generated by part i). As such, there exists an integer n such that A_n contains every generator of B. We have $B = A_n$ and $A_n = A_{n+1} = \cdots$. Hence, A is s-Noetherian.

Conversely, assume A is s-Noetherian. This statement holds trivially if $A = \varnothing$. If $A \neq \varnothing$, we can take $a_1 \in A$. Let A_1 be the M-subset generated by a_1. We are done if $A_1 = A$. Otherwise, there exists an element $a_2 \in A$ such that $a_2 \notin A_1$. Denote by A_2 the M-subset generated by a_1 and a_2. We are done if $A_2 = A$. Otherwise we continue. This process will stop after a finite number of steps by our assumption. Hence, A is finitely generated.

iii) Let A_i, $i \in I$ be a family of M-subsets. Choose $i_1 \in I$ and consider the corresponding M-subset A_{i_1}. We are done if A_1 is maximal. Otherwise, there exists $i_2 \in I$ such that $A_{i_1} \subsetneq A_{i_2}$. We obtain an ascending sequence of M-subsets in this way, which must stabilise by part ii). Hence, $A_n = A_{n+1} = \cdots$ and as such, A_n is maximal. $\mathcal{Q.E.D}$

Lemma 7.2.7. *Let M be a monoid.*

i) Let $f : M \to N$ be a surjective homomorphism of monoids. If M is s-Noetherian, then N is s-Noetherian.

ii) If M is s-Noetherian, then $M \times \mathbb{N}$ is also s-Noetherian.

iii) M is s-Noetherian if and only if $M_{\mathsf{red}} = M/M^\times$ is s-Noetherian.

iv) M is s-Noetherian if it is commutative and finitely generated.

Proof. i) Take a right ideal $\mathbf{n}$ of N. The preimage $f^{-1}(\mathbf{n})$ is a right ideal of M and hence, finitely generated. The images of these generators generate $\mathbf{n}$. The result follows from Lemma 7.2.4.

ii) Consider the canonical projection $\pi : M \times \mathbb{N} \to M$, given by $\pi(m,t) = m$, $m \in M$ and $t \in \mathbb{N}$. Let $\mathfrak{m}$ be a right ideal of $M \times \mathbb{N}$. Its image $\pi(\mathfrak{m})$ is a right ideal of M since π is surjective. By assumption, there are elements

$$(m_1,t_1),\cdots,(m_k,t_k) \in \mathfrak{m}$$

such that $m_1,\cdots,m_k$ generate $\pi(\mathfrak{m})$. Let $t = \max\{t_1,\cdots,t_k\} \in \mathbb{N}$. For a natural number $s \leq t$, we consider the subset $\mathfrak{m}_s$ of $\mathfrak{m}$ given by

$$\mathfrak{m}_s = \{m \in M \mid (m,s) \in \mathfrak{m}\}.$$

It is not only a subset of M, but also a subideal of $\pi(\mathfrak{m})$, because if $(m,s) \in \mathfrak{m}$ and $m' \in M$, then $(mm',s) = (m,s)(m',0) \in \mathfrak{m}$ (recall that we use additive notations for $\mathbb{N}$). Denote by $m_{1s},\cdots,m_{l_s}$ the generators of this ideal. We claim that the elements

$$(m_i,t_i),(m_{js},s),\ i = 1,\cdots,k, 0 \leq s \leq t, 1 \leq j \leq l_s$$

generate $\mathfrak{m}$. To prove this claim, let $\mathbf{n} \subseteq \mathfrak{m}$ denote the ideal generated by these elements. We have to show that $\mathfrak{m} \subseteq \mathbf{n}$. Take an element $(m,r) \in \mathfrak{m} \subseteq M \times \mathbb{N}$. If $r \leq t$, then $m \in \mathfrak{m}_r$. So, we can write $m = m_{jr}m'$ for some j and $m' \in M$. This gives us $(m,r) = (m_{jr},r)(m',0)$. We have shown that $(m,r) \in \mathbf{n}$. Assume now that $r > t$. Since $m \in \pi(\mathfrak{m})$, we can write $m = m_i m'$, for some $i \in \{1,\cdots,k\}$ and $m' \in M$. We have

$$(m,r) = (m_i,t_i)(m',r-t_i) \in \mathbf{n},$$

and we are done.

iii) By part i), we only have to show that M is s-Noetherian if M_{red} is s-Noetherian. Consider the canonical projection $\pi : M \to M_{\mathsf{red}}$. The image $\pi(\mathfrak{m})$ of any ideal $\mathfrak{m} \subseteq M$ is a

finitely generated ideal of M_{red}. There exists a finite collection of elements $m_1, \cdots, m_k \in \mathfrak{m}$ such that $\pi(m_1), \cdots, \pi(m_k)$ generate $\pi(\mathfrak{m})$. Take any element $a \in \mathfrak{m}$. There is an $i \in \{1, \cdots, k\}$ and an $m \in M$ such that $\pi(a) = \pi(m_i)\pi(m)$. Hence, $ag = m_i m$ for some $g \in M^\times$. We have $a = m_i(mg^{-1})$ and hence, $m_1, \cdots, m_k$ generate $\mathfrak{m}$.

iv) By Example 7.2.5, $\mathbb{N}$ is s-Noetherian. It follows from part ii) that the monoid $\mathbb{N}^d$ is s-Noetherian for all $d \geq 0$. We can use part i) to finish the proof. $\mathcal{Q.E.D}$

To see why we are working with locally s-Noetherian topoi, rather than s-Noetherian, observe the following example:

Example 7.2.8. *Let $M = \mathbb{N}$ be the monoid of natural numbers. The category $\mathscr{S}_{\mathbb{N}}$ is a locally s-Noetherian topos by Lemma 7.2.2. Clearly, $A = \mathbb{N}$ is a free $\mathbb{N}$-set with one generator. Thus, it is s-Noetherian. However, $B = A \times A = \mathbb{N}^2$ is not finitely generated as an $\mathbb{N}$-set and as such, not s-Noetherian.*

7.2.1 Subobject classifier of s-Noetherian monoids

We will show that under the s-Noetherianness assumption, the functor

$$f_! : \mathscr{S}_M \to \mathscr{S}_{S^{-1}M};\ f_!(A) = S^{-1}A$$

respects the subobject classifier. This is due to the following Proposition:

Proposition 7.2.9. *Let M be a commutative and s-Noetherian monoid. The canonical map*

$$f_{*S} : S^{-1}\Omega^M \to \Omega^{S^{-1}M}$$

is an isomorphism of $S^{-1}M$-sets for any submonoid $S \subseteq M$.

Proof. The above map is surjective by Corollary 7.1.7. Before we prove injectivity, we need to prove two auxiliary claims.

Our first claim is that for any ideal $\mathfrak{m}$, there is an element $s_{\mathfrak{m}}$ such that

$$f^* f_*(\mathfrak{m}) = (\mathfrak{m} : s_{\mathfrak{m}}).$$

We have $f^* f_*(\mathfrak{m}) = \bigcup_{s \in S}(\mathfrak{m} : s)$ by part iii) Lemma 7.1.6. Since M is s-Noetherian, the ideal $\bigcup_{s \in S}(\mathfrak{m} : s)$ is finitely generated, say by $x_1 \in (\mathfrak{m} : s_1), \cdots, x_k \in (\mathfrak{m} : s_k)$. Take $s_{\mathfrak{m}} = s_1 \cdots s_k$. Then $x_i \in (\mathfrak{m} : s_{\mathfrak{m}})$ for all $i = 1, \cdots, k$. Hence, $(\mathfrak{m} : s_{\mathfrak{m}})$ contains the ideal generated by $x_1, \cdots, x_k$, which is $f^* f_*(\mathfrak{m})$.

For any ideal $\mathfrak{m}$ of M, we have $\mathfrak{m} \in \Omega^M$, which is an M-set. So, we can also consider $\mathfrak{m}1$ as an element of $S^{-1}\Omega^M$.

Our second claim is that the equality

$$\frac{\mathfrak{m}}{1} = \frac{f^* f_*(\mathfrak{m})}{1}$$

holds in $S^{-1}\Omega^M$. To show this fact, observe that by Claim 1 and part iv) of Lemma 7.1.6, we have

$$(\mathfrak{m} : s_{\mathfrak{m}}) = f^* f_*(\mathfrak{m}) = (f^* f_*(\mathfrak{m}) : s_{\mathfrak{m}})$$

and the second claim follows.

This second claim shows that the composite of the map $f^* : \Omega^{S^{-1}M} \to \Omega^M$ with the canonical morphism $\Omega^M \to S^{-1}\Omega^M$ is the inverse of $f_{*S} : S^{-1}\Omega^M \to \Omega^{S^{-1}M}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.2.1.1 The necessity of s-Noetherianness

More generally, if $f = (f_*, f^*) : \mathscr{T} \to \mathscr{K}$ is a geometric morphism from a topos $\mathscr{T}$ to a topos $\mathscr{K}$, there exist a canonical morphism $\tau : f^*(\Omega^{\mathscr{K}}) \to \Omega^{\mathscr{T}}$, classifying the monomorphism $f^*(t) : f^*(1) \to f^*(\Omega^{\mathscr{T}})$. According to [33], f is called *sub-open* if τ is a monomorphism. Hence, if M is an s-Noetherian monoid, the localisation $S^{-1} : \mathscr{S}_M \to \mathscr{S}_{S^{-1}M}$ is the inverse image part of a sub-open geometric morphism $\mathscr{S}_M \to \mathscr{S}_{S^{-1}M}$. It should be point-out that the s-Noetherian assertion is essential.

In fact, take $M = \mathbb{N}^{(m)}$ to be the multiplicative monoid of the natural numbers. We take $S = M$. We will show that the map τ is not a monomorphism. Firstly, observe that $S^{-1}M$ is a group. Hence $B = \Omega^{S^{-1}M}$ has only two element; the empty set and B. Now consider the morphism $\alpha : A \to B$, where $A = \Omega^M$, $B = \Omega^{S^{-1}M}$ and $\alpha(I) = S^{-1}I$ for any $I \in \Omega^M$. We will use the lemma below. In fact we take $a_1, a_2 \in \Omega^M$ to be the following elements: $a_1 = M$, while $a_2 = J$ consists of those natural numbers $= \prod_p p^{v_p}$, for which either $v_p = 0$ or $v_p \geq p$. Here, p runs through the set of all prime numbers. In this case, $\alpha(a_1) = \alpha(a_2)$ since non of these are empty and B has only two elements, one of them being $\varnothing$. We claim that there is no natural number s for which $M = J : s$. The Lemma below will imply the non-injectivity of τ. To prove the assertion, assume that such an s exists. Choose a prime p, such that p does not divide s. Then $p \in M$, but $p \neq sn$, where $n \in J$.

Lemma 7.2.10. *Let M be a commutative monoid and $S \subset M$ a submonoid. Moreover, let A be an M-set and B an $S^{-1}M$-set, which is considered as an M-set via the canonical homomorphism $M \to S^{-1}M$. Denote by $\beta : A \to S^{-1}A$ the canonical map. For a morphism of M-sets $\alpha : A \to B$, the following two conditions are equivalent*

i) The morphism of $S^{-1}M$-sets $\tau : S^{-1}A \to B$ is a monomorphism. Here, $\tau(\frac{a}{s}) = \frac{\alpha(a)}{s}$.

ii) If $\alpha(a_1) = \alpha(a_2)$, there exist $s \in S$ such that $a_1 s = a_2 s$.

Proof. i) $\Longrightarrow$ ii). If $\alpha(a_1) = \alpha(a_2)$, then $\beta(a_1) = \beta(a_2)$. This follows from the injectivity of τ since $\alpha = \tau\beta$. It follows that $\frac{a_1}{1} = \frac{a_2}{1}$ and $a_1 s = a_2 s$ for some $s \in S$.

ii) $\Longrightarrow$ i). Assume $\tau(\frac{a_1}{s_1}) = \tau(\frac{a_2}{s_2})$. Thus, $\frac{\alpha(a_1)}{s_1} = \frac{\alpha(a_2)}{s_2}$. So, $\alpha(a_1 s_2 s) = \alpha(a_2 s_1 s)$ for some $s \in S$. By ii), this implies $a_1 s_2 st = a_2 s_1 st$ for some $t \in S$ and $\frac{a_1}{s_1} = \frac{a_2}{s_2}$. The injectivity of τ follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.3 The points of the topos $\mathscr{S}_M$

7.3.1 On filtered M-sets

7.3.1.1 General properties

Recall that a *point* of a Grothendieck topos $\mathscr{T}$ is a geometric morphism $\mathsf{p} = (\mathsf{p}_*, \mathsf{p}^*) : \mathscr{S} \to \mathscr{T}$ from the topos of sets $\mathscr{S}$ to $\mathscr{T}$. The inverse image functor $\mathsf{p}^* : \mathscr{T} \to \mathscr{S}$ preserves colimits and finite limits. It is also well-known that, conversely, for any functor $\mathsf{f} : \mathscr{T} \to \mathscr{S}$ which preserves colimits and finite limits, one has $\mathsf{f} = \mathsf{p}^*$ for a uniquely defined point p. We let $\mathbf{Pts}(\mathscr{T})$ be the category of points of $\mathscr{T}$ and $\mathcal{F}(\mathscr{T})$ the isomorphism classes of the category $\mathbf{Pts}(\mathscr{T})$.

Instead of $\mathbf{Pts}(\mathscr{S}_M)$ and $\mathsf{F}(\mathscr{S}_\mathsf{M})$, we write $\mathbf{Pts}(M)$ and F_M. By Diaconescu's theorem [42], (see also Theorem 6.1.10) the category $\mathbf{Pts}(M)$ is equivalent to the category of filtered left M-sets. Recall that a left M-set A is called *filtered*, provided the functor

$$(-) \otimes_M A : {}_M\mathscr{S} \to \mathscr{S}$$

commutes with finite limits. The collection of all filtered left M-sets is of course a full subcategory of the topos ${}_M\mathscr{S}$, which we denote by ${}_M\mathcal{F}$.

The topos point of $\mathscr{S}_M$, corresponding to a filtered left M-set A, is denoted by $\mathsf{p}_A = (\mathsf{p}_{A*}, \mathsf{p}_A^*)$. The inverse image functor $\mathsf{p}_A^* : \mathscr{S}_M \to \mathscr{S}$ is given by

$$\mathsf{p}_A^*(X) = X \otimes_M A,$$

while the direct image functor $\mathsf{p}_{A*} : \mathscr{S} \to \mathscr{S}_M$ sends a set Y to $\mathrm{Hom}_{\mathscr{S}}(A, Y)$. The latter is considered as a right M set via

$$(\alpha m)(a) := \alpha(ma).$$

Here, $\alpha \in \mathrm{Hom}_{\mathscr{S}}(A, X)$, $a \in A$ and $m \in M$.

The following well-known fact [46, p.24], Theorem 6.1.10 is a very useful tool for checking whether a given M-set is filtered.

Lemma 7.3.1. *A left M-set A is filtered if and only if the following three conditions hold:*

(F1) $A \neq \varnothing$.

(F2) If $m_1, m_2 \in M$ *and* $a \in A$ *satisfies the condition*

$$m_1 a = m_2 a,$$

there exist $m \in M$ *and* $\tilde{a} \in A$ *such that* $m\tilde{a} = a$ *and* $m_1 m = m_2 m$.

(F3) If $a_1, a_2 \in A$, *there are* $m_1, m_2 \in M$ *and* $a \in A$, *such that* $m_1 a = a_1$ *and* $m_2 a = a_2$.

Example 7.3.2. *Clearly, $A = M$ is always filtered and corresponds to the canonical point, denoted by* p_M. *Thus,* p_M^* *is the forgetful functor* $\mathscr{S}_M \to \mathscr{S}$.

For a localisation $\mathscr{T}$ of $\mathscr{T}$ and $\mathsf{p} = (\mathsf{p}_*, \mathsf{p}^*) : \mathscr{S} \to \mathscr{T}$ a topos point of $\mathscr{T}$, we write $\mathsf{p} \pitchfork \mathscr{T}$ if $\mathsf{p}_*(S) \in \mathscr{T}$ for every set $S \in \mathscr{S}$.

Lemma 7.3.3. *Let $\mathcal{F}$ be a topology on a monoid M and A a filtered left M-set. Then*

$$\mathsf{p}_A \pitchfork \mathsf{Sh}(M, \mathcal{F})$$

if and only if for any $\mathfrak{a} \in \mathcal{F}$, the canonical map $\mathfrak{a} \otimes_M A \to A$ is an isomorphism. Here, p_A denotes the point of $\mathscr{S}_M$ corresponding to A.

Proof. Since the direct image functor $\mathscr{S} \to \mathscr{S}_M$ corresponding to the point p_A is given by $S \mapsto \mathsf{Hom}_{\mathscr{S}}(A, S)$, we see that $\mathsf{p}_A \pitchfork \mathsf{Sh}(M, \mathcal{F})$ if and only if $\mathsf{Hom}_{\mathscr{S}}(A, S)$ is an $\mathfrak{F}$-sheaf. This means that for all $\mathfrak{a} \in \mathcal{F}$, the canonical map

$$\mathsf{Hom}_{\mathscr{S}}(A, S) \to \mathsf{Hom}_{\mathscr{S}_M}(\mathfrak{a}, \mathsf{Hom}_{\mathscr{S}}(A, S))$$

is an isomorphism. The map in question is the same as

$$\mathsf{Hom}_{\mathscr{S}}(A, S) \to \mathsf{Hom}_{\mathscr{S}}(\mathfrak{a} \otimes_M A, S).$$

Since S is any set, Yoneda's lemma implies that, this happens if and only if $\mathfrak{a} \otimes_M A \to A$ is an isomorphism. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.3.1.2 Elementary examples

We give the following basic examples to help illustrate the category of topos points of $\mathscr{S}_M$.

Example 7.3.4. *Let $M = \{1\}$ be the trivial monoid. Our topos is just the category of sets in this case. As such, condition (F2) always holds. Any filtered M-set A must have at least one element by (F1), and by (F3), it can have at most one. Thus, $\mathbf{Pts}(\mathscr{S})$ is equivalent to the category with one object and one morphism. So, $\mathscr{S}$ has a unique point, up to a canonical isomorphism.*

Example 7.3.5. *Let G be a group and A a filtered G-set. We have the G-set homomorphism*

$$\lambda_a : G \to A, \quad \lambda_a(g) = ga$$

for any $a \in A$. Assume $\lambda_a(g) = \lambda_a(h)$. That is, $ga = ha$. It follows from (F2), by taking $m_1 = g$ and $m_2 = h$, that $gm = hm$ for some $m \in G$. We obtain $g = h$ since G is a group and hence, λ_a is injective.

Take $x \in A$ and use condition (F3) for $a_1 = a$ and $a_2 = x$. There are $m_1, m_2 \in G$ and $a_0 \in B$ such that $m_1 a_0 = a$ and $m_2 a_0 = x$. Thus, $x = m_2 m_1^{-1} b = \lambda_b(m_2 m_1^{-1})$ and λ_b is surjective. We have shown that the category $\mathbf{Pts}(\mathscr{S}_G)$ is equivalent to G, considered as a one object category.

Example 7.3.6. *Let us consider M as a left M-set. The action is obviously given by the multiplication in M. Condition (F1) holds since $1 \in M$. To see that (F2) is satisfied, take $m = a$ and $\tilde{a} = 1$. Similarly, we can take $m_1 = a_1, m_2 = a_2$ and $a = 1$ for (F3). We call this the* trivial point *of $\mathscr{S}_M$.*

Example 7.3.7. *Let M_1 and M_2 be monoids. The topos $\mathscr{S}_{M_1 \times M_2}$ is the 2-categorical product (often called 2-product or pseudo-product) of $\mathscr{S}_{M_1}$ and $\mathscr{S}_{M_2}$ in the 2-category of topoi, see [42, p.419, Exercise VII.14]. Hence,*

$$\mathbf{Pts}(\mathscr{S}_{M_1 \times M_2}) \simeq \mathbf{Pts}(\mathscr{S}_{M_1}) \times \mathbf{Pts}(\mathscr{S}_{M_2})$$

and therefore

$$\mathsf{F}_{\mathsf{M}_1 \times \mathsf{M}_2} \simeq \mathsf{F}_{\mathsf{M}_1} \times \mathsf{F}_{\mathsf{M}_2}.$$

7.3.1.3 Induced points

Recall the following well-known fact:

Lemma 7.3.8. *Let $f : M \to N$ be a homomorphism of monoids. If A is a filtered M-set, then*

$$f_!(A) \coloneqq A \otimes_M N$$

is a filtered N-set. Here, N is considered as an M-set via the homomorphism f. This yields a functor

$$f_! : \mathcal{F}_M \to \mathcal{F}_N,$$

and hence a functor

$$f_* : \mathbf{Pts}(\mathscr{S}_M) \to \mathbf{Pts}(\mathscr{S}_N).$$

Proof. Take X to be an N-set. We need to show that the functor

$$X \mapsto f_!(A) \otimes_N X$$

commutes with finite limits. But this functor is the same as

$$X \mapsto (A \otimes_M N) \otimes_N X \simeq A \otimes_M f^*(X).$$

Both $f^* :_N \mathscr{S} \to_M \mathscr{S}$ and $(-) \otimes_M A :_M \mathscr{S} \to \mathscr{S}$ commute with finite limits. As such, the composite functor also commutes. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

The M'-set constructed in the Lemma is said to be *induced from A via the homomorphism f*. The functor

$$\mathbf{Pts}(f) : \mathbf{Pts}(M) \to \mathbf{Pts}(M'),$$

induces the map

$$\mathsf{F}_\mathsf{f} : \mathsf{F}_\mathsf{M} \to \mathsf{F}_{\mathsf{M}'}.$$

Corollary 7.3.9. *Let $f : M \to N$ be an injective monoid homomorphism. Any filtered M-set is isomorphic to an M-subset of a filtered N-set.*

Proof. Let A be a filtered M-set. By Lemma 7.3.8, $A \otimes_M N$ is a filtered N-set. The functor $A \otimes_M (-)$ respects injective morphisms since A is filtered. We apply it to the inclusion $M \to N$ to obtain that

$$A \simeq A \otimes_M M \longrightarrow A \otimes_M N$$

is injective and we are done. $\mathcal{Q.E.D}$

7.3.2 Idempotents of M and points of $\mathscr{S}_M$

In this section, we show that any idempotent of a monoid M gives rise to a point of the topos $\mathscr{S}_M$. This fact was recently observed in [60], [30]. We will prove that if M is finite, any point of $\mathscr{S}_M$ comes in this way, see Theorem 7.4.15 below. We will also recall the category of idempotents, see for example [67], as it will play an important role in Section 7.4.

7.3.2.1 Points corresponding to idempotents

We first consider the case when $M = \{1, t\}$, $t^2 = t$. One easily sees that the singleton is a filtered M-set (see also Corollary 7.4.5 for a significantly more general statement).

Take an arbitrary monoid M. Any idempotent $e \in M$ induces a homomorphism of monoids $\eta : \{1, t\} \to M$, where $t^2 = t$, given by $\eta(t) = e$. The singleton, which is filtered over $\{1, t\}$, induces a left filtered M-set, see Section 7.3.1.3. One easily sees that this is isomorphic to Me.

As a reminder M^{sl} is the associated semi-lattice of M. That is, $M^{\mathsf{sl}} = M/\sim$, where for all $a, b \in M$, $ab \sim ba$ and $a^2 \sim a$. Denote by $q : M \to M^{\mathsf{sl}}$ the canonical quotient map. As we will see, the induced map

$$\mathsf{F_q} : \mathsf{F_M} \to \mathsf{F_{M^{sl}}}$$

is bijective if M is finitely generated and commutative. In the non-commutative setting, this induced map is not a bijection in general, even if M is finite. However, we will show that $|\mathsf{F_M}|$ is finite if M is finite, see Theorem 7.4.3 below. We also have the following:

Proposition 7.3.10. *Let M be a finite monoid. The canonical map*

$$\mathsf{F_q} : \mathsf{F_M} \to \mathsf{F_{M^{sl}}}$$

is surjective. In particular $|\mathsf{F_M}| \geq |\mathsf{M^{sl}}|$.

Proof. Denote by $\mathsf{Idem}(M)$ the set of all idempotents of M. Then $\mathsf{Idem}(M) \ni e \mapsto Me$ yields the map $\mathsf{Idem}(M) \to \mathsf{F_M}$. Since $\mathsf{Idem}(M^{\mathsf{sl}}) = M^{\mathsf{sl}}$, the functoriality yields the following commutative diagram

$$\begin{array}{ccc} \mathsf{Idem}(M) & \longrightarrow & \mathsf{F_M} \\ \downarrow & & \downarrow \\ M^{\mathsf{sl}} & \xrightarrow{\sim} & \mathsf{F_{M^{sl}}}. \end{array}$$

The bottom arrow is a bijection and the left vertical map is surjective, thanks to [67, Lemma 1.6]. It follows that the right vertical map is surjective as well. The last statement follows from Theorem 7.5.10 below, and the fact that for finitely generated commutative monoids, $\mathsf{Spec}(M) \simeq M^{\mathsf{sl}}$, as argued in Subsection 4.1.1.3. $\mathcal{Q.E.D}$

7.3.2.2 Category $\mathcal{I}(M)$

Let $m \in M$ be an element. We have a natural left ideal Mm, which can also be considered as a left M-set. We have the following well-known fact, see for example [67, Proposition 1.8].

Lemma 7.3.11. *Let $e \in M$ be an idempotent. For any left M-set X, we have a bijection*

$$eX \cong \mathrm{Hom}_M(Me, X),$$

which sends an element $ex \in eX$ to the homomorphism $\alpha_x : Me \to X$, given by $\alpha_x(me) = mex$.

Proof. Take any morphism of M-sets $\beta : Me \to X$. We have $\beta(e) = \beta(ee) = e\beta(e)$. Thus, $\beta(e) \in eX$ and $\beta \mapsto \beta(e)$ defines a map $\mathrm{Hom}_M(Me, X) \to eX$, which is obviously inverse to the map $ex \mapsto \alpha_x$. $\mathcal{Q.E.D}$

Corollary 7.3.12. *Let $e, f \in M$ be idempotents. We have $\mathrm{Hom}_M(Me, Mf) = eMf$.*

We define the category $\mathcal{I}(M)$ as follows: Objects are idempotents of M and a morphism from e to f is an element of the form $fme \in M$, $m \in M$. An other way of saying that is that morphisms are equivalence classes of elements of M, where $m \sim n$ if $fme = fne$. We have

$$\mathrm{Hom}_{\mathcal{I}(M)}(e, f) = fMe.$$

The composition is given by the multiplication in M, i.e. $(gnf) \circ (fme) = gnfme$. That is, the composite of arrows

$$e \xrightarrow{fme} f \xrightarrow{gnf} g$$

is equal to $e \xrightarrow{gnfme} g$. The identity morphism of e is just $e = e1e$. It is clear that

$$\mathcal{I}(M^{op}) = (\mathcal{I}(M))^{op}.$$

Recall the Green relations $\mathscr{L}, \mathscr{R}, \mathscr{J}$. By definition, we have $e\mathscr{L}e'$ provided $Me = Me'$, $e\mathscr{R}e'$ if $eM = e'M$ and $e\mathscr{J}e'$ if $MeM = Me'M$. We let $\mathrm{Idem}_{\mathscr{K}}(M)$ be the corresponding quotient set, where $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{J}\}$.

It is clear that if $e\mathscr{L}e'$ and $f\mathscr{R}f'$, then

$$\mathrm{Hom}_{\mathcal{I}(M)}(e, f) = \mathrm{Hom}_{\mathcal{I}(M)}(e', f').$$

Lemma 7.3.13. *The assignment $e \mapsto Me$ induces a contravariant equivalence between the category $\mathcal{I}(M)$ and the full subcategory of ${}_M\mathscr{S}$, consisting of objects of the form Me.*

Proof. This is a direct consequence of Corollary 7.3.12. $\mathcal{Q.E.D}$

Lemma 7.3.14. *Idempotents e and f are isomorphic in $\mathcal{I}(M)$ if and only if there are $a, b \in M$ such that $ab = e$ and $ba = f$. This happens if and only if $e\mathscr{J}f$. Thus, the set of iso-classes of the category $\mathcal{I}(M)$ is bijective to the set $\mathrm{Idem}_{\mathscr{J}}(M)$.*

Proof. A consequence of [67, Theorem 1.11]. $\mathcal{Q.E.D}$

7.3.3 The points and localisations of the topos $\mathscr{S}_M$, when M is a left zero monoid

7.3.4 Left zero monoids

In this section, we investigate the points and localisations of the topoi $\mathscr{S}_M$ and ${}_M\mathscr{S}$, where M is a left zero monoid. Let us recall that we can define the product in any set S by

$$st = s, \quad s, t \in S.$$

In this way, we obtain a semigroup and we let S_+ be the corresponding monoid, which is obtained by adding the unite element 1 to S. Thus $S_+ = S \cup \{1\}$. Such monoids are known as *left zero monoids*. The opposite monoids are known as *right zero* monoids. It is immediate that S_+ is commutative if and only if $|S| = 0, 1$.

7.3.4.1 Points of $\mathscr{S}_{S_+}$

We will assume that $S \neq \varnothing$. Each element of S_+ is an idempotent. If $s \in S$, then clearly $S_+ s = S$. It follows that $\mathrm{Idem}_{\mathscr{J}}(S_+)$ is a two element set given by 1 and any $s \in S$. Thus, we have at least two non-isomorphic points of $\mathscr{S}_{S_+}$, corresponding to the left filtered S_+-sets S_+ and $S = S_+ s$.

Our immediate goal is to prove that there are no other non-isomorphic points of the topos $\mathscr{S}_{S_+}$. The proof will use the following fact:

Lemma 7.3.15. *Let M be a monoid and $M^\times$ the subgroup of all invertible elements of M. We set*

$$I := M \setminus M^\times.$$

Any filtered left M-set A with $A \neq IA$ is isomorphic to M as a left M-set.

Proof. Choose an element $a \in A \setminus IA$ and define a morphism of left M-sets $f : M \to A$ by

$$f(m) = ma.$$

We claim that f is an isomorphism. For surjectivity, take an element $b \in A$. By F3 (see Lemma 7.3.1), there exist elements $c \in A$ and $m, n \in M$ such that $a = mc$ and $b = nc$. Since $a \notin IA$, we see that $m \notin I$ and hence, it is invertible. So, $c = m^{-1}a$ and $b = nm^{-1}a = f(nm^{-1})$. This proves the surjectivity of f. For injectivity, assume $f(m) = f(n)$, that is $ma = na$. By F2, there are $\tilde{m} \in M$ and $b \in A$ such that $a = \tilde{m}b$ and $m\tilde{m} = n\tilde{m}$. Again, $\tilde{m}$ is invertible since $a \notin IA$. Hence, we have $m = n$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Theorem 7.3.16. *If A is a filtered left S_+-set, then A is isomorphic to either S_+ or S.*

Proof. We set

$$\hat{A} = A \setminus \bigcup_{s \in S} sA.$$

Claim 1. $|\hat{A}| \leq 1$. Take $x, y \in \hat{A}$. By condition F3 of Lemma 7.3.1, there exist $z \in A$, $m_1 \in S_+$ and $m_2 \in S_+$ such that $x = m_1 z$ and $y = m_2 z$. Since $x, y \in \hat{A}$, we must have $m_1 = 1 = m_2$. It follows that $x = z = y$, which implies the claim.
Claim 2. If $sA \cap tA \neq \varnothing$, then $s = t$. Assume $sa = tb$, for some $a, b \in A$. By F3, we have $a = m_1 c$, $b = m_2 c$ and thus,

$$sc = sm_1 c = sa = tb = tm_2 c = tc.$$

By F2, there exist $m \in S_+$ and $d \in A$ such that $md = c$ and $sm = tm$. So, we have $s = sm = tm = t$ and the claim follows.
Claim 3. For any $a, b \in A$, one has $sa = sb$. Once again, F3 says that there are $m_1, m_2 \in S_+$ and $c \in A$ such that $a = m_1 c$ and $b = m_2 c$. Hence, we have

$$sa = sm_1 c = sc = sm_2 c = sb.$$

Claim 4. If $\hat{A} = \varnothing$, then A is isomorphic to S. By assumption, $A = \bigcup_{s \in S} sA$. Claim 3 implies that each sA consists of exactly 1 element and Claim 2 states that the union must be disjoint. This implies the forth claim.

To finish the proof of the theorem, observe that if A is not isomorphic to S, then $\hat{A}$ is not empty, by Claim 4. It follows from Claim 1 that $\hat{A}$ is a set with one element. We can apply Lemma 7.3.15 to finish the proof. $\mathcal{Q.E.D}$

Corollary 7.3.17. *We have*

$$\mathcal{F}(\mathscr{S}_{S_+}) \cong \mathsf{Idem}_{\mathscr{J}}(S_+).$$

In particular, both sets have exactly two elements.

7.3.4.2 Points of ${}_{S_+}\mathscr{S}$

In this case, the singleton is a filtered right S_+-set: While conditions F1 and F3 hold for any monoid, condition F2 is easily verified in this circumstance, see also Lemma 7.4.5. We will show that, conversely, any filtered right S_+-set is isomorphic to either S_+ or the singleton. Since $sS_+ = \{s\}$ and $S_+ sS_+ = S_+ tS_+$ for all $s, t \in S$, it follow that any point of the topos of left S_+-sets comes from an idempotent. In particular, the topos ${}_{S_+}\mathscr{S}$ has exactly two non-isomorphic points and we have a bijection

$$\mathcal{F}({}_{S_+}\mathscr{S}) \cong \mathsf{Idem}_{\mathscr{J}}(S_+).$$

Lemma 7.3.18. *Any filtered right S_+-set A is either isomorphic to S_+ or a singleton.*

Proof. Let A be a filtered right S_+-set containing at least two distinct elements $a_1 \neq a_2 \in A$. By Lemma 7.3.15, it suffices to show that $A \neq AS$. Assume $A = AS$. By F3, we can write $a_i = bm_i$, $i = 1, 2$.

Assume $m_1 = 1$, then $a_2 = a_1 m_2$. Since $a_1 \in AS$, we have $a_1 = ct$ for some $t \in S$. We obtain $a_2 = a_1 m_2 = ctm_2 = ct = a_1$, contradicting our assumption and hence, implying $m_1 \in S$. The same reasoning implies $m_2 \in S$. Since $b \in A = AS$, we can write $b = ds$ for some $s \in S$. It

follows

$$a_1 = bm_1 = dsm_1 = ds = dsm_2 = bm_2 = a_2,$$

once again giving a contradiction. The result follows. $\mathcal{Q.E.D}$

7.3.4.3 Localisations of $\mathscr{S}_{S_+}$ and ${}_{S_+}\mathscr{S}$

Recall that there is a one-to-one correspondence between localisations of $\mathscr{S}_M$ and Grothendieck topologies on M. Here, M is any monoid. Recall also that there are the minimal and maximal Grothendieck topologies on M, denoted respectively by $\mathcal{F}_{\mathsf{min}}$ and $\mathcal{F}_{\mathsf{max}}$ and defined as $\mathcal{F}_{\mathsf{min}} = \{M\}$, $\mathcal{F}_{\mathsf{max}} = \Omega^M$. Their corresponding sheaf categories are $\mathscr{S}_M$ and the trivial category (with one object and morphism).

We wish to understand the Grothendieck topologies $\mathcal{F}$, with $\mathcal{F} \neq \mathcal{F}_{\mathsf{min}}$ and $\mathcal{F} \neq \mathcal{F}_{\mathsf{max}}$. Throughout this section, we will assume $M = S_+$. For any set B, we can consider the right M-act

$$i_*(B) := \mathsf{Maps}(S, B),$$

where the action of S_+ on $i_*(B)$ is given as follows: Take $i_*(B) \ni \alpha : S \to B$ and $s \in S$. Define the map $\alpha \cdot s : S \to B$ by

$$(\alpha \cdot s)(t) = \alpha(st).$$

Since $\alpha(st) = \alpha(s)$, we have

$$\alpha \cdot s = \mathfrak{c}_{\alpha(s)},$$

where $\mathfrak{c}_b$ denotes the constant function $S \to B$ with value b. This is known as the *cofree M-set cogenerated by B*. We can extend this to a functor through composition in the usual way. We will not need it but let us mention that $i_* : \mathscr{S} \to \mathscr{S}_M$ is a push-forward part of a topos point $i = (i_*, i^*) : \mathscr{S} \to \mathscr{S}_M$, corresponding to the filtered S_+-set S, see Section 7.3.4.1. The next lemma claims that the functor $i_* : \mathscr{S} \to \mathscr{S}_M$ is full and faithful.

Lemma 7.3.19. *Let B, C be sets and $\psi : i_*(B) \to i_*(C)$ a morphism of S_+-sets. There exists a unique map $f : B \to C$ such that $\psi = i_*(f)$. In other words, for any $\alpha \in i_*(B) = \mathsf{Maps}(S, B)$, one has $\psi(\alpha) = f \circ \alpha$.*

Proof. Since ψ is a morphism of S_+-sets, we have

$$\psi(\alpha \cdot s) = \psi(\alpha) \cdot s = \mathfrak{c}_{\psi(\alpha)(s)}$$

for all $\alpha : S \to B$ and $s \in S$. By setting $\alpha = \mathfrak{c}_b$, we see that ψ sends constant functions to constant ones. It follows that there is a unique map $f : B \to C$ such that

$$\psi(\mathfrak{c}_b) = \mathfrak{c}_{f(b)}.$$

In fact, one must have

$$f(b) = \psi(\mathfrak{c}_b)(s),$$

for any $s \in S$ and any $b \in B$.

For any $\alpha : S \to B$ and any $s \in S$, we have

$$\mathfrak{c}_{f(\alpha(s))} = \psi(\mathfrak{c}_{\alpha(s)}) = \psi(\alpha \cdot s) = \psi(\alpha) \cdot t = \mathfrak{c}_{\psi(\alpha)(t)}.$$

It follows that $f \circ \alpha = \psi(\alpha)$ and we are done. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Before we state our next result, observe that if X is a subset of S, then $XS \subset X$. Thus, X is a right ideal of $M = S_+$ and

$$\Omega^M = \{M\} \cup \mathfrak{B}(S). \tag{7.1}$$

Here, $\mathfrak{B}(S)$ is the set of all subsets of S.

Proposition 7.3.20. *Let $\mathcal{F}$ be a Grothendieck topology on $M = S_+$ such that $\mathcal{F} \neq \mathcal{F}_{\mathsf{min}}$, $\mathcal{F} \neq \mathcal{F}_{\mathsf{max}}$. Then*

$$\mathcal{F} = \{M, S\}.$$

A right S_+-module A is a sheaf over the topology $\mathcal{F}$ if and only if A is a cofree S_+-set and the functor i_* induces an equivalence of categories

$$\mathscr{S} \cong \mathsf{Sh}(M, \mathcal{F}).$$

Proof. The last statement is a consequence of the previous Lemma. Next, we observe that $\{M, S\}$ is a Grothendieck topology, thanks to Lemma 7.4.17, because S is a two sided idempotent ideal of S_+. If $\mathcal{F} \neq \mathcal{F}_{\mathsf{min}}$, $\mathcal{F} \neq \mathcal{F}_{\mathsf{max}}$ and $\mathcal{F} \neq \{M, S\}$, there must exist an $X \in \Omega^M$ such that X is a proper subset of S. Choose any $s \in S \setminus X$. Then $\varnothing = (X : s) \in \mathcal{F}$. It follows that $\mathcal{F} = \mathcal{F}_{\mathsf{max}}$ and we obtained a contradiction. Hence, $\mathcal{F} = \{M, S\}$ and the first assertion is proved.

Observe that an S_+-set A is a sheaf if and only if the restriction map

$$A = \mathsf{Hom}_{S_+}(S_+, A) \to \mathsf{Hom}_{S_+}(S, A)$$

is bijective. Set

$$B = \{a \in A \mid sa = a, s \in S\}.$$

For any morphism $f : S \to A$ of S_+-sets, we have $f(st) = f(s)t$. Equivalently, $f(s) = f(s)t$ for all $s, t \in S$. If follows, that the image of f lies in B. This shows that

$$\mathsf{Hom}_{S_+}(S, A) = \mathsf{Maps}(S, B) = i_*(B),$$

and hence, A is a sheaf if and only if $A \cong i_*(B)$, finishing the proof. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Consider the category ${}_{S_+}\mathscr{S}$. Denote by ${}_+S$ the opposite of the monoid S_+. Clearly ${}_{S_+}\mathscr{S} = \mathscr{S}_{{}_+S}$.

Proposition 7.3.21. *i) We have*

$$\Omega^{{}^+S} = \{\varnothing, S, {}_+S\}.$$

ii) Let $\mathcal{F}$ be a Grothendieck topology on ${}_+S$ such that $\mathcal{F} \neq \mathcal{F}_{\min}$, $\mathcal{F} \neq \mathcal{F}_{\max}$. Then

$$\mathcal{F} = \{{}_+S, S\}.$$

iii) A left S_+-set A is a sheaf on the topology $\mathcal{F}$ if and only if the action of S_+ on A is trivial, that is $sa = a$ for all $a \in A$ and $s \in S$. Thus

$$\mathsf{Sh}({}_+S, \mathcal{F}) \cong \mathscr{S}.$$

Proof. i) Assume I is a non-empty left ideal of S_+. For any $t \in I$, we have $S = St \subset I$. Thus, either $I = S$ or $I = S_+$.

ii) Since $\mathcal{F}$ is not maximal, $\varnothing \notin \mathcal{F}$, as otherwise by the first property of Grothendieck topologies, it would have to contain everything. We also have $S \in \mathcal{F}$, otherwise $\mathcal{F}$ would be $\mathcal{F}_{\min}$. Using property i) of Grothendieck topologies again yields $\mathcal{F} = \{{}_+S, S\}$. The fact that this really is a Grothendieck topology follows from the fact that S is a two sided idempotent ideal of ${}_+S$, thanks to Lemma 7.4.17.

iii) Observe that for any left S_+-set A, the left S_+-set structure on $\mathsf{Hom}_{{}_+S}(S, A)$ is given by $(s \cdot \alpha)(t) = \alpha(ts)$, where $s, t \in S$ and $\alpha \in \mathsf{Hom}_{{}_+S}(S, A)$. Since $\alpha(ts) = \alpha(t)$, we see that $s \cdot \alpha = \alpha$. Thus, $\mathsf{Hom}_{{}_+S}(S, A)$ is always a trivial left S_+-set. Recall that A being a sheaf over the topology $\mathcal{F}$ means that the canonical map

$$A = \mathsf{Hom}_{{}_+S}({}_+S, A) \to \mathsf{Hom}_{{}_+S}(S, A)$$

is an isomorphism of left S_+-sets. Hence, if A is a sheaf, the action of S_+ on A must be trivial.

Conversely, assume the action of S_+ on A is trivial. The map $\alpha : S \to A$ is a morphism of left S_+-sets if and only if $\alpha(st) = s\alpha(t)$. Since $st = s$ and $sa = a$ for all $a \in A$, we see that $\alpha(s) = \alpha(t)$. Thus, α is constant and it follows that $A \to \mathsf{Hom}_{{}_+S}(S, A)$ is an isomorphism, implying the proposition. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.4 The points of the topos $\mathscr{S}_M$, when M is finite

7.4.1 M-congruences

For an equivalence relation ρ on a set S, we denote by q the canonical map $q : S \to S/\rho$.

Let M be a monoid and A a left M-set. An equivalence relation $\sim_\rho$ on A is called an *M-congruence* if $a \sim b$ implies $ma \sim mb$ for all $m \in M$. It is clear that in this case, the quotient A/ρ has a unique left M-set structure such that $q : A \to A/\rho$ is an M-set map.

We will use this terminology to distinguish between congruences on a monoid M in the world of monoids, and congruences on M, considered as a left M-set, using the multiplication in M.

Lemma 7.4.1. *Let $\sim_\rho$ be an M-congruence on a left M-set A. For any $a \in A$, the subset*

$$K_\rho(a) = \{x \in M | a \sim xa\}$$

is a submonoid of M.

Proof. Since $\sim$ is an equivalence relation, we have $a \sim a = 1 \cdot a$. Thus, $1 \in K$. Assume $x, y \in K(a)$. That is, $a \sim ya$ and $a \sim xa$. Since $\sim$ is M-congruence, we have $xa \sim xya$. It follows that $a \sim xa \sim xya$. Hence, $xy \in K(a)$. $\mathcal{Q.E.D}$

7.4.2 Finiteness of points

We start with the following observation:

Lemma 7.4.2. *Let A be a filtered left M-set and $a_1, \cdots, a_k$ elements of A. There are $m_1, \cdots, m_k \in M$ and $c \in A$ such that $a_i = m_i c$ for all $1 \leq i \leq k$.*

Proof. The case $n = 1$ is clear and $n = 2$ is just condition F3. We proceed by induction. Assume there are $n_1, \cdots, n_{k-1} \in M$ and $b \in A$ such that $a_i = n_i b$ for all $1 \leq i \leq n-1$. By the case $n = 2$, we can choose $c \in A$ and $n', m_k \in M$ such that $n'c = b$ and $m_k c = a_k$. We put $m_i = n' n_i$ for $i = 1, \cdots k-1$ to obtain $m_i c = a_i$ for all $1 \leq i \leq k$. $\mathcal{Q.E.D}$

Theorem 7.4.3. *Let M be a finite monoid.*

i) Any filtered M-set A is cyclic, that is, generated by a single element. In particular, we have $|A| \leq |M|$.

ii) The set $\mathcal{F}_M$ is finite.

Proof. i) Assume there are k distinguished elements of A, say $a_1, \cdots, a_k$. By Lemma 7.4.2, we can find $m_1, \ldots, m_k \in M$ and $c \in A$ such that $a_i = m_i c$ for all $1 \leq i \leq k$. If $|M| < k$, we see that there are $i \neq j$ such that $m_i = m_j$. It follows that

$$a_i = m_i c = m_j c = a_j.$$

This contradicts our assumptions on the a_i's. Hence, $k \leq |M|$. This implies $|A| \leq |M|$. Moreover, by taking $a_1, \cdots, a_k$ to be all the elements of A, we see that A is generated by c and hence, it is cyclic.

ii) This is an obvious consequence of i). $\mathcal{Q.E.D}$

7.4.3 Right-collapsible monoids and right-collapsible submonoids

Definition 7.4.4. *We call a monoid M a* right-collapsible monoid, *if for any $m, n \in M$, there exists an $x \in M$ such that*

$$mx = nx.$$

For more on this subject, see [37].

Lemma 7.4.5. *Let M be a monoid. The terminal object of the category ${}_M\mathscr{S}$ (i.e., the singleton) is filtered if and only if M is a right-collapsible monoid.*

Proof. For such a set, conditions F1 and F3 always hold. Condition F2 in this case says that for any $m_1, m_2 \in M$, there exists an $m \in M$ such that $m_1 m = m_2 m$. This means that M is a right-collapsible monoid. $\mathcal{Q.E.D}$

Clearly, if M is a semilattice, then M is a right-collapsible monoid (we can take $x = mn$). Another class of right-collapsible monoids (which is a generalisation of semilattices in the finite case) are monoids that have a right zero. That is, an element ϱ such that $x\varrho = \varrho$ for all $x \in K$. Clearly, if such a ϱ exists, it is unique and an idempotent. Our next goal is to show that the converse is also true, if M is finite. We will need the following lemma.

Lemma 7.4.6. *Let K be a right-collapsible monoid. For any finite collection of elements $m_1, \cdots, m_k$ of K, there exists an $x \in K$ such that $m_i x = x$ for all $i = 1, \cdots, k$.*

Proof. We proceed by induction. Let $k = 1$. In this case, the assertion follows directly from the definition of a right-collapsible monoid, by taking $m = m_1$ and $n = 1$, where $m, n \in M$ are as in Definition 7.4.4. Next, consider the case $k = 2$. Since the assertion is true for $k = 1$, we can find y_i, $i = 1, 2$ such that $m_i y_i = y_i$ and $i = 1, 2$. As K is a right-collapsible monoid, there exists a $z \in K$ such that $y_1 z = y_2 z$. For $x = y_1 z = y_2 z$, we have

$$m_i x = m_i y_i z = y_i z = x, \quad i = 1, 2.$$

Let $k > 2$. By the induction assumption, there exists a $y \in K$, for which $m_i y = y$, for all $i = 1, \cdots, k-1$. Since the result is also true for $k = 2$, we can apply it for y and m_k, and conclude that there exists a $x \in K$, for which $m_k x = x$ and $yx = x$. We have $m_i x = m_i yx = yx = x$, $i < k$ and thus, for all $1 \leq i \leq k$. This finishes the proof. $\mathcal{Q.E.D}$

Corollary 7.4.7. *A finite monoid is a right-collapsible monoid if and only if it has a right zero element.*

A submonoid K of a monoid M is called an *right-collapsible submonoid* if K is a right-collapsible monoid.

7.4.4 Saturated submonoids

A submonoid $K \subseteq M$ of a monoid M is called *saturated* if for any $a, b \in M$, the condition $ab \in K$ implies $a \in K$, $b \in K$. Note that for commutative monoids, saturated submonoids are exactly the complements of prime ideals. In the case when M is an F-monoid, the saturated condition can be simplified to the following: For any $m \in M$ for which $mx = x$, for some $x \in K$, one has $m \in K$. It is clear, that to any submonoid K, there is a smallest saturated submonoid containing K. This is the intersection of all saturated submonoids containing K. (The fact that the intersection of saturated submonoids is again saturated is readily checked). We denote this associated saturated submonoid by $\widehat{K}$ and sometimes refer to it as the *saturation* of K.

Example 7.4.8. *Let $M = \{1, 0, a, b, ab\}$, where $a^2 = a, b^2 = b$ and $ba = 0$. The only saturated right-collapsible submonoids of M are $\{1\}, \{1, a\}, \{1, b\}$ and M itself. On the other hand, $\{1, 0\}, \{1, 0, a\}, \{1, 0, b\}$ and $\{1, 0, ab\}$ are non-saturated right-collapsible submonoids.*

7.4.5 Quotient by right-collapsible submonoids

Let $H \subseteq A$ be a submonoid of a monoid A. We can define a relation $\sim_H$ on A by setting $a \sim_H b$ if and only if there exist $x, y \in H$ such that $ax = by$. This relation, however, need not be an equivalence relation as the transitivity property need not hold. If it does, however, $\sim_H$ is an M-congruence and the quotient, denoted by M/K, is a natural M-set.

Lemma 7.4.9. *Let K be a right-collapsible submonoid of M and $m, n \in M$. The induced relation $\sim_K$ is an M-congruence and $m \sim_K n$ if and only if there exists an $x \in K$ such that $mx = nx$.*

Proof. First, we show that the two relations are the same. One side is clear by letting $x = y$. For the other side, let $x, y \in K$ be elements for which $mx = ny$. Since K is a right-collapsible submonoid, we can find a $z \in K$ such that $xz = yz$. As $mx = ny \Rightarrow mxz = nyz$, the result follows.

To see that $\sim_K$ is an M-congruence, let $m \sim_K n$ and $n \sim_K k$. That is, we have $x, y \in K$ such that $mx = mx$ and $ny = ky$. As K is an right-collapsible submonoid, there exists a $z \in K$ such that $xz = yz$. We have

$$m(xz) = (mx)z = (nx)z = n(xz) = n(yz) = (ny)z = (ky)z = k(yz).$$

Hence, $m \sim_K k$ with the first definition and so, transitivity holds. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

In general, the M-congruence $\sim_K$ is not a congruence (that is, the quotient M/K need not be a monoid), even if K is a right-collapsible submonoid. In fact, take once again $M = \{1, 0, a, b, ab\}$, where $a^2 = a, b^2 = b$, $ba = 0$ and let $K = \{1, b\}$. In this case, we have three equivalence classes $1 \sim_K b$, $a \sim_K ab$ and 0. Since $1 \cdot a = a \not\sim_K ba = 0$, we see that $\sim_K$ is not a congruence relation.

We have shown that M/K is a left M-set. More is true, however.

The following can already be inferred from [37] and [30]. In [37, Lemma 14.13 (ii) and Proposition 16.6], it is shown that condition F2 (called condition E in said monograph) is satisfied by M/K and that this condition agrees with "pullback-flatness" for cyclic M-sets. Further, it is shown in [30, Proposition 1.10 (iii)] that in this case, pullback-flatness is the same as being filtered.

Proposition 7.4.10. *Let K be a right-collapsible submonoid of M. The quotient M/K is a filtered left M-set.*

Proof. The fact that M/K is a left M-set is just Lemma 7.4.9. To see that M/K is in addition filtered, consider the canonical surjective map $q : M \to M/K$. Since M is a monoid, M/K is non-empty and condition F1 holds.

To show F2, assume $m_1 q(a) = m_2 q(a)$ with $a, m_1, m_2 \in M$. We have to find $m, \tilde{a} \in M$ such that $m_1 m = m_2 m$ and $mq(\tilde{a}) = q(a)$. To this end, observe that

$$q(m_1 a) = m_1 q(a) = m_2 q(a) = q(m_2 a).$$

Hence, $m_1ax = m_2ax$ for an element $x \in K$. Since K is a right-collapsible submonoid, there exists an element $y \in K$ such that $xy = y$. So, we can take $m = ax$ and $\tilde{a} = q(1)$. We have $m_1m = m_1ax = m_2ax = m_2m$ and $mq(1) = q(ax) = q(a)$, because $axy = ay$ and $ax \sim_K a$.

To show F3, take $q(a_1), q(a_2) \in M/K$. Then $a_1q(1) = q(a_1)$ and $a_2q(1) = q(a_2)$. Thus, F3 holds with $a = q(1)$. $\mathcal{Q.E.D}$

We have the following "inverse statement" to the above:
Part i) of this Proposition can be found in [37, Lemma 14.14]

Proposition 7.4.11. *Let ρ be an M-congruence on M such that $M/\sim_\rho$ is a filtered left M-set and define*

$$K_\rho = \{m \in M \mid 1 \sim_\rho m\}.$$

i) The subset K_ρ is a saturated right-collapsible submonoid of M.

ii) We have $M/\sim_\rho \simeq M/K_\rho$.

Proof. i) By Lemma 7.4.1, K_ρ is a submonoid of M. Take $x, y \in K_\rho$. We have $x \sim_\rho 1 \sim_\rho y$, by the definition of K_ρ. Thus $q(x) = q(y)$, where $q : M \to M/\sim_\rho$ is the canonical map. It follows that $xq(1) = yq(1)$. Since $M/\sim_\rho$ is filtered, there are $m, \tilde{a} \in M$ such that $xm = ym$ and $mq(\tilde{a}) = q(1)$. The last condition implies that $z := m\tilde{a} \in K_\rho$. We have $xz = xm\tilde{a} = ym\tilde{a} = yz$. This shows that K_ρ is a right-collapsible submonoid.

Take $m \in M$ and $x \in K_\rho$ such that $mx = x$. Since $1 \sim_\rho x$, it follows that $m \sim_\rho mx = x \sim_\rho 1$. Thus, $m \in K_\rho$ and K_ρ is saturated.

ii) Assume $m \sim_\rho n$. It follows that $mq(1) = nq(1)$. By F2, there are $x, a \in M$ such that $mx = nx$ and $xq(a) = q(1)$. The last condition implies $y = xa \in K$. Since $my = mxa = nxa = ny$, we see that $m \sim_K n$. Conversely, let $m \sim_K n$. That is $mx = nx$ for $x \in K$. We have $q(mx) = mq(x) = mq(1) = q(m)$ and $q(nx) = q(n)$. This yields $q(m) = q(n)$, which implies $m \sim_\rho n$. This finishes the proof. $\mathcal{Q.E.D}$

Corollary 7.4.12. *Let K be a right-collapsible submonoid of M and ρ be the congruence $\sim_K$, corresponding to K. Then*

i) $K_\rho = \{m \in M | mx = x, \text{ for an element } x \in K\} = \widehat{K}$, where $\widehat{K}$ is the saturation of K.

ii) $\widehat{K}$ is a right-collapsible submonoid.

iii) If K is saturated, we have $\widehat{K} = K$.

Proof. i) By definition, $m \in K_\rho$ if and only if $1 \sim_\rho m$. This happens exactly if there exists an $x \in K$ such that $mx = x$. This proves the first equality. Next, take $m \in K$. Since K is a right-collapsible monoid, there exists an $x \in K$ such that $mx = x$. Hence, $1 \sim_\rho m$ and $m \in K_\rho$. This shows that $K \subseteq K_\rho$. Since K_ρ is saturated by Proposition 7.4.11, we have $\widehat{K} \subseteq K_\rho$. To show the opposite inclusion, take any $m \in K_\rho$. Thus, $1 \sim_\rho m$. So, $m = mx$ for some $x \in K$. Since $\widehat{K}$ is saturated and $x \in K \subseteq \widehat{K}$, we see that $m \in \widehat{K}$. It follows that $K_\rho = \widehat{K}$.

ii) This Follows from part i) by virtue of Proposition 7.4.11 and part iii) holds by definition. $\mathcal{Q.E.D}$

Theorem 7.4.13. *Let M be a finite monoid. Any filtered left M-set is isomorphic to M/K, for a suitable saturated right-collapsible submonoid K of M.*

Below (see Theorem 7.4.15), we will prove a much stronger result.

Proof. Any filtered M-set is cyclic by Theorem 7.4.3. Hence, it is isomorphic to M/ρ, where ρ is an M-congruence. Part ii) of Proposition 7.4.11 says that $K = K_\rho$ is a saturated right-collapsible submonoid of M. It remains to show that $M/\sim_\rho \simeq M/K_\rho$, which we already did in Proposition 7.4.11. $\mathcal{Q.E.D}$

Corollary 7.4.14. *Let K and L be right-collapsible submonoids of M. Then $M/K \simeq M/L$ if and only if $\widehat{K} = \hat{L}$. In particular, $M/K \simeq M/\widehat{K}$.*

7.4.6 Some examples

i) Assume M is a right-collapsible monoid and take $K = M$. In this case, M/M is a singleton. So, the terminal object of $\mathscr{S}_M$ is filtered. Conversely, if the terminal object of $\mathscr{S}_M$ is filtered, then M is a right-collapsible monoid. In fact, the terminal object is $M/\sim_\rho$, where $x \sim_\rho y$ for all $x, y \in M$. In this case, $K = M$ and hence, M is a right-collapsible monoid, thanks to the proof of Theorem 7.4.13. In particular, the single element set is filtered if M has a right zero or M is a semilattice.

The above example i) can already be found in [37, Exercise 14.3 (4)].

ii) Let $e \in M$ be an idempotent. Then $K = \{1, e\}$ is a right-collapsible submonoid. In this case, $\sim_K$ is the equivalence relation on M, defined by $a \sim_K b$ if and only if $ae = be$. The left M-set $M/K \simeq Me$ is filtered.

The following was indirectly shown in [3, Exercise 3.12 (b)] and [60]. In the first, it is shown that every point of the topos of M-sets is essential, for a finite monoid M. In the second, it is shown essential points correspond directly to idempotents.

Theorem 7.4.15. *Let M be a finite monoid. Any filtered M-set is of the form Me, for an idempotent $e \in M$. Thus, $e \mapsto Me$ yields an equivalence of categories*

$$(\mathcal{I}(M))^{op} \cong \mathbf{Pts}(M).$$

Proof. We have already proven (see Theorem 7.4.13) that any filtered M-set is of the form M/K, where K is a saturated right-collapsible submonoid of M. By Corollary 7.4.7, K has a right zero ϱ. Take $L = \{1, \varrho\}$. We have $L \subseteq K$ and $K \subseteq \widehat{L}$. Since K is saturated, we have $K = \widehat{L}$. By Corollary 7.4.14, we have $\sim_K = \sim_L$. Thus, we can take $e = \varrho$. $\mathcal{Q.E.D}$

Corollary 7.4.16. *Let M be a finite monoid, then*

i) $|\mathbf{Pts}(M)| = |\mathbf{Pts}(M^{op})|$.

ii) If p *is a point corresponding to an idempotent e, one has an isomorphism of monoids*

$$\mathsf{End}(\mathsf{p}) \cong (eMe)^{\mathrm{op}}.$$

iii) There is a bijection

$$\mathsf{F_M} \cong \mathsf{Idem}_{\mathscr{J}}(\mathsf{M}).$$

7.4.7 Localisations of $\mathscr{S}_M$, for finte M

We assume that M is finite throughout this section. We will show that there is a one-to-one correspondence between localisations of $\mathscr{S}_M$ and idempotent ideals of M. This fact allows us to prove that the topology on $\mathcal{F}_M$, defined by localisations of the topos $\mathscr{S}_M$, is homeomorphic to the order topology of the poset of principal idempotent ideals.

7.4.8 Idempotent ideals

Lemma 7.4.17. *Let $\mathfrak{m}$ be a two-sided ideal of a monoid M such that $\mathfrak{m} = \mathfrak{m}^2$. The set*

$$\mathcal{F}_\mathfrak{m} = \{\mathfrak{F} \mid \mathfrak{m} \subseteq \mathfrak{F}\}$$

of right ideals containing $\mathfrak{m}$ is a Grothendieck topology on M.

Proof. Condition (T1) is obvious. For (T2), assume $\mathfrak{m} \subseteq \mathfrak{F}$ and $m \in M$. For any $x \in \mathfrak{m}$, $mx \in M\mathfrak{m} = \mathfrak{m} \subseteq \mathfrak{F}$. Hence $x \in (\mathfrak{F} : m)$. It follows that $\mathfrak{m} \subseteq (\mathfrak{F} : m)$ and $(\mathfrak{F} : m) \in \mathcal{F}_\mathfrak{m}$. So (T2) holds. For (T3), take $\mathfrak{B} \in \mathcal{F}_\mathfrak{m}$. Assume $\mathfrak{a}$ is a right ideal such that $(\mathfrak{a} : b) \in \mathcal{F}_\mathfrak{m}$ for any $b \in \mathfrak{B}$. By assumption, $\mathfrak{m} \subseteq \mathfrak{B}$ and $\mathfrak{m} \subseteq (\mathfrak{a} : b)$ for all $b \in \mathfrak{B}$. Take any $x \in \mathfrak{m}$. Since $\mathfrak{m} = \mathfrak{m}^2$, we can write $x = yz$, where $y, z \in \mathfrak{m}$. Since $z \in \mathfrak{m} \subseteq (\mathfrak{a} : y)$, we see that $x = yz \in \mathfrak{a}$. Thus, $\mathfrak{m} \subseteq \mathfrak{a}$ and $\mathfrak{a} \in \mathcal{F}_\mathfrak{m}$, from which (T3) follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Proposition 7.4.18. *Let $\mathcal{F}$ be a Grothendieck topology on a finite monoid M. There exists a two-sided ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\mathcal{F} = \mathcal{F}_\mathfrak{m}$.*

Proof. As M is finite, $\mathcal{F}$ is finite as well. We can see that $\mathcal{F}$ contains the smallest element $\mathfrak{m}$ since $\mathcal{F}$ is closed with respect to finite intersections. Take $x \in M$. Since $(\mathfrak{m} : x) \in \mathcal{F}$, it follows that $\mathfrak{m} \subseteq (\mathfrak{m} : x)$. Equivalently, $x\mathfrak{m} \subseteq \mathfrak{m}$, which implies that $\mathfrak{m}$ is a two-sided ideal of M. We also know that $\mathcal{F}$ is closed with respect to the product. Hence $\mathfrak{m}^2 \in \mathcal{F}$. By the minimality of $\mathfrak{m}$, we have $\mathfrak{m} \subseteq \mathfrak{m}^2$ and hence, $\mathfrak{m}^2 = \mathfrak{m}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

It should be pointed out that in [3, Exercise 9.1.12 (c)], it was shown that for finite M, every localisation is essential. In [36, Theorem 4.4], it was shown that this, in turn, corresponds to idempotents ideals.

We will use the following well-known fact, see [67, Proposition 1.23].

Lemma 7.4.19. *Any two-sided idempotent ideal of a finite monoid M has the form*

$$\bigcup_{i \in I} Me_iM,$$

for a (finite) family of idempotents $(e_i)_{i \in I}$.

Lemma 7.4.20. *Let e be an idempotent of M and $\mathfrak{m} \subseteq M$ a two-sided ideal, such that $\mathfrak{m} = \mathfrak{m}^2$. Then*

$$\mathsf{p}_{Me} \nmid \mathsf{Sh}(M, \mathcal{F}_\mathfrak{m})$$

if and only if $e \in \mathfrak{m}$.

Proof. Assume $\mathsf{p}_{Me} \nVdash \mathsf{Sh}(M, \mathcal{F}_{\mathfrak{m}})$. The canonical map

$$\mu_{\mathfrak{m}} : \mathfrak{m} \otimes_M Me \to Me$$

is an isomorphism by Lemma 7.3.3. The surjectivity of this map implies that $e = xme$, for some $x \in \mathfrak{m}$, $m \in M$. So, $e \in \mathfrak{m}M = \mathfrak{m}$. Conversely, assume $e \in \mathfrak{m}$. We have to show that for any right ideal $\mathfrak{F}$ containing $\mathfrak{m}$, the canonical map $\mu_{\mathfrak{F}} : \mathfrak{F} \otimes_M Me \to Me$ is an isomorphism. For any $m \in M$, we have $me \in \mathfrak{m} \subseteq \mathfrak{F}$. Since $\mu(me \otimes e) = me$, we see that μ is surjective. Injectivity of μ follows directly from the fact that $- \otimes_M Me$ commutes with finite limits, hence respects monomorphisms. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Example 7.4.21. *i) Let M once again be the monoid $\{1, 0, a, b, ab\}$, $a^2 = a$, $b^2 = b$ and $ba = 0$. It has 4 idempotents $1, 0, a, b$. Since $M1 = M$, $M0 = \{0\}$, $Ma = \{0, a\}$ and $Mb = \{0, b, ab\}$, we see that they all define non-isomorphic idempotents. Thus, M has 4 non-isomorphic points, corresponding to the filtered left M-sets $\{0\}, \{0, a\}, \{0, b, ab\}$ and M. Since $M1M = M$, $M0M = \{0\}$, $MaM = \{0, a, ab\}$, $MbM = \{0, b, ab\}$, we see that there are 6 idempotent ideals*

$$\varnothing, \{0\}, \{0, a, ab\}, \{0, b, ab\}, \{0, a, b, ab\}, M.$$

ii) Let $M = T_3$ be the monoid of all endomorphic maps $\{1, 2, 3\} \to \{1, 2, 3\}$. It has $3^3 = 27$ elements. If f is a such map, the cardinality of the image of f is known as the rank of f and is denoted by $rk(f)$. There are 3 maps of rank 1 (i.e. constant maps) and all of them are idempotents. There are 6 idempotent elements of rank 2 and only one idempotent of rank 3. All together, we have 10 idempotents. However, $\mathsf{Idem}_{\mathfrak{I}}(M)$ has only three elements, as two idempotents are $\mathfrak{I}$-equivalent if and only if they have the same rank. All two-sided ideals of T_3 are idempotent and they are

$$\varnothing \subseteq I_1 \subseteq I_2 \subseteq I_3 = M,$$

where $I_k = \{f \in T_3 \mid rk(f) \leq k\}$, $k = 1, 2, 3$. Observe that in this case M^{com} is isomorphic to the multiplicative monoid $\{0, 1, -1\}$. In fact, for a map $f : \{1, 2, 3\} \to \{1, 2, 3\}$, denote by $[f]$ the class of f in M^{com}. Since non-bijective self maps form an ideal of M. Thus, we have a well-defined surjective homomorphism $M^{\mathsf{com}} \to \{0, 1, -1\}$, given by $[f] \mapsto 0$, if f is a non-bijective self map and $[f] \mapsto \mathsf{sgn}(f)$, if f is a bijection. It remains to show that all non-bijective maps define the same element in M^{com}. Denote by $\mathfrak{c}_i$ the constant map with value i. For all i, j, we have

$$[\mathfrak{c}_i] = [\mathfrak{c}_i \circ \mathfrak{c}_j] = [\mathfrak{c}_j \circ \mathfrak{c}_i] = [\mathfrak{c}_j].$$

This shows that all constant maps have the same image in M^{com}. Moreover, since $f \circ \mathfrak{c}_i = \mathfrak{c}_{f(i)}$ and $\mathfrak{c} \circ f = \mathfrak{c}$, we see that the class of constant maps define the zero element of M^{com}. Denote by $s, t \in T_3$ the maps given by $s(1) = 1 = s(2)$, $s(3) = 2$ and $t(1) = 1, t(2) = 2 = t(3)$. We have

$$[\mathfrak{c}_1] = [s \circ t] = [t \circ s] = [s].$$

Since any non-constant and non-bijective map can be written as $f \circ s \circ g$, with f and g bijective,

we see that all non-bijective maps define the same classes in M^{com}. *Since* $\{0, 1, -1\}$ *has two idempotents, we see that, in this case,* $3 = |\mathsf{F}(\mathscr{S}_{\mathsf{M}})| \neq |\mathsf{F}(\mathscr{S}_{\mathsf{M}^{\mathsf{com}}})| = 2$.

7.4.9 Distributivity of the lattice $\mathfrak{Loc}(\mathscr{S}_M)$

Denote by $\mathscr{II}(M)$ the set of two-sided idempotent ideals of a monoid M. Our interest in $\mathscr{II}(M)$ comes from the bijection

$$\mathfrak{Loc}(\mathscr{S}_M) \cong \mathscr{II}(M), \tag{7.2}$$

which is true for all finite M. This follows from Lemmas 7.4.17 and 7.4.18. In particular, $|\mathfrak{Loc}(\mathscr{S}_M)|$ is finite and

$$|\mathfrak{Loc}(\mathscr{S}_M)| = |\mathfrak{Loc}(\mathscr{S}_{M^{\mathrm{op}}})| \tag{7.3}$$

holds.

Since the union of two-sided idempotent ideals of M is again a two-sided idempotent ideal, we see that $\mathscr{II}(M)$ is a join-semilattice, with $I \vee J = I \cup J$. Its greatest element is M and its least element is $\varnothing$. In general, the intersection of two-sided idempotent ideals is not an idempotent ideal (see Part i) of Example 7.4.21).

It is well-known that any finite join-semilattice L with greatest element is a lattice, where

$$a \wedge b = \bigvee_x x,$$

with $a, b \in L$ and x running through all the elements of the set

$$\{x \in L \mid x \leq a \quad \text{and} \quad x \leq b\}.$$

It follows that for finite M, the set $\mathscr{II}(M)$ is a lattice.

Lemma 7.4.22. *Let* I *and* J *be two-sided idempotent ideals. Then*

$$I \wedge J = \bigcup_e MeM,$$

where e *runs through all the idempotents of* $I \cap J$.

Proof. The right hand side is a two-sided ideal, generated by idempotents. Hence, it is an element of $\mathscr{II}(M)$. If $e \in I \cap J$, it follows that $MeM \subseteq I$ and $MeM \subseteq J$. Thus, $\bigcup_e MeM \subseteq I \wedge J$. Conversely, assume K is a two-sided idempotent ideal such that $K \subseteq I$ and $K \subseteq J$. By Lemma 7.4.19, we can write $K = \bigcup_{i \in I} Me_iM$ for some idempotents e_i. We have $e_i \in I \cap J$ by assumption, which implies that $I \wedge J$ is also a subset of $\bigcup_e MeM$. The result follows. $\mathcal{Q.E.D}$

Corollary 7.4.23. *If* M *is finite, then* $\mathscr{II}(M)$ *is a distributive lattice.*

Proof. Take $I, J, K \in \mathscr{II}(M)$. We need to show that

$$I \wedge (J \cup K) = (I \wedge J) \cup (I \wedge K).$$

By Lemma 7.4.22, we see that $I \wedge (J \cup K)$ is generated as a two-sided ideal by the idempotent elements of $I \cap (J \cup K)$, while $(I \wedge J) \cup (I \wedge K)$ is generated (as a two-sided ideal) by the idempotent elements of $I \cap J$ and $I \cap K$. The result follows. $\mathcal{Q.E.D}$

7.4.10 Topology on $\mathsf{F_M}$

We start by recalling the well-known relationship between (finite) distributive lattices, posets and topologies.

Any poset P has a natural topology, called the *order topology*, where a subset $S \subseteq P$ is open if $y \in P$ and $x \leq y$ implies $x \in P$. Thus, $\mathfrak{Off}(P)$ is a distributive lattice and it is finite if P is finite. It is well-known that any finite distributive lattice L is of this form, for a uniquely defined P (see for example [66, p.106]). Specifically, for $P = \mathsf{Irr}(L)$, the subset of irreducible elements of L (an element $x \in L$ is irreducible if $x = y \vee z$ implies $x = y$ or $x = z$).

The poset of our interest is $\mathsf{F_M}$, the iso-classes of the topos points of M, where M is finite. According to Corollary 7.4.16, we have

$$\mathsf{F_M} \cong \mathsf{Idem}_{\mathscr{J}}(\mathsf{M}),$$

which allows us to work with $\mathsf{Idem}_{\mathscr{J}}(M)$ instead of $\mathsf{F_M}$, interchangeably. The set $\mathsf{Idem}_{\mathscr{J}}(M)$ has a canonical order

$$e \leq_{\mathscr{J}} f \quad \text{if} \quad MeM \subseteq mfM.$$

Thus, we have a canonical order topology on $\mathsf{Idem}_{\mathscr{J}}(M)$ and as such, on $\mathsf{F_M}$.

Proposition 7.4.24. *One has a bijection*

$$\mathscr{IJ}(M) \to \mathfrak{Off}(\mathsf{Idem}_{\mathscr{J}}(M)).$$

Proof. It suffices to show that irreducible elements of $\mathscr{IJ}(M)$ are exactly ideals of the form MeM, where e is an idempotent. This follows form the facts that for any element $I \in \mathscr{IJ}(M)$, one has $I = \bigcup_{e \in I} MeM$, where e is an idempotent. If $MeM = J \cup K$, then $e \in K$, or $e \in J$. We get that $MeM = J$, or $MeM = K$. $\mathcal{Q.E.D}$

The following corollary can already be found as [3, Exercise 9.1.12(c)].

Corollary 7.4.25. *Let M be a finite monoid. The topology on $\mathsf{F_M}$, defined using localisations, is homeomorphic to the order topology of the poset of principal idempotents ideals.*

We will see that if M is a finitely generated commutative monoid, the topology on $\mathsf{F_M}$, defined using localisations, is homeomorphic to $\mathsf{Spec}(M)$.

7.5 The points of the topos $\mathscr{S}_M$, when M is commutative

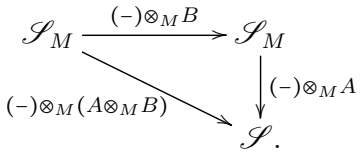
7.5.1 General remarks and relation with prime ideals

In this section, we will assume that M is a commutative monoid. As such, we will no longer make any distinction between left and right M-sets. The category of M-sets will be written

as $\mathscr{S}_M$, and the action of M on A will be written as ma.

Proposition 7.5.1. *Let M be a monoid and A, B filtered M-sets. Then $A \otimes_M B$ is a filtered M-set as well.*

Proof. Consider the commutative diagram of topoi:



The composition $(-) \otimes_M (A \otimes_M B)$ commutes with finite limits and all colimits since both functors $(-) \otimes_M A$ and $(-) \otimes_M B$ do so. $\mathcal{Q.E.D}$

Now we will construct a map

$$\gamma_M : \mathsf{Spec}(M) \to \mathsf{F_M}$$

from the set of prime ideals of M to the set of isomorphism classes of points of the topos $\mathscr{S}_M$.

Lemma 7.5.2. *Let $\mathfrak{p}$ be a prime ideal of M. The localisation $M_\mathfrak{p}$ is a filtered M-set, where $M_\mathfrak{p}$ is considered as an M-set via the canonical monoid homomorphism $M \to M_\mathfrak{p}$.*

Proof. The set $M_\mathfrak{p}$ is not empty since $1 \in M$. Hence, (F1) holds. Assume $m_1 a = m_2 a$ for $m_1, m_2 \in M$ and $a \in M_\mathfrak{p}$. We can write $a = \frac{m_3}{s}$, where $m_3 \in M$ and $s \in M \setminus \mathfrak{p}$. There is an element $t \notin \mathfrak{p}$ such that $m_1 m_3 st = m_2 m_3 st$. This is true since $\frac{m_1 m_3}{s} = \frac{m_2 m_3}{s}$. Take $m = m_3 st$ and $\tilde{a} = \frac{1}{s^2 t}$. We see that condition (F2) holds.

To show (F3), take two elements $a_1 = \frac{m_1'}{s_1}$ and $a_2 = \frac{m_2'}{s_2}$ in $M_\mathfrak{p}$. Put $a = \frac{1}{s_1 s_2}$, $m_1 = m_1' s_2$ and $m_2 = m_2' s_1$. We see that $m_1 a = a_1$ and $m_2 a = a_2$, proving the assertion. $\mathcal{Q.E.D}$

This lemma shows that we have a map

$$\gamma_M : \mathsf{Spec}(M) \to \mathsf{F_M}$$

from the prime ideals of a commutative monoid M to the isomorphism classes of the topos points of $\mathscr{S}_M$. As previously shown [48], $\mathsf{Spec}(M)$ is in fact a lattice. As such, it has two intertwining operations. On their own, both of them make it into a commutative monoid. One is simply the union (recall that in the monoid world, the union of ideals is again an ideal) and the other is defined as follows: For two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, define $\mathfrak{p} \Cap \mathfrak{q}$ to be the union of all the prime ideals contained in both $\mathfrak{p}$ and $\mathfrak{q}$. That is,

$$\mathfrak{p} \Cap \mathfrak{q} := \bigcup_i \mathfrak{p}_i \mid \mathfrak{p}_i \subseteq \mathfrak{p} \cap \mathfrak{q}. \tag{7.4}$$

The combination of Lemma 7.5.1 and Example 7.3.6 show that $\mathsf{F_M}$ is a commutative monoid. It is straightforward to see that $M_\mathfrak{p} \otimes_M M_\mathfrak{q} \simeq M_{\mathfrak{p} \Cap \mathfrak{q}}$. Hence, γ_M is a monoid homomorphism.

By ii) Lemma 7.5.4, $M_\mathfrak{p}$ and $M_\mathfrak{q}$ are isomorphic as M-sets if and only if $\mathfrak{p} = \mathfrak{q}$. We sum up our discussion in the following proposition.

Proposition 7.5.3. *Let M be a commutative monoid. There exists an injective homomorphism of commutative monoids*

$$\gamma_M : \mathsf{Spec}(M) \to \mathsf{F_M},$$

given by $\mathfrak{p} \mapsto M_\mathfrak{p}$.

7.5.2 The case when $M/M^\times$ is finitely generated. From topos points to prime ideals

The aim of this subsection is to prove that the homomorphism γ_M, defined above, is an isomorphism if $M/M^\times$ is finitely generated. Here, $M^\times$ denotes the subgroup of invertible elements. Recall that for an M-set A and an element $m \in M$, we defined the homomorphism $l_m : A \to A$ by $l_m(a) = ma$.

Lemma 7.5.4. *Let A be an M-set.*

i) The set $\mathfrak{p}_A$, or simply $\mathfrak{p}$ when there is no ambiguity, defined by

$$\mathfrak{p} = \{m \in M \mid l_m : A \to A \textit{is not an isomorphism}\},$$

is a prime ideal of M. Moreover, the action of M on A extends to an action of $M_\mathfrak{p}$ on A. Hence, A is an $M_\mathfrak{p}$-set.

ii) If $A = M_\mathfrak{p}$, then $\mathfrak{p}_A = \mathfrak{p}$.

Proof. i) Take $m \in \mathfrak{p}$ and $x \in M$. Assume l_{xm} is an isomorphism and let $g : A \to A$ be the inverse of l_{xm}. Then $l_x \circ g = g \circ l_x$ since g is a map of M-sets and $g \circ l_{xm} = \mathsf{Id} = l_{xm} \circ g$. We have

$$l_m \circ l_x \circ g = l_{mx} \circ g = l_{xm} \circ g = \mathsf{Id}$$

and

$$l_x \circ g \circ l_m = g \circ l_x \circ l_m = g \circ l_{xm} = \mathsf{Id}$$

since $l_{xm} = l_x \circ l_m$. Hence, l_m is an isomorphism, which contradicts our assumption. This proves that $\mathfrak{p}$ is an ideal.

To see that it is prime, take $m, n \notin \mathfrak{p}$. The composition $l_{mn} = l_m \circ l_n$ is an isomorphism since both l_m and l_n are isomorphisms. Thus, $mn \notin \mathfrak{p}$.

For the last statement, take an element $a \in A$ and $x = \frac{m}{s} \in M_\mathfrak{p}$. The map l_s is an isomorphism since $s \notin \mathfrak{p}$. Put

$$xa := ml_s^{-1}(a).$$

It is straightforward to check that this defines an action of $M_\mathfrak{p}$ on A.

ii) Take $s \notin \mathfrak{p}$. The map l_s is an isomorphism. Hence, $s \notin \mathfrak{p}_A$ and we see that $\mathfrak{p}_A \subseteq \mathfrak{p}$. Assume, there exists $x \in \mathfrak{p}$ such that l_x is an isomorphism. Then $1 \in \mathsf{Im}(l_x)$. So, $x \cdot \frac{y}{s} = 1$ in

$M_{\mathfrak{p}}$ for some $y \in M$ and $s \notin \mathfrak{p}$. It follows that $xyt \notin \mathfrak{p}$ for some $t \notin \mathfrak{p}$. We see that $x \notin \mathfrak{p}$ since $\mathfrak{p}$ is prime. This contradiction implies the result. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.5.2.1 Conservative M-sets

We call an M-set A *conservative*, provided any $m \in M$, for which $l_m : A \to A$ is an isomorphism, is an invertible element of M.

Lemma 7.5.5. *Let A be an M-set. Then A is conservative as an $M_{\mathfrak{p}}$-set, where $\mathfrak{p} = \mathfrak{p}_A$ is the prime ideal defined in Lemma 7.5.4.*

Proof. Any element $x \in M_{\mathfrak{p}}$ can be written as $x = \frac{m}{s}$, where $s \notin \mathfrak{p}$. We have $l_x = l_s^{-1} \circ l_m$. Assume $l_x : A \to A$ is an isomorphism. Then $l_m = l_s \circ l_x$ is also an isomorphism. Thus, $m \notin \mathfrak{p}$ and therefore, x is invertible in $M_{\mathfrak{p}}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.5.2.2 Source

Let A be an M-set. We call an element $a \in A$ a *source*, provided $a = mb$ implies that $l_m : A \to A$ is an isomorphism. Here, $m \in M$ and $b \in A$.

Lemma 7.5.6. *Let $\mathfrak{p}$ be a prime ideal of M and assume that $M_{\mathfrak{p}}$ is a conservative M-set. Then $M_{\mathfrak{p}} \simeq M$ and 1 is a source of $M_{\mathfrak{p}}$.*

Proof. Take $s \in M \setminus \mathfrak{p}$. The homomorphism $l_s : M_{\mathfrak{p}} \to M_{\mathfrak{p}}$, $x \mapsto sx$ is an isomorphism since s is invertible in $M_{\mathfrak{p}}$. The definition of conservativeness implies that s is already invertible in M and as such, $M \simeq M_{\mathfrak{p}}$. It is now clear that the unit, indeed every invertible element, is a source. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 7.5.7. *Let A be a filtered and conservative M-set and take an element $a \in A$. The map*

$$\lambda_a : M \to A, \quad \lambda_a(m) = ma$$

is an isomorphism if a is a source of A.

Proof. Take $b \in A$. By Property (F3), there are $c \in A$, $m_1, m_2 \in M$ such that $a = m_1 c$ and $b = m_2 c$. It follows that m_1 is invertible since A is a conservative M-set and a is a source. Hence, $b = m_2 m_1^{-1} a$ and λ_a is surjective.

Assume $\lambda_a(m_1) = \lambda_a(m_2)$, that is $m_1 a = m_2 a$. By (F2), there are elements $b \in A$ and $m \in M$ such that $a = mb$ and $m_1 m = m_2 m$. Since a is a source and A conservative, m must be invertible and thus, $m_1 = m_2$. This shows that λ_a is injective and hence, an isomorphism. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 7.5.8. *Let A be a filtered M-set and $m \in M$. The map $l_m : A \to A$, given by $a \mapsto ma$, is an isomorphism if it is surjective.*

Proof. Let $a_1, a_2 \in A$ and assume $ma_1 = ma_2$ holds. Property (F3) says that there exist $a \in A$ and $m_1, m_2 \in M$ such that $m_1 a = a_1, m_2 a = a_2$. Thus, $mm_1 a = mm_2 a$. By (F2), there exist

an element $b \in A$ and $m_3 \in M$ such that $a = m_3 b$ and $mm_1m_3 = mm_2m_3$. There exist $c \in A$ such that $b = mc$ since l_m is surjective. We have

$$a_1 = m_1 a = m_1 m_3 b = m_1 m_3 mc = m_2 m_3 mc = m_2 m_3 b = m_2 a = a_2.$$

Hence, l_m is injective and as such, an isomorphism. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

We will say that a commutative monoid M is *almost finitely generated* if there is a finite number of non-invertible elements $c_1, \cdots, c_k \in M \setminus M^\times$ such that any $m \in M$ can be written as a product $m = g e_1^{\alpha_1} \cdots e_k^{\alpha_k}$. Here, $g \in M^\times$ is invertible and $\alpha_1, \cdots, \alpha_k \in \mathbb{N}$ are natural numbers. The elements $e_1, \cdots, e_k$ are called *almost generators*. Clearly, this class encompasses any group, as well as any finitely generated monoid. Moreover, if M is almost finitely generated and $\mathfrak{p}$ is a prime ideal of M, then $M_\mathfrak{p}$ is also almost finitely generated. Indeed, this is true for all localisations since every localisation in a monoid is isomorphic to a localisation with a prime ideal, see [18].

An other way of saying that M is almost finitely generated is that $M/M^\times$ is finitely generated.

Lemma 7.5.9. *Let M be an almost finitely generated monoid and A a filtered and conservative M-set. There exists an element $a \in A$ such that the canonical map $\lambda_a : M \to A$, $m \mapsto ma$ is an isomorphism of M-sets.*

Proof. Let $e_1, \cdots, e_k$ be almost generators of M. The map $l_{e_i} : A \to A$ is not an isomorphism for any $i = 1, \cdots, k$ since A is conservative. It follows from Lemma 7.5.8 that the maps l_{e_i} are not surjective. Hence, there is an element $a_i \in A$ for every i such that a_i is not in the image of l_{e_i}. This means that the equations $e_i x = a_i$ have no solutions in A. We know from (F3) that there exists an element $a \in A$ and a finite collection $m_1, \cdots, m_k \in M$ such that

$$a_i = m_i a, \quad i = 1, \cdots, k.$$

We claim that a is a source of A. Let $a = mb$, where $b \in A$ and $m \in M$. Our assumption that M is almost finitely generated implies that the element m can be written in the form

$$m = g e_1^{\alpha_1} \cdots e_k^{\alpha_k}, \quad \alpha_i \in \mathbb{N}.$$

Assume there exists an $i \in \{1, \cdots, k\}$ such that $\alpha_i > 0$. We have $m = e_i m'$ and thus,

$$a_i = m_i a = m_i m b = m_i (e_i m') b = e_i x,$$

where $x = m_i m' b$. This contradicts the assumption that $a_i = e_i x$ has no solution. As such, $\alpha_1 = \cdots = \alpha_k = 0$ and $m = g$ is invertible in M. It follows that a is a source. Since M is conservative, $\lambda_a : M \to A$ is an isomorphism by Lemma 7.5.7. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

We are now in position to prove our main theorem of this section.

Theorem 7.5.10. *Let M be an almost finitely generated monoid. There exists an isomorphism of commutative monoids*

$$\gamma_M : \mathsf{Spec}(M) \to \mathsf{F_M},$$

given by $p \mapsto M_p$. In particular, there exist a prime ideal $\mathfrak{p}$ for any point $f : \mathscr{S} \to \mathscr{S}_M$ of the topos $\mathscr{S}_M$ such that the functors

$$f^* : \mathscr{S}_M \to \mathscr{S} \text{ and } (-)_{\mathfrak{p}} : \mathscr{S}_M \to \mathscr{S}$$

are isomorphic.

Proof. Let A be a filtered M-set and consider the prime ideal $\mathfrak{p} = \mathfrak{p}_A$ defined in Lemma 7.5.2. Then A becomes an $M_{\mathfrak{p}}$-set, which is conservative and filtered over $M_{\mathfrak{p}}$. Hence, Lemma 7.5.9 implies that A is isomorphic to $M_{\mathfrak{p}}$ and we are done. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.5.3 $\Delta(P)$: Ordering of prime ideals as topos points

Let $\mathscr{T}$ be a topos and $P = (P^\bullet, P_\bullet) \in \mathbf{Pts}(\mathscr{T})$ a point. Denote by $\mathsf{End}(P)$ the monoid of endomorphisms of P, meaning the monoid of natural transformations of the inverse image functor $P^\bullet$.

Let X be an object of $\mathscr{T}$, $x \in P^\bullet(X)$ and $\alpha \in \mathsf{End}(P)$. The set $P^\bullet(X)$ has a natural left $\mathsf{End}(P)$-set structure, given by

$$\alpha \cdot x := \alpha(X)(x).$$

It follows that any topos point P has a canonical decomposition

$$\begin{array}{ccccc} \mathscr{S} & \xrightarrow{\mathcal{U}} & \mathscr{S}_{\mathsf{End}(P)} & \xrightarrow{\hat{P}} & \mathscr{T}. \\ & \searrow & & \nearrow & \\ & & P & & \end{array}$$

Here, $\mathcal{U}$ is the trivial topos point of $\mathscr{S}_{\mathsf{End}(P)}$, whose inverse image functor is the forgetful functor and $\hat{P}$ is a geometric morphism. Let $Q : \mathscr{S} \to \mathscr{S}_{\mathsf{End}(P)}$ be a topos point of $\mathscr{S}_{\mathsf{End}(P)}$. The composition $\hat{P} \circ Q$ defines a topos point of $\mathscr{T}$. The subcategory of $\mathbf{Pts}(\mathscr{T})$ obtained in this way will be denoted by $\Delta(P)$. These are the topos points of $\mathscr{T}$ that factor through $\mathscr{S}_{\mathsf{End}(P)}$. We clearly have $P \in \Delta(P)$. This can be thought of as the '*canonical neighbourhood*' of the topos point P in the 'space' of all points of $\mathscr{T}$.

We aim to show that this agrees with the Zariski topology for almost finitely generated commutative monoids.

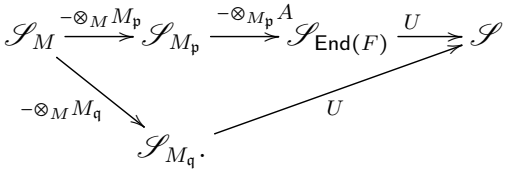
Let M be a commutative monoid and $\mathfrak{p}, \mathfrak{q} \in \mathsf{Spec}(M)$ its prime ideals. Denote by $P, Q : \mathscr{S} \to \mathscr{S}_{M_{\mathfrak{p}}}$ the topos points associated to these prime ideals. In other words, we have $P^\bullet = - \otimes_M M_{\mathfrak{p}}$ and $Q^\bullet = - \otimes_M M_{\mathfrak{q}}$.

Lemma 7.5.11. *Let $F : \mathscr{S} \to \mathscr{S}_{M_{\mathfrak{p}}}$ be a topos point such that the diagram*

$$\begin{array}{ccccc} \mathscr{S}_M & \xrightarrow{-\otimes_M M_{\mathfrak{p}}} & \mathscr{S}_{M_{\mathfrak{p}}} & \xrightarrow{F^\bullet} & U \\ & \searrow^{-\otimes_M M_{\mathfrak{q}}} & & \nearrow_{U} & \\ & & \mathscr{S}_{M_{\mathfrak{q}}} & & \end{array}$$

commutes. Here, U denotes the forgetful functor. We have $F^\bullet = - \otimes_{M_\mathfrak{p}} M_\mathfrak{q}$, where we identify $\mathfrak{q} = \mathfrak{q}M_\mathfrak{p} \in \mathrm{Spec}(M_\mathfrak{p})$.

Proof. We know that the inverse image $F^\bullet$ of F is given by $- \otimes_{M_\mathfrak{p}} A$, where A is a filtered $M_\mathfrak{p}$-set, and simultaneously, an $\mathrm{End}(F)$-set. This yields the following commutative diagram



Here, U denotes the forgetful functor. Commutativity implies that

$$U((- \otimes_M M_\mathfrak{p}) \otimes_{M_\mathfrak{p}} A) \simeq U(- \otimes_M M_\mathfrak{q}).$$

In particular

$$(M \otimes_M M_\mathfrak{p}) \otimes_{M_\mathfrak{p}} A \simeq M \otimes_M M_\mathfrak{q},$$

and hence, $A \simeq M_\mathfrak{q}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

This Lemma has the following direct corollary.

Corollary 7.5.12. *We have $Q \in \Delta(P)$ if and only if $\mathfrak{q} \subseteq \mathfrak{p}$.*

The implications of this are as follows: Saying $Q \leq P$ if $Q \in \Delta(P)$ defines a partial ordering on F, the isomorphism classes of topos points of $\mathscr{S}_M$. This ordering agrees with the natural order of $\mathrm{Spec}(M)$, if we identify $\mathrm{Spec}(M)$ as a part of $\mathrm{F_M}$ via the natural inclusion defined in Proposition 7.5.3.

We will also mention the following little fact, thought it is not needed for the purposes of this dissertation.

Lemma 7.5.13. *We have $Q \in \Delta(P)$ if and only if $\Delta(Q) \subseteq \Delta(P)$.*

Proof. One side is clear. Assume $Q \in \Delta(P)$ and let $Q_1 \in \Delta(Q)$. There exists a geometric morphism $G : \mathscr{S} \to \mathscr{S}_{M_\mathfrak{q}}$ such that $G^\bullet \circ (- \otimes_M M_\mathfrak{q}) = Q_1^\bullet$. Since $Q \in \Delta(P)$, we know from Lemma 7.5.11 and Corollary 7.5.12 that there exists a topos point $F : \mathscr{S} \to \mathscr{S}_{M_\mathfrak{p}}$, with $F^\bullet = - \otimes_{M_\mathfrak{p}} M_\mathfrak{q}$. In particular, we have $F^\bullet \circ (- \otimes_M M_\mathfrak{p}) \simeq - \otimes_M M_\mathfrak{q}$. Hence,

$$Q_1^\bullet = G^\bullet \circ (- \otimes_M M_\mathfrak{q}) = (G^\bullet \circ F^\bullet) \circ (- \otimes_M M_\mathfrak{p}),$$

showing that $Q_1 \in \Delta(P)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Let M be a commutative monoid. Recall the associated semilattice M^{sl}, given by $M^{\mathsf{sl}} = M/\sim$, where $\sim$ is the congruence generated by $m^2 = m$ for every $m \in M$. The explicit relation was given in Formula (4.1) on page 103.

Proposition 7.5.14. *Let M be a commutative monoid. There are injective homomorphisms of commutative monoids*

$$M^{\mathsf{sl}} \xrightarrow{\alpha_M} \mathsf{Spec}(M) \xrightarrow{\gamma_M} \mathsf{F_M},$$

which are isomorphisms when M is finitely generated. The monoid structures are induced by the quotient, the operation $\Cap$ and the tensor product, respectively.

We recall that Spec is a contravariant functor, whereas sl and F are both covariant functors. The composite map $\gamma_M \circ \alpha_M$ sends an element $[f] \in M^{\mathsf{sl}}$ (which can be thought of as a class of an element $f \in M$) to the point of the topos of M-sets corresponding to the filtered M-set M_f. Based on this fact, one easily sees that $\gamma_M \circ \alpha_M$ defines a natural transformation from the functor $M \mapsto M^{\mathsf{sl}}$ to the functor $M \mapsto \mathsf{F_M}$.

We will see in Section 7.6.5 that when M is not finitely generated, neither α_M nor γ_M are isomorphisms in general.

7.6 Infinitely generated commutative monoids

We will now investigate the case of non-finitely commutative generated monoids. We will give a description of the topos points of free commutative monoids, and show that in this case, the prime ideals are no longer enough to classify them all.

7.6.1 Cancellative monoids

Recall that a monoid M is called cancellative if for any elements a, b, c of M, $ab = ac$ implies that $b = c$. Condition $(F2)$ in Lemma 7.3.1 can be simplified as saying that if $m_1 a = m_2 a$, then $m_1 = m_2$.

Lemma 7.6.1. *Let M be a cancellative monoid and A be a filtered M-set.*

i) Let $a, b \in A$ and $m \in M$. If $ma = mb$, then $a = b$.

ii) Let $Ma = Mb$, where $a, b \in A$. There exists an invertible element $m \in M$ such that $a = mb$.

Proof. i) There are elements $c \in A$, $m_1, m_2 \in M$ by (F3) such that $m_1 c = a$ and $m_2 c = b$. So, $m_1 mc = m_2 mc$. Hence, $m_1 m = m_2 m$ and $m_1 = m_2$.

ii) We have $a = mb$ and $b = na$ for some $m, n \in M$. As such, $a = mna$ and $mn = 1$. This finishes the proof. $\mathcal{Q.E.D}$

Let M be a commutative monoid. The universal (or Grothendieck) group of M will be denoted by M^{Gr}. Recall that it consists of elements of the form $\frac{a}{b}$, $a, b \in M$. Two elements $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent in M^{Gr} if there exists an element $s \in M$ such that $sad = scb$.

Lemma 7.6.2. *Let M be a cancellative monoid.*

i) Any filtered M-set is isomorphic to an M-subset of M^{Gr}.

ii) Assume A and B are filtered M-subsets of M^{Gr} and $\alpha : A \to B$ is an M-set homomorphism. There exists $b \in \mathrm{M}^{\mathrm{Gr}}$ such that $\alpha(a) = ab$ for all $a \in A$.

Proof. i) This is a direct consequence of Corollary 7.3.9 and Example 7.3.5.

ii) It follows from the proof of Corollary 7.3.9 and Example 7.3.5 that we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_B} \\ A \otimes_M \mathrm{M}^{\mathrm{Gr}} & \xrightarrow{\alpha_*} & B \otimes_M \mathrm{M}^{\mathrm{Gr}} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{M}^{\mathrm{Gr}} & \xrightarrow[\beta]{} & \mathrm{M}^{\mathrm{Gr}}. \end{array}$$

Here, i_A and i_B are injective maps. We have $\beta(ax) = a\beta(x)$ for all $a, x \in \mathrm{M}^{\mathrm{Gr}}$ since β is a M^{Gr}-map. Take $x = 1$ to obtain $\beta(a) = ab$, where $b = \beta(1)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 7.6.3. *Let M be a cancellative monoid and A and B filtered M-sets. Any morphism of M-sets $\alpha : A \to B$ is injective.*

Proposition 7.6.4. *Let M be a commutative monoid and P a topos point of $\mathscr{S}_M$. Assume M is cancellative or almost finitely generated. The monoid* $\mathrm{End}(P)$ *of endofunctors of P is commutative.*

Proof. Let A be a filtered M-set corresponding to the point P. We know that there is an isomorphism of monoids $\mathrm{End}_{\mathscr{S}_M}(A) \simeq \mathrm{End}_M(P)$.

Assume M is cancellative and let $\phi, \psi \in \mathrm{End}_{\mathscr{S}_M}(A)$. By Lemma 7.6.2, we can assume that A is an M-subset of M^{Gr} and that $\phi(a) = s_\phi \cdot a$, $\psi(a) = s_\psi \cdot a$, for all $a \in A$. Here, s_ψ and s_ϕ are the appropriate elements in M^{Gr}. It follows that

$$\phi\psi(a) = s_\psi s_\phi \cdot a = \psi\phi(a).$$

Hence, $\mathrm{End}_{\mathscr{S}_M}(A)$ is commutative in this case.

If M is almost finitely generated, we have $A \simeq M_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$, by Theorem 7.5.10. One easily sees that $\mathrm{End}_{\mathscr{S}_M}(A) \simeq M_{\mathfrak{p}}$ as monoids. This implies the result. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.6.2 Free commutative Monoids

In what follows, M will denote a free commutative monoid with a basis set $\mathcal{P}$. The aim is to investigate $\mathbf{Pts}(\mathscr{S}_M)$, the points of the topos of M-sets for arbitrary $\mathcal{P}$. We will describe the isomorphism classes of $\mathbf{Pts}(\mathscr{S}_M)$ in terms of equivalence classes of certain functions $\mathcal{P} \to \mathbb{Z} \coprod \{-\infty\}$. Our result is equivalent to the one obtained in [16], when $\mathcal{P}$ is countably infinite. Our methods and description, however, are rather different.

Recall that we denote the universal group of M by M^{Gr}. In our case, it is the free abelian group with basis $\mathcal{P}$. Thus, any element $x \in \mathrm{M}^{\mathrm{Gr}}$ has a unique presentation as

$$x = \prod_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}^{\nu_{\mathfrak{p}}(x)}.$$

Here, $\mathsf{v}_\mathfrak{p}(x) \in \mathbb{Z}$ and $\mathsf{v}_\mathfrak{p}(x) = 0$ for all but a finite number of $\mathfrak{p}$. It is also clear that $x \in M$ if and only if $\mathsf{v}_\mathfrak{p}(x) \geq 0$ for all $\mathfrak{p}$.

7.6.3 The set $\Sigma_*(M)$

Denote by $\Sigma(M)$ the set of all functions

$$f : \mathcal{P} \to \mathbb{Z} \coprod \{-\infty\}$$

for which $f(\mathfrak{p}) \leq 0$ for all but finitely many $\mathfrak{p} \in \mathcal{P}$. We define a congruence relation on $\Sigma(M)$, where $f \sim g$ provided: $f(\mathfrak{p}) = -\infty$ if and only if $g(\mathfrak{p}) = -\infty$, and $f(\mathfrak{p}) = g(\mathfrak{p})$ for all but finitely many $\mathfrak{p} \in \mathcal{P}$. In other words, if the sum of all their differences $\sum_{\mathfrak{p}\in\mathcal{P}} f(\mathfrak{p}) - g(\mathfrak{p})$ is finite.

Denote by $\Sigma_*(M)$ the set of equivalence classes $\Sigma(M)/\sim$ under this equivalency. The obvious addition in $\mathbb{Z} \coprod \{-\infty\}$ induces a monoid structure on $\Sigma_*(M)$.

Theorem 7.6.5. *Let M be a free monoid. There is an isomorphism of monoids*

$$\mathsf{F}_\mathsf{M} \simeq \Sigma_*(\mathsf{M}).$$

Recall that we denoted by F_M the isomorphism classes of the topos points of M-sets. The proof is based on several auxiliary results and is given in Subsection 7.6.4.

7.6.4 Auxiliary results

Let A be a filtered M-subset of M^{Gr} and $\mathfrak{p} \in \mathcal{P}$. That is to say, A is a subset of M^{Gr} which is additionally filtered as an M-set. We know by part i) of Lemma 7.6.2 that all M-sets are of this form. Denote

$$\mathsf{v}_\mathfrak{p}(A) = \inf\{\mathsf{v}_\mathfrak{p}(a) \mid a \in A\}.$$

We have $\mathsf{v}_\mathfrak{p}(A) \neq +\infty$ since $A \neq \varnothing$. Thus, $\mathsf{v}_\mathfrak{p}(A) \in \mathbb{Z} \coprod \{-\infty\}$. We obtain the function

$$\mathsf{v}(A) : \mathcal{P} \to \mathbb{Z} \coprod \{-\infty\}, \quad \mathfrak{p} \mapsto \mathsf{v}_\mathfrak{p}(A)$$

by varying $\mathfrak{p} \in \mathcal{P}$.

Proposition 7.6.6. *Let M be a free commutative monoid on a basis set $\mathcal{P}$.*

i) Let A be a filtered M-subset of M^{Gr}. Then $\mathsf{v}(A) \in \Sigma(M)$ and

$$A = \{x \in \mathrm{M}^{\mathrm{Gr}} \mid \mathsf{v}_\mathfrak{p}(x) \geq \mathsf{v}_\mathfrak{p}(A) \quad \text{for all} \quad \mathfrak{p} \in \mathcal{P}\}.$$

ii) Let $f \in \Sigma(M)$ and set

$$A(f) = \{x \in \mathrm{M}^{\mathrm{Gr}} \mid \mathsf{v}_\mathfrak{p}(x) \geq f(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \mathcal{P}\}.$$

Then $A(f)$ is a filtered M-subset of M^{Gr}.

Proof. i) Assume $\mathsf{v}(A) \notin \Sigma(M)$. We have $\mathsf{v}_\mathfrak{p}(A) > 0$ for infinitely many $\mathfrak{p} \in \mathcal{P}$. Let $a \in A$. We have $\mathsf{v}_\mathfrak{p}(a) \geq \mathsf{v}_\mathfrak{p}(A) > 0$ for infinitely many $\mathfrak{p}$. This contradicts the fact that $\mathsf{v}_\mathfrak{p}(a) = 0$ for all but finitely many $\mathfrak{p}$ since $A \neq \varnothing$. Hence, $\mathsf{v}(A) \in \Sigma(M)$.

For the next assertion, observe that

$$A \subseteq \{x \in \mathrm{M}^{\mathsf{Gr}} \mid \mathsf{v}_\mathfrak{p}(x) \geq \mathsf{v}_\mathfrak{p}(A) \quad \text{for all} \quad \mathfrak{p} \in \mathcal{P}\}$$

by the definition of $\mathsf{v}(A)$. To see the other side, take an element $x \in \mathrm{M}^{\mathsf{Gr}}$ such that $\mathsf{v}_\mathfrak{p}(x) \geq \mathsf{v}_\mathfrak{p}(A)$ for all $\mathfrak{p} \in \mathcal{P}$. We have to show that $x \in A$. We can choose elements $a_\mathfrak{p} \in A$, $\mathfrak{p} \in \mathcal{P}$ for which

$$\mathsf{v}_\mathfrak{p}(x) \geq \mathsf{v}_\mathfrak{p}(a_\mathfrak{p}), \tag{7.5}$$

by our assumption on x. Moreover, we can assume $\mathsf{v}_\mathfrak{p}(a_\mathfrak{p}) \leq 0$ if $\mathsf{v}_\mathfrak{p}(A) \leq 0$. Define

$$\sigma(x) = \{\mathfrak{p} \in \mathcal{P} \mid \mathsf{v}_\mathfrak{p}(x) \neq 0\}.$$

Observe that $\sigma(x)$ is a finite set. It follows from the filteredness axiom (F3) that there exists an element $a \in A$ and a finite set $\{m_\mathfrak{p} \in M \mid \mathfrak{p} \in \sigma(x)\}$ such that $a_\mathfrak{p} = m_\mathfrak{p} a$ for all $\mathfrak{p} \in \sigma(x)$. Likewise, we introduce the finite set

$$\pi(a) = \{\mathfrak{p} \in \mathcal{P} \mid \mathfrak{p} \notin \sigma(x) \text{ and } \mathsf{v}_\mathfrak{p}(a) > 0\}.$$

We use axiom (F3) once again to deduce that there exist elements $b \in A$, $m \in M$ and a finite set $\{m_\mathfrak{r} \in M \mid \mathfrak{r} \in \pi(a)\}$ such that $a = mb$ and $a_\mathfrak{r} = m_\mathfrak{r} b$ for all $\mathfrak{r} \in \pi(a)$.

We claim that

$$\mathsf{v}_\mathfrak{p}(x) \geq \mathsf{v}_\mathfrak{p}(b) \quad \text{for all} \quad \mathfrak{p} \in \mathcal{P}.$$

To check the claim, we first observe that

$$\mathsf{v}_\mathfrak{p}(a) = \mathsf{v}_\mathfrak{p}(m) + \mathsf{v}_\mathfrak{p}(b) \geq \mathsf{v}_\mathfrak{p}(b) \quad \text{for all} \quad \mathfrak{p} \in \mathcal{P}.$$

This is true since $m \in M$ and so, $\mathsf{v}_\mathfrak{p}(m) \geq 0$. There are three cases to be considered.

Case 1: $\mathfrak{p} \in \sigma(x)$. By Equation (7.5), we have

$$\mathsf{v}_\mathfrak{p}(x) \geq \mathsf{v}_\mathfrak{p}(a_\mathfrak{p}) = \mathsf{v}_\mathfrak{p}(m_\mathfrak{p}) + \mathsf{v}_\mathfrak{p}(a) \geq \mathsf{v}_\mathfrak{p}(a) \geq \mathsf{v}_\mathfrak{p}(b).$$

Case 2: $\mathfrak{p} \notin \sigma(x)$ and $\mathfrak{p} \notin \pi(a)$. We have $\mathsf{v}_\mathfrak{p}(x) = 0$ and $\mathsf{v}_\mathfrak{p}(a) \leq 0$ by assumption. This implies that

$$\mathsf{v}_\mathfrak{p}(x) = 0 \geq \mathsf{v}_\mathfrak{p}(a) \geq \mathsf{v}_\mathfrak{p}(b).$$

Case 3: $\mathfrak{p} \in \pi(a)$. Equation (7.5) and our assumption implies that we have

$$0 = \mathsf{v}_\mathfrak{p}(x) \geq \mathsf{v}_\mathfrak{p}(a_\mathfrak{p}) = \mathsf{v}_\mathfrak{p}(m_\mathfrak{p}) + \mathsf{v}_\mathfrak{p}(b) \geq \mathsf{v}_\mathfrak{p}(b).$$

This proves the claim. Write $x = nb$, where $n = xb^{-1} \in \mathrm{M}^{\mathrm{Gr}}$. We have

$$\mathsf{v}_{\mathfrak{p}}(n) = \mathsf{v}_{\mathfrak{p}}(x) - \mathsf{v}_{\mathfrak{p}}(b) \geq 0, \ \mathfrak{p} \in \mathcal{P}.$$

It follows that $n \in M$. We see that $x \in A$ since $x = nb$, $b \in A$, $n \in M$ and A is an M-subset of M^{Gr}.

ii) If $x \in A(f)$ and $m \in M$, then $\mathsf{v}_{\mathfrak{p}}(mx) = \mathsf{v}_{\mathfrak{p}}(m) + \mathsf{v}_{\mathfrak{p}}(x) \geq \mathsf{v}_{\mathfrak{p}}(x) \geq f(\mathfrak{p})$ and $mx \in A(f)$. It follows that $A(f)$ is an M-subset of M^{Gr}. To see that it is filtered, we have to check axioms (F1)-(F3). Axiom (F2) holds automatically as $A(f) \subseteq \mathrm{M}^{\mathrm{Gr}}$ and M^{Gr} is a group and in particular, cancellative.

To see that $A(f)$ is non-empty, take an element $a \in \mathrm{M}^{\mathrm{Gr}}$ such that

$$\mathsf{v}_{\mathfrak{p}}(a) = \begin{cases} 0, \text{ if } f(\mathfrak{p}) \leq 0, \\ f(\mathfrak{p}), \text{ if } f(\mathfrak{p}) > 0. \end{cases}$$

Such an element exists because f takes positive values for only a finite number of $\mathfrak{p} \in \mathcal{P}$. It is clear that $a \in A(f)$. This implies $(F1)$.

Take two elements $b, c \in A(f)$. We choose a finite subset $S \subseteq \mathcal{P}(M)$ such that $f(\mathfrak{p}) \leq 0$ and $\mathsf{v}_{\mathfrak{p}}(b) = 0 = \mathsf{v}_{\mathfrak{p}}(c)$ for all elements $\mathfrak{p} \notin S$. Take $d \in \mathrm{M}^{\mathrm{Gr}}$ such that

$$\mathsf{v}_{\mathfrak{p}}(d) = \begin{cases} 0, \text{ if } \mathfrak{p} \notin S \\ f(\mathfrak{p}), \text{ if } \mathfrak{p} \in S. \end{cases}$$

We have $d \in A(f)$. Define $m = bd^{-1}$ and $n = cd^{-1}$. Then $\mathsf{v}_{\mathfrak{p}}(m), \mathsf{v}_{\mathfrak{p}}(n) \geq 0$ for all $\mathfrak{p} \in \mathcal{P}$ and thus, $m, n \in M$. Since $b = md, c = nd$, (F3) follows as well. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

We have shown in the above proposition, in conjunction with part i) of Lemma 7.6.2, that elements of $\Sigma(M)$ define filtered M-sets, and that every filtered M-set is of that form. We will now investigate when two two functions $f, g \in \Sigma(M)$ define isomorphic M-set.

Lemma 7.6.7. *Let $f, g \in \Sigma(M)$ and assume*

$$\mathsf{Hom}_{\mathscr{S}_M}(A(f), A(g)) \neq \varnothing.$$

Then $f(\mathfrak{p}) = -\infty$ implies $g(\mathfrak{p}) = -\infty$. Moreover, $f(\mathfrak{p}) \geq g(\mathfrak{p})$ for all but a finite number of $\mathfrak{p} \in \mathcal{P}(M)$.

Proof. Let $\psi : A(f) \to A(g)$ be an M-set homomorphism. There exists an element $a \in \mathrm{M}^{\mathrm{Gr}}$ such that $\psi(x) = ax$, $x \in A(f)$. This follows directly from Lemma 7.6.2. Hence, $\mathsf{v}_{\mathfrak{p}}(a) + f(\mathfrak{p}) \geq g(\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{P}$ and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 7.6.8. *Let $f, g \in \Sigma(M)$. The isomorphism of M-sets*

$$\alpha : \mathrm{M}^{\mathrm{Gr}} \otimes_M \mathrm{M}^{\mathrm{Gr}} \to \mathrm{M}^{\mathrm{Gr}},$$

given by $\alpha(a \otimes b) = ab$*, restricts to an isomorphism of* M*-sets*

$$A(f) \otimes_M A(g) \to A(f+g).$$

Proof. Take $a \in A(f)$ and $b \in A(g)$. We have $\mathsf{v}_\mathfrak{p}(ab) = \mathsf{v}_\mathfrak{p}(a) + \mathsf{v}_\mathfrak{p}(b) \geq f(\mathfrak{p}) + g(\mathfrak{p})$. This shows that the image of $A(f) \otimes_M A(g)$ under α lies in $A(f+g)$. We get a commutative diagram

$$\begin{array}{ccc} A(f) \otimes_M A(g) & \xrightarrow{\beta} & A(f+g) \\ \downarrow & & \downarrow \\ \mathrm{M}^{\mathsf{Gr}} \otimes_M \mathrm{M}^{\mathsf{Gr}} & \xrightarrow[\alpha]{} & \mathrm{M}^{\mathsf{Gr}}. \end{array}$$

The vertical arrows are injective and β is a restriction of α. Since α is bijective, β is bijective if and only if it is surjective. Let $c \in A(f+g)$ and define

$$S = \{\mathfrak{p} \in \mathcal{P} \mid f(\mathfrak{p}) \leq 0, g(\mathfrak{p}) \leq 0 \text{ and } \mathsf{v}_p(c) = 0\}.$$

We can choose $a \in \mathrm{M}^{\mathsf{Gr}}$ such that

$$\mathsf{v}_\mathfrak{p}(a) = \begin{cases} f(\mathfrak{p}), \text{ if } \mathfrak{p} \notin S \\ 0, \text{ if } \mathfrak{p} \in S \end{cases}$$

since $\mathcal{P} \setminus S$ is finite. We have $a \in A(f)$. Define $b = ca^{-1} \in \mathrm{M}^{\mathsf{Gr}}$. We claim that $b \in A(g)$. Assume $\mathfrak{p} \notin S$. We have

$$\mathsf{v}_\mathfrak{p}(b) = \mathsf{v}_\mathfrak{p}(c) - \mathsf{v}_\mathfrak{p}(a) \geq f(\mathfrak{p}) + g(\mathfrak{p}) - f(\mathfrak{p}) = g(\mathfrak{p}).$$

On the other hand, if $\mathfrak{p} \in S$, one has

$$\mathsf{v}_\mathfrak{p}(b) = \mathsf{v}_\mathfrak{p}(c) - \mathsf{v}_\mathfrak{p}(a) = 0 \geq g(\mathfrak{p}).$$

This shows that $\beta(a \otimes b) = c$ and hence, β is surjective. This finishes the proof. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Proof of Theorem 7.6.5 We are now in position to give the proof of the main theorem of this section.

Proof. We know by part ii) of Proposition 7.6.6 that there exists a filtered M-set $A(f)$ for every $f \in \Sigma(M)$. This association induces a map $\Sigma(M) \to \mathsf{F}_\mathsf{M}$. This is surjective by part i) of Proposition 7.6.6. Assume $f \sim g$ and define

$$S = \{\mathfrak{p} \in \mathcal{P} \mid f(\mathfrak{p}) \neq g(\mathfrak{p})\}.$$

There exist an element $b \in \mathrm{M}^{\mathsf{Gr}}$ such that

$$\mathsf{v}_\mathfrak{p}(b) = \begin{cases} 0, \text{ if } \mathfrak{p} \notin S \\ \mathsf{v}_\mathfrak{p}(g) - \mathsf{v}_\mathfrak{p}(f), \text{ if } \mathfrak{p} \in S \end{cases}$$

since S is a finite set. As such, $\psi : A(f) \to A(g)$ is an isomorphism of M-sets. Here,

$$\psi(a) = ab,\ a \in A(f).$$

This shows that the above map $\Sigma(M) \to \mathsf{F}_{\mathsf{M}}$ factors through $\Sigma_*(M) \to \mathsf{F}_{\mathsf{M}}$, which is naturally surjective as well.

Take $f, g \in \Sigma(M)$ and assume that $A(f)$ and $A(g)$ are isomorphic M-sets. It follows from Lemma 7.6.7 that $f \sim g$. This proves the bijectivity. It respects the monoid structure according to Lemma 7.6.8. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

7.6.5 On the maps α_M and γ_M

Let us return to the injective homomorphisms of commutative monoids

$$M^{\mathsf{sl}} \xrightarrow{\alpha_M} \mathsf{Spec}(M) \xrightarrow{\gamma_M} F_M,$$

see Proposition 7.5.14. We assume that M is a free monoid generated by the infinite set $\mathcal{P}$.

To see that α_M is not an isomorphism, we first observe that $\mathsf{Card}(M) = \mathsf{Card}(\mathcal{P}) = \mathsf{Card}(M^{\mathsf{sl}})$. As previously shown, we have an isomorphism

$$\mathsf{Spec}(M) \simeq \mathsf{Hom}(M, \mathbb{I}).$$

Here, $\mathbb{I} = \{0, 1\}$ with obvious multiplication. It follows that $|\mathsf{Spec}(M)| = 2^{\mathcal{P}}$ since M is free with basis $\mathcal{P}$. As such, $\mathsf{Card}(\mathsf{Spec}(M)) \gneqq \mathsf{Card}(M^{\mathsf{sl}})$ is strictly bigger and α_M can not be an isomorphism.

Lemma 7.6.9. *The homomorphism*

$$2^{\mathcal{P}} \simeq \mathsf{Spec}(M) \xrightarrow{\gamma_M} \mathsf{F}_{\mathsf{M}} \simeq \Sigma_*(\mathsf{M})$$

has the following description, according to Theorem 7.6.5: Let T be a subset of $\mathcal{P}(M)$. Then $\gamma_M(T) \in \Sigma_*(M)$ is represented by the function $\gamma(T) : \mathcal{P}(M) \to \mathbb{Z} \coprod -\infty$, given by

$$\gamma(T)(\mathfrak{p}) = \begin{cases} -\infty, & \text{if } \mathfrak{p} \notin T \\ 0, & \text{if } \mathfrak{p} \in T. \end{cases}$$

Proof. It is clear that the prime ideal $\mathfrak{p}_T$ corresponding to T is given by

$$\mathfrak{p}_T = \{x \in M \mid \mathsf{v}_{\mathfrak{p}}(x) > 0 \text{ for all } \mathfrak{p} \in T\}.$$

Therefore, the localisation of M at the prime ideal $\mathfrak{p}_T$ is

$$\{x \in \mathrm{M}^{\mathsf{Gr}} \mid \mathsf{v}_{\mathfrak{p}}(x) \geq 0 \text{ for all } \mathfrak{p} \in T\} = A(\gamma(T)).$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}$

It is clear from the description that γ_M is not an isomorphism for infinite $\mathcal{P}$: Indeed, let $f \in \Sigma$ be any function for which $-\infty < f(\mathfrak{p}) < 0$ for infinitely many $\mathfrak{p}$. It defines an element in $\Sigma_*(M)$, which is not in the image of γ_M.

Chapter 8

Monoid schemes and topoi

In this chapter, we will study the category $\mathfrak{Qc}(X)$ of quasi-coherent sheaves over a monoid scheme X. Recall that these are sheaves of $\mathcal{O}_X$-sets, which locally (meaning over an open subscheme Spec(M)) look like $\tilde{A}$, where A is an M-set. We prove that if X is an s-Noetherian monoid scheme, the category $\mathfrak{Qc}(X)$ is a locally s-Noetherian topos (see Theorem 8.1.5). Further, we will show that any point $x \in X$ defines a topos point Stalk$(x) : \mathfrak{Qc}(X) \to \mathscr{S}$, which sends a sheaf $\mathcal{A}$ to the stalk $\mathcal{A}_x$. Our next goal will be to study the category $\mathfrak{Qc}(X)$, for $X = \mathrm{Proj}(M_*)$, a graded monoid M_*. This is achieved by representing $\mathfrak{Qc}(X)$ as a localisation of the category $\mathscr{S}_{\mathrm{gr}M}$, of graded M_*-sets, see Proposition 8.3.6. Under some finiteness conditions, we will be able to classify all points of the topos $\mathscr{S}_{\mathrm{gr}M}$, see Theorem 8.2.6. Based on this, we will finally prove that under some condition on X, all points of the topos $\mathfrak{Qc}(X)$ have the form Stalk(x), see Theorem 8.3.9. Thus, from the category $\mathfrak{Qc}(X)$, one can reconstruct the set X as set of isomorphic classes of the topos points of $\mathfrak{Qc}(X)$. Our next goal will be to prove that one can reconstruct underlying topological space and structural sheaf as well. We can summarise our main results in the following main theorem:

Theorem 8.0.1. *Let X be a Noetherian monoid scheme. The following results hold:*

i) The category $\mathfrak{Qc}(X)$ is a locally s-Noetherian topos.

ii) The map $\Phi_X : \mathfrak{Off}(X) \to \mathfrak{Loc}(\mathfrak{Qc}(X))$, from the set of open subsets of X to the localisations of $\mathfrak{Qc}(X)$ given by

$$\Phi_X(U) = \mathfrak{Qc}(U),$$

is an isomorphism of posets.

iii) If U is an open subset of X and $\mathscr{D} = \Phi_X(U)$ is corresponding localisation of $\mathfrak{Qc}(X)$, then for the centre of $\mathscr{D}$ we have an isomorphism

$$\mathcal{O}_X(U) = \mathcal{Z}(\mathscr{D}).$$

iv) For any point $x \in X$, the functor

$$\mathfrak{Qc}(X) \to \mathscr{S}, \quad \mathcal{A} \mapsto \mathcal{A}_x,$$

which assigns to a quasi-coherent sheaf $\mathcal{A}$ the stalk $\mathcal{A}_x$ at x, is the inverse part of a point of the topos $\mathfrak{Qc}(X)$. Furthermore, the corresponding map

$$\Pi : X \to \mathfrak{F}(\mathfrak{Qc}(X))$$

from the underlying topological space of X to the isomorphism classes of the topos points of $\mathfrak{Qc}(X)$, is a bijection if X is quasi-projective.

v) If $x \in X$ and U is an open subset of X, then $x \in U$ if and only if $\Pi(x) \Vdash \Phi_X(U)$.

Part i) appears as Theorem 8.1.5 below, part iv) will be proven in Section 8.3.5. We will prove part ii) in the affine case at the end of Subsection 8.4 (see Corollary 8.4.11), part iii), in full generality, is proven in Section 8.5.2 (Lemma 8.5.3) and finally, part v) and the general case of part ii) is proven at the very end of this section, in Subsection 7.3.9.

8.1 Quasi Coherent Sheaves form a topos

The aim of this section is to prove the part i) of the Theorem 8.0.1. That is, we wish to show that the category $\mathfrak{Qc}(X)$ of quasi-coherent sheaves is a locally s-Noetherian topos under some finiteness conditions on X. We refer to Section 4 for more on monoids and monoid schems, especially Subsection 4.1.3.

We will proceed to study quasi-coherent sheaves in more detail, proving some analogues of the well-known facts on quasi-coherent sheaves over classical schemes. These facts allow us to use our results on the gluing of locally Noetherian topoi (see Section 6.2), to prove that if X is an s-Noetherian monoid scheme, the category $\mathfrak{Qc}(X)$ is a locally s-Noetherian topos (see Theorem 8.1.5).

Henceforth, all monoids are assumed to be commutative and written multiplicatively.

8.1.1 Sheaves of $\mathcal{O}_X$-sets

It is well-known that for an M-set A, there exist a unique sheaf of sets $\tilde{A}$ over Spec(M) such that

$$\Gamma(D(f), \tilde{A}) = A_f.$$

The section of a sheaf $\mathcal{F}$ on an open subset U (i.e. $\mathcal{F}(U)$) will also be denoted by $\Gamma(U, \mathcal{F})$. As with monoids, the localisation of A with respect to a submonoid of M generated by f will be denoted by A_f. The stalk of $\tilde{M}$ at the point $\mathfrak{p}$ is the localisation $A_{\mathfrak{p}}$ of A by the submonoid $M \setminus \mathfrak{p}$. We use the notation $\mathcal{O}_M$ for the sheaf $\tilde{M}$ in the special case when $A = M$. In this manner, we get sheaves of monoids, while the sheaf $\tilde{A}$ becomes a sheaf of $\mathcal{O}_M$-sets, for any M-set A.

We wish to consider such objects in the non-affine setting as well. For this, let X be a monoid scheme and $\mathcal{A}$ a sheaf of $\mathcal{O}_X$-sets. That is to say, $\mathcal{A}$ is a sheaf of sets, together with an action of the monoid $\mathcal{O}_X(U)$ on the set $\mathcal{A}(U)$, for all open $U \subseteq X$. Further, these actions are required to be compatible for varying U. One denotes by $\mathcal{O}_X$-Sets the category of sheaves of $\mathcal{O}_X$-sets.

Lemma 8.1.1. *Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-sets. One has the following isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{A}) \simeq \Gamma(X, \mathcal{A}).$$

Proof. Recall that a morphism $\alpha : \mathcal{O}_X \to \mathcal{A}$ of the category $\mathcal{O}_X$-Sets is a collection of maps $\alpha_U : \mathcal{O}_X(U) \to \mathcal{A}(U)$, where U is running through the open subsets of X. One easily sees that the assignment $\alpha \mapsto \alpha_X(1)$ induces the isomorphism in question. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

It is well-known that the category $\mathcal{O}_X$-Sets is a topos. The *tensor product* of two $\mathcal{O}_X$-sets $\mathcal{A}$ and $\mathcal{B}$ is the sheafification of the presheaf of $\mathcal{O}_X$-sets given by $U \mapsto \mathcal{A}(U) \otimes_{\mathcal{O}_X} \mathcal{B}(U)$. Here, $U \subseteq X$ is an open subset of X.

Let $(f, f^{\#}) : X \to Y$ be a morphism of monoid schemes. As already mentioned, the continuous map f induces a geometric morphism of topoi $(f^{\bullet}, f_{\bullet}) : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. The functor $f_{\bullet}$ respects inverse limits. As such, the sheaf $f_{\bullet}\mathcal{O}_X$ is a sheaf of monoids on Y and for any $\mathcal{O}_X$-set $\mathcal{A}$, the sheaf $f_{\bullet}\mathcal{A}$ is a sheaf of $f_{\bullet}\mathcal{O}_X$-sets. The morphism of monoid sheaves $f^{\#} : \mathcal{O}_Y \to f_{\bullet}\mathcal{O}_X$ allows us to define an action of $\mathcal{O}_Y$ on $f_{\bullet}\mathcal{A}$. We obtain the functor

$$f_{\bullet} : \mathcal{O}_X\text{-Sets} \to \mathcal{O}_Y\text{-Sets}, \quad \text{given by} \quad \mathcal{A} \mapsto f_{\bullet}\mathcal{A}.$$

The functor $f_{\bullet}$ has a left adjoint functor f^*, which is constructed as follows. Take an $\mathcal{O}_Y$-set $\mathcal{B}$. Since $f^{\bullet} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ respects finite limits, $f^{\bullet}\mathcal{B}$ has an $f^{\bullet}\mathcal{O}_X$-set structure.

On the other hand, the morphism of monoid sheaves $f^{\#} : \mathcal{O}_Y \to f_{\bullet}\mathcal{O}_X$ induces (by adjointness) the morphism of monoid sheaves $f^{\bullet}\mathcal{O}_Y \to \mathcal{O}_X$. Hence, $\mathcal{O}_X$ can be considered as an $f^{\bullet}\mathcal{O}_Y$-set and one has

$$f^*(\mathcal{B}) := f^{\bullet}\mathcal{B} \otimes_{f^{\bullet}\mathcal{O}_Y} \mathcal{O}_X.$$

Lemma 8.1.2. *Let U be an open subscheme of a monoid scheme X. The inclusion $j : U \to X$ induces a localisation of topoi $j = (j^*, j_*) : \mathcal{O}_U$-Sets $\to \mathcal{O}_X$-Sets. That is to say, $j^* : \mathcal{O}_X$-Sets $\to \mathcal{O}_U$-Sets is a localisation of categories.*

Proof. In this case, $j^*\mathcal{O}_X = \mathcal{U}_X$. Hence $j^* = j^{\bullet}$, and the result follows from Lemma 6.1.7. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

We will usually write f_* instead of $f_{\bullet}$, in order to synchronise the notation with f^*. Whenever used, we will always have $f_* := f_{\bullet}$.

8.1.2 Quasi-coherent sheaves

Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-set. It is said to be *quasi-coherent* [15] if there is an open and affine subscheme $U = \mathrm{Spec}(M)$ for any point $x \in X$ such that $x \in U$ and the restriction of $\mathcal{A}$ on U is isomorphic to a sheaf of the form $\tilde{A}$. Assume $\mathcal{A}$ is quasi-coherent. It is well-known [15, Theorem 3.12], that the restriction of $\mathcal{A}$ on any open affine subscheme $U = \mathrm{Spec}(M)$, is isomorphic to a sheaf of the form $\tilde{A}$, for some M-set A.

The category of quasi-coherent sheaves over X is denoted by $\mathfrak{Q}c(X)$. It posses all colimits and finite limits since the localisation functor respects them [15, Lemma 3.16]. If $X = \mathrm{Spec}(M)$ is affine, then $A \mapsto \tilde{A}$ defines an equivalence of categories $\mathfrak{Q}c(X) \simeq M$-Sets, see [15, Corollary 3.13].

Lemma 8.1.3. *Let $f : X \to Y$ be a morphism of monoid schemes.*

i) If $\mathcal{B}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-sets, then $f^*\mathcal{B}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-sets.

ii) Assume X is s-Noetherian. If $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-sets, then $f_*\mathcal{A}$ is a quasi-coherent sheaf of $f_*\mathcal{O}_X$-sets.

Proof. i) Take $x \in X$ and consider $f(x) \in Y$. Since $\mathcal{B}$ is quasi-coherent, there is a on open and affine subscheme $V = \mathrm{Spec}(N) \subseteq Y$ such that $f(x) \in V$ and the restriction of $\mathcal{B}$ on V is of the form $\tilde{B}$, for some N-set B. Choose an open and affine subscheme $U = \mathrm{Spec}(M)$ of X such that $f(U) \subseteq V$. The restriction of f gives a morphism $U \to V$. This is determined by a monoid homomorphism $\phi : N \to M$. The result follows since $f^*(\mathcal{B}) = \widetilde{M \otimes_N B}$.

ii) First, consider the case when both $X = \mathrm{Spec}(M)$ and $Y = \mathrm{Spec}(N)$ are affine. In this case, f is induced by $\phi : N \to M$. Moreover, $\mathcal{A} = \tilde{A}$ for an M-set A and $f_*\mathcal{A} = \tilde{A}$. Here, N acts on A via ϕ and the result follows.

Let V be an open subset of a general monoid scheme Y. Then $U = f^{-1}(V)$ is s-Noetherian by Lemma 4.1.11. Hence, we can cover it by a finite number of open affine subschemes U_i, $i = 1, \cdots, n$. Moreover, we can cover each $U_i \cap U_j$ with a finite number of open affine subschemes U_{ijk}. The restriction of $f_*\mathcal{A}$ on V is the equaliser of the diagram

$$\prod_i f_*(\mathcal{A}_{|U_i}) \rightrightarrows \prod_{ijk} f_*(\mathcal{A}_{|U_{ijk}}),$$

as $\mathcal{A}$ is a sheaf. Since the statement is true in affine case, $f_*(\mathcal{A}_{|U_i})$ and $f_*(\mathcal{A}_{|U_{ijk}})$ are quasi-coherent sheaves. We also know that a finite limit of quasi-coherent sheaves is again quasi-coherent [15, Lemma 3.16]. It follows that $f_*\mathcal{A}$ is quasi-coherent since the product, as well as the equaliser, are specific types of limits. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 8.1.4. *Let X be an s-Noetherian monoid scheme and U an open subscheme. The inclusion $j : U \to X$ induces a localisation of categories $j^* : \mathfrak{Qc}(X) \to \mathfrak{Qc}(U)$.*

Proof. The restriction of $j = (j^*, j_*) : \mathcal{O}_U$-Sets $\to \mathcal{O}_X$-Sets on quasi-coherent sheaves yields a well-defined morphism of topoi, by Lemma 8.1.3. The morphism $j^* : \mathcal{O}_U$-Sets $\to \mathcal{O}_X$-Sets is a localisation by Lemma 8.1.2. Hence, the same is true for $j^* : \mathfrak{Qc}(X) \to \mathfrak{Qc}(U)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

The following theorem could be of independent interest, as it introduces a previously unknown class of topoi, which is very closely related to the topoi of M-sets.

Theorem 8.1.5. *Let X be an s-Noetherian monoid scheme. The category $\mathfrak{Qc}(X)$ of quasi-coherent sheaves is a locally s-Noetherian topos.*

Proof. Let $X = \mathrm{Spec}(M)$ be an affine monoid scheme. In this case, $\mathfrak{Qc}(X)$ is equivalent to the category of M-sets. The result follows from Lemma 7.2.2.

For a general monoid scheme X, observe that if U and V are open monoid subschemes of X, the category $\mathfrak{Qc}(U \cup V)$ is equivalent to the gluing of $\mathfrak{Qc}(U)$ and $\mathfrak{Qc}(V)$, along $\mathfrak{Qc}(U \cap V)$. An obvious induction argument, together with Theorem 6.2.8, implies the result. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Remark 8.1.6. *The subobject classifier [42] of the topos $\mathfrak{Qc}(X)$, where X is an s-Noetherian monoids scheme, is obtained by gluing the subobject classifiers of the topoi of quasi-coherent sheaves over affine subschemes. A similar statement for the exponential objects is not true in general. We will not need these facts. Hence, we omit the arguments.*

8.1.3 Line bundles

Recall that a quasi-coherent sheaf $\mathcal{L}$ over a monoid scheme X is called a *line bundle*, if it is locally isomorphic to $\mathcal{O}_X$ [49]. Let X be a monoid scheme and $x \in X$ a point of its underlying space. Denote the maximal ideal of $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$. This is the set of non-invertible elements of the monoid $\mathcal{O}_{X,x}$.

Let $\mathcal{L}$ be a line bundle over X and $f \in \Gamma(X, \mathcal{L})$. We define

$$X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x \mathcal{L}_x\},$$

where f_x is the restriction of f on $\mathcal{O}_{X,x}$. Observe that X_f is an open subset of X. We have $X_f = D(f)$ when $X = \mathrm{Spec}(M)$ is affine and $\mathcal{L} = \mathcal{O}_X$.

We will make use of the obvious paring in the following lemma:

$$\Gamma(X, \mathcal{A}) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}).$$

The image of $a \otimes b$ under this map is denoted by ab.

Lemma 8.1.7. *Let X be an s-Noetherian monoid scheme, $\mathcal{L}$ a line bundle on X and $f \in \Gamma(X, \mathcal{L})$. Moreover, let $\mathcal{F}$ be a quasi-coherent sheaf over X.*

i) Let $s_1, s_2 \in \Gamma(X, \mathcal{F})$ be sections such that restricted on X_f, they are equal. There exist a natural number $k > 0$ such that $f^k s_1 = f^k s_2$ as elements of $\Gamma(X, \mathcal{L}^{\otimes k} \otimes \mathcal{F})$.

ii) For any $t \in \Gamma(X_f, \mathcal{F})$, there exists $n > 0$ such that $f^n t \in \Gamma(X_f, \mathcal{L}^{\otimes n} \otimes \mathcal{F})$ is in the image of the restriction map

$$\Gamma(X, \mathcal{L}^{\otimes n} \otimes \mathcal{F}) \to \Gamma(X_f, \mathcal{L}^{\otimes n} \otimes \mathcal{F}).$$

Proof. First consider the case when $X = \mathrm{Spec}(M)$ is affine. Any line bundle (or, more generally, any vector bundle) is trivial in this case, [49]. Hence, $\mathcal{L} \simeq \mathcal{O}_X$ and so, we can assume that $f \in M$ and $\mathcal{F} = \tilde{A}$ for an M-set A. Since $X_f = D(f)$, the restriction of $\mathcal{F}$ on X_f is $\tilde{A}_f$, the quasi-coherent sheaf associated to the M_f-set A_f. The result now follows from the basic properties of localisation.

For the general case, we choose a finite open cover of X by affine subschemes $X_1, \cdots X_m$. In this case, $X_f = X_{1f} \cup \cdots \cup X_{mf}$.

To see part i), observe that for each $1 \leq i \leq m$, there exists a k_i, such that the restrictions of $f^{k_i} s_1$ and $f^{k_i} s_2$ on X_{if} coincide. Let $k = \max_{1 \leq i \leq m} k_i$ be the maximum of the k_i-s. We see that $f^k s_1$ and $f^k s_2$ agree with each other on every X_{if}. Hence, they agree on X_f. This proves the first assertion.

For the second part, choose a natural number n_i for each $1 \leq i \leq m$ such that $f^{n_i} t$ has an extension to X_i. We take n' to be the maximal among $n_1, \cdots, n_m$. There exists an element $t_i \in \Gamma(X_i, \mathcal{L}^{\otimes n'} \otimes \mathcal{F})$ for each $1 \leq i \leq m$, such that $t_i = f^{n'} t$ restricted to X_{if}. The first part of this lemma says that there exists an integer $n > 0$ such that $f^n t, \cdots, f^n t_m$ glue to a single global element, which restricts to $f^n t$. $\mathcal{Q.E.D}$

8.2 Graded monoids and topoi

From the point of view of monoid schemes, commutative monoids are the dual objects of affine monoid schemes. One can hope that Theorem 7.5.10 and other facts of Section 7.5 are just the affine versions of far more general results about monoid schemes.

We will see in the next section that this is the case, at least for monoid schemes which are open subschemes of 'projective monoid schemes'. The latter is based on the theory of graded monoids, much like in classical algebraic geometry. Hence, this section aims at collecting some basic results on graded monoids.

8.2.1 Graded sets

Let $\{X_i\}_{i\in\mathbb{Z}}$ be a set indexed by the integers. This is called a *graded set* and is dented by X_*. The disjoint union $\mathrm{Tot}(X_*) := \coprod_{i\in\mathbb{Z}} X_i$ is called the *total* or *underlying set* of the graded set X_*.

A *graded map* between X_* and Y_*, denoted by $f_* : X_* \to Y_*$, is given by a family of maps $\{f_i : X_i \to Y_i\}_{i\in\mathbb{Z}}$. Any such family induces a well defined map $\mathrm{Tot}(f_*) : \mathrm{Tot}(X_*) \to \mathrm{Tot}(Y_*)$ of underlying sets, which sends an element $x \in X_i$ to $f_i(x)$.

The category of graded sets and graded maps is denoted by ${}_{\mathrm{gr}}\mathscr{S}$. It is the product of $\mathbb{Z}$-copies of $\mathscr{S}$. Clearly, $\mathrm{Tot} :{}_{\mathrm{gr}}\mathscr{S} \to \mathscr{S}$ is the forgetful functor.

The category ${}_{\mathrm{gr}}\mathscr{S}$ is a topos. The limits and colimits are computed degreewise. Observe that the forgetful functor Tot preserves colimits. But it does not preserves finite products. The terminal object of ${}_{\mathrm{gr}}\mathscr{S}$ is denoted by $\mathbf{1}_*$. The i-th set $\mathbf{1}_i$ is the one element set, say $\{i\}$, for any $i \in \mathbb{Z}$. It is clear that $\mathrm{Tot}(\mathbf{1}_*) = \mathbb{Z}$.

8.2.1.1 Graded product

Let X_* and Y_* be graded sets. We define the graded product $X_* \boxtimes Y_*$ by

$$(X_* \boxtimes Y_*)_i = \coprod_{j+k=i} X_j \times Y_k.$$

We see that $\boxtimes$ defines a symmetric monoidal structure on the category ${}_{\mathrm{gr}}\mathscr{S}$. The unit object for $\boxtimes$ is $\mathbb{I}_*$, where

$$\mathbb{I}_i = \begin{cases} \varnothing, & i \neq 0, \\ \{1\}, & i = 0. \end{cases}$$

One easily checks that

$$\mathrm{Tot}(X_* \boxtimes Y_*) \simeq \mathrm{Tot}(X_*) \times \mathrm{Tot}(Y_*)$$

and $\mathrm{Tot}(\mathbb{I}_*) \simeq \{1\}$. In other words, the functor Tot is symmetric monoidal.

8.2.1.2 Shifting

The category ${}_{\mathsf{gr}}\mathscr{S}$ posses a special class of endofunctors called *shiftings*. Let X_* be a graded set and i an integer. Denote by $X_*(i)$ the graded set given by

$$(X_*(i))_j = X_{i+j}.$$

This yields endofunctors $S_i : {}_{\mathsf{gr}}\mathscr{S} \to {}_{\mathsf{gr}}\mathscr{S}$, given by $X_* \mapsto X_*(i)$ for all $i \in \mathbb{Z}$. Observe that $\mathsf{Tot}(X_*(i)) = \mathsf{Tot}(X_*)$.

8.2.2 Graded monoids

A monoid object in the symmetric monoidal category $({}_{\mathsf{gr}}\mathscr{S}, \boxtimes, \mathbb{I})$ is called a *graded monoid*. Note that this terminology crashes with the term of graded group. Explicitly, a graded monoid is a graded set M_*, together with a distinguished element $1 \in M_0$ and product maps $M_i \times M_j \to M_{i+j}$, $i, j \in \mathbb{Z}$. Further, these must satisfy the associativity and the unit condition. It follows that $\mathsf{Tot}(M_*)$ is a monoid, called the *underlying monoid* of the graded monoid M_*. Let M'_* and M''_* be graded monoids. Their graded product $M_* = M'_* \boxtimes M''_*$ has the structure of a graded monoid given by

$$(M'_* \boxtimes M''_*) \boxtimes (M'_* \boxtimes M''_*) \simeq (M'_* \boxtimes M'_*) \boxtimes (M''_* \boxtimes M''_*) \to (M'_* \boxtimes M''_*).$$

The last map is induced by the graded monoid structures on M'_* and M''_*.

Let M_* and M'_* be graded monoids. A graded map $f_* : M_* \to M'_*$ is called a *homomorphism of graded monoids*, or *graded morphism* for short, if one has $f_{i+j}(xy) = f_i(x)f_j(y)$ for all $i, j \in \mathbb{Z}$, $x \in M_i$ and $y \in M_j$. It is clear that $M_* \mapsto \mathsf{Tot}(M_*)$ defines a covariant functor from the category of graded monoids to the category of monoids.

Let M_* be a graded monoid and $m \in \mathsf{Tot}(M_*)$. Denote by $|m|$ the unique integer for which $m \in M_{|m|}$. This gives us a map $|-| : M_* \to \mathbb{Z}$, called the *degree map*. We have $|m_1 m_2| = |m_1| + |m_2|$, making it a monoid homomorphism.

Conversely, let M be a monoid and $\mu : M \to \mathbb{Z}$ a monoid homomorphism. This allows us to see M as the underlying monoid of a graded monoid M_* for which $M_i = \mu^{-1}(i)$.

We can regard $\mathbb{Z}$ and $\mathbb{N}$ as the underlying monoids of graded monoids by considering the identity and the inclusion map respectively. The graded product $\boxtimes$ allows us to do the same for $\mathbb{Z}^d$ and $\mathbb{N}^d$, $d \geq 1$ as well.

A graded map $f_* : M_* \to M'_*$ is a *morphism of graded monoids* if and only if the diagram

$$\begin{array}{ccc} M_* & \xrightarrow{\quad f \quad} & M'_* \\ & {\scriptstyle |-|} \searrow \quad \swarrow {\scriptstyle |-|} & \\ & \mathbb{Z} & \end{array}$$

commutes.

It is quite useful to consider a monoid M as a one object category. Covariant (resp. contravariant) functors over this category are nothing but left (resp. right) M-sets. One can

define a category $\mathsf{Cat}(\mathsf{M}_*)$ in a similar vain by starting with a graded monoid M_*. Objects of $\mathsf{Cat}(\mathsf{M}_*)$ are integers. The set of homomorphism $\mathsf{Hom}_{\mathsf{Cat}(\mathsf{M}_*)}(i,j)$ between two objects (integers) $i,j \in \mathbb{Z}$ is M_{j-i}. In particular, $\mathsf{Hom}_{\mathsf{Cat}(\mathsf{M}_*)}(i,j) = \varnothing$ if $i > j$. The composition is induced by the monoid structure of M_*.

8.2.2.1 Graded M_*-sets

Let M_* be a graded monoid. A *graded left M_*-set* is a graded set X_*, together with a morphism $M_* \boxtimes X_* \to X_*$ satisfying the obvious axioms. Equivalently, it is a collection of maps $M_i \times X_j \to X_{i+j},\ (m,x) \mapsto mx$ satisfying the properties:

$$1x = x \quad \text{and} \quad (mn)x = m(nx) \quad \text{for any} \quad x \in X_i,\ m \in M_k,\ n \in M_j.$$

Graded right M_*-sets are similarly defined.

Let X_* and Y_* be graded left M_*-sets. A morphism $f_* : X_* \to Y_*$ of graded sets is called a *morphism of graded left M_*-sets* if $f_{i+j}(mx) = mf_j(x)$ for any $m \in M_i$, $x \in X_j$. We let ${}_{M_*}\mathscr{S}$ be the category of graded left M_*-sets and graded morphisms of left M_*-sets. Accordingly, $\mathsf{Hom}_{{}_{\mathrm{gr}M_*}\mathscr{S}}(X_*, Y_*)$ denotes the set of all graded morphisms of left M_*-sets from X_* to Y_*. Since Tot is a symmetric monoidal functor, we see that the set $\mathsf{Tot}(X_*)$ has a natural left $\mathsf{Tot}(M_*)$-set structure for any graded left M_*-set X_*. This gives us a functor

$${}_{\mathrm{gr}M_*}\mathscr{S} \to {}_{\mathsf{Tot}(M_*)}\mathscr{S}$$

from graded left M_*-sets to left $\mathsf{Tot}(M_*)$-sets. Likewise also for graded right M_*-sets.

Lemma 8.2.1. *Assume M_* is a graded monoid.*

i) Let X_* be a graded left M_*-set. Then $i \mapsto X_i$ extends to a covariant functor

$$\mathfrak{f}(X_*) : \mathsf{Cat}(\mathsf{M}_*) \to \mathscr{S}.$$

Moreover, this assignment is an equivalence of categories:

$$\mathfrak{f} : {}_{\mathrm{gr}M_*}\mathscr{S} \xrightarrow{\sim} \mathsf{Funct}(\mathsf{Cat}(\mathsf{M}_*), \mathscr{S}).$$

ii) Let X_* be a graded right M_*-set. Then $i \mapsto X_{-i}$ extends to a contravariant functor

$$\mathfrak{f}^{op}(X_*) : \mathsf{Cat}(\mathsf{M}_*) \to \mathscr{S}.$$

Moreover, this assignment is an equivalence of categories:

$$\mathfrak{f}^{op} : \mathscr{S}_{\mathrm{gr}M_*} \xrightarrow{\sim} \mathsf{Funct}(\mathsf{Cat}(\mathsf{M}_*)^{\mathsf{op}}, \mathscr{S}).$$

Proof. i) We already described $\mathfrak{f}(X_*)$ on objects:

$$(\mathfrak{f}(X))(i) = X_i.$$

It act on morphisms of $\mathsf{Cat}(\mathsf{M}_*)$ as follows: Let $m \in M_{j-i}$. The map

$$(\mathfrak{f}(X))(m) : (\mathfrak{f}(X))(i) = X_i \to X_j = (\mathfrak{f}(X))(j)$$

is given by

$$X_i \ni x \mapsto mx \in X_j.$$

Conversely, take a covariant functor $F : \mathsf{Cat}(\mathsf{M}_*) \to \mathscr{S}$ and define $\mathfrak{x}(F)_*$ to be the graded set $(F(i))_{i\in\mathbb{Z}}$. Moreover, the product mx is defined as $F(m)(x) \in F(j)$ for $m \in M_{j-i}$ and $x \in F(i)$. We obtain the functor

$$\mathfrak{x} : \mathsf{Funct}(\mathsf{Cat}(\mathsf{M}_*), \mathscr{S}) \to {}_{\mathsf{gr}_*}\mathscr{S},$$

which is the inverse of $\mathfrak{f}$.

ii) This is quite similar to i). We just remark that the right multiplication by $r \in M_{j-i}$ defines a map $X_{-j} \to X_{-i}$ when considered as a morphism $i \to j$. $\mathcal{Q.E.D}$

The category $\mathscr{S}_{\mathsf{gr}M_*}$ is a topos, as it is a category of presheaves. In particular, it has limits and colimits, which are given componentwise. For example, if X_* and Y_* are graded right M_*-sets, then

$$(X_* \coprod Y_*)_i = X_i \coprod Y_i, \quad (X_* \times Y_*)_i = X_i \times Y_i.$$

We see that the obvious forgetful functor

$$\mathsf{Tot} : \mathscr{S}_{\mathsf{gr}M_*} \to \mathscr{S}_{M_*}$$

respects the coproduct. It does, however, not preserve the product in general.

8.2.2.2 The bifunctor $\boxtimes_{M_*}$

Let M_* be a graded monoid, X_* a graded right M_*-set and Y_* a graded left M_*-set. Define $X_* \boxtimes_{M_*} Y_*$ to be the coequaliser of the diagram

$$X_* \boxtimes M_* \boxtimes Y_* \rightrightarrows X_* \boxtimes Y_*.$$

Here, the first arrow is induced by the right action of M_* on X_*. The second arrow is induced by the left action of M_* on Y_*. The 0-th component of $X_* \boxtimes_{M_*} Y_*$ is denoted by $X_* \odot_{M_*} Y_*$. Thus, $X_* \odot_{M_*} Y_*$ is the set of equivalence classes of pairs (x, y), where $x \in X_i$ and $y \in Y_{-i}$. The equivalence relation is generated by $(um, v) \sim (u, mv)$. Here, $u \in X_i$, $m \in M_j$ and $v \in Y_{-i-j}$.

Lemma 8.2.2. *Let M_*, X_* and Y_* be as above. One has the following isomorphism of sets:*

$$i) \qquad \mathsf{Tot}(X_* \boxtimes_{M_*} Y_*) = \mathsf{Tot}(X_*) \otimes_{\mathsf{Tot}(M_*)} \mathsf{Tot}(Y_*).$$

$$ii) \qquad X_* \odot_{M_*} Y_* \simeq \mathfrak{f}^{op}(X_*) \otimes_{\mathsf{Cat}(\mathsf{M}_*)} \mathfrak{f}(Y_*).$$

The right hand side of the second equivalence is the tensor product in the category of functors [42, p.256].

Definition 8.2.3. *A graded left M_*-set X_* is called* filtered *if the following conditions hold:*

i) There is an integer i such that $X_i \neq \varnothing$.

ii) Let $a \in X_i$ and $b \in X_j$. There exists an integer k and elements $u \in M_{i-k}$, $v \in M_{j-k}$, $c \in M_k$ such that $uc = a$ and $vc = b$.

iii) Let $a \in X(i)$ and $u, v \in M_j$ such that $ua = va$. There is an integer k and elements $b \in X(k)$, $w \in M_{i-k}$ such that $uw = vw$ and $wb = a$.

In other words, X_* is filtered if and only if $\mathfrak{f}(X)$ is a filtered functor, where $\mathfrak{f}(X_*) : \mathrm{Cat}(\mathsf{M}_*) \to \mathscr{S}$ is the functor defined in Lemma 8.2.1. [42, p. 386].

Lemma 8.2.4. *Assume X_* is a filtered graded left M_*-set. The graded left M_*-set $X_*(i)$ is filtered for any integer i.*

The classical version of Diaconescu theorem implies the following result.

Proposition 8.2.5. *The category of points of the topos $\mathscr{S}_{\mathrm{gr}M}$ is equivalent to the category of graded filtered left M_*-sets. The inverse image functor $P_{X_*} : \mathscr{S}_{\mathrm{gr}M_*} \to \mathscr{S}$ of the point corresponding to a graded filtered left M_*-set X_* is given by $(-) \odot_{M_*} X_*$.*

8.2.2.3 Localisations of a commutative graded monoid

We will henceforth assume that M_* is commutative, and will no longer make any distinction between left and right M_*-sets. Let M_* be a graded monoid and S a submonoid of $\mathrm{Tot}(M_*)$. One sets $S_i = S \cap M_i$, $i \in \mathbb{Z}$. This gives us the graded set S_*, which is in fact a graded submonoid of M_*. We can form the localisation $S^{-1}(\mathrm{Tot}(M_*))$.

The degree map $\mathrm{Tot}(M_*) \to \mathbb{Z}$ has an extension

$$S^{-1}(\mathrm{Tot}(M_*)) \to \mathbb{Z},$$

where

$$\left|\frac{m}{s}\right| = |m| - |s|.$$

Thus, $S^{-1}(\mathrm{Tot}(M_*))$ is an underlying monoid of a graded monoid $S_*^{-1}M_*$. Here, $(S_*^{-1}M_*)_i$ is the set of all fractions $\frac{m}{s}$, $m \in M_{|m|}$, $s \in S_{|s|}$ for which $|m| = |s| + i$.

This construction has an obvious continuation to graded M_*-sets. If A_* is a graded M_*-set, then $S_*^{-1}A_*$ is a graded $S_*^{-1}M_*$-set. The i-th degree consists of all fractions $\frac{a}{s}$, $a \in A_{|m|}$, $s \in S_{|s|}$ for which $|a| = |s| + i$.

Of special interest are the cases when $S = \{f^n \mid n \in \mathbb{N}\}$ and when S is the complement of a prime ideal $\mathfrak{p}$ of $\mathrm{Tot}(M_*)$. Their corresponding localisations are written as A_{*f} and $A_{*\mathfrak{p}}$. We will denote the 0-th degree of A_{*f} and $A_{*\mathfrak{p}}$ by $A_{*(f)}$ and $A_{*(\mathfrak{p})}$ respectively. For example, $A_{*(f)} = \{\frac{m}{f^k}; |m| = k|f|\}$ and $A_{*(\mathfrak{p})} = \{\frac{m}{r}; |m| = |r|\}$. It follows from the definition that $A_{*(f)} \simeq A_* \odot_{M_*} M_{*f}$ and similarly, $A_{*(\mathfrak{p})} \simeq A_* \odot_{M_*} M_{*\mathfrak{p}}$.

8.2.2.4 Positively graded monoids

A graded monoid M_* is called *positively graded* if $M_i = \varnothing$ for $i < 0$. Henceforth, all graded monoids are going to be assumed to be positively graded. Denote

$$M_+ = \coprod_{d>0} M_d.$$

Then M_+ is a prime ideal of $\mathrm{Tot}(M_*) = \coprod_{d\geq 0} M_d$.

Let $\mathfrak{a}$ be an ideal of $\mathrm{Tot}(M_*)$ and set $\mathfrak{a}_d = \mathfrak{a} \cap M_d$. Then $\mathfrak{a}_0$ is an ideal of M_0 and $\mathfrak{a} = \coprod_{d\geq 0} \mathfrak{a}_d$ is also an ideal of $\mathrm{Tot}(M_0)$. We have $\mathfrak{a} = \mathfrak{a}_0 \coprod \mathfrak{a}_+$, where $\mathfrak{a}_+ = \coprod_{d>0} \mathfrak{a}_d$.

We omit the proof of the following theorem as it is identical to the proof of Theorem 7.5.10.

Theorem 8.2.6. *Let M_* be a commutative and positively graded monoid. Assume that M_0 is a finitely generated monoid, M_1 is a finitely generated M_0-set and M_0 and M_1 generate* $\mathrm{Tot}(M_*)$ *as a monoid. Any filtered M_*-set X_* is of the form $(M_*(i))_{\mathfrak{p}}$. Here, $i \in \mathbb{Z}$ is an integer and $\mathfrak{p}$ is a prime ideal of* $\mathrm{Tot}(M_*)$*. The corresponding point of the topos $\mathscr{S}_{\mathrm{gr}M_*}$ is given by $A_* \mapsto (A_*(i))_{(\mathfrak{p})}$.*

8.3 The topos of quasi-coherent sheaves over the monoid scheme Proj(M_*)

The proofs and constructions of this subsection follows [25] and [28]. As such, many proofs are only partially given.

Let X be a monoid scheme. We have seen (Theorem 8.1.5) that the category $\mathfrak{Q}c(X)$ of quasi-coherent sheaves over X is an s-Noetherian topos when X is s-Noetherian. The stalk at a point $x \in X$ defines a point of the topos $\mathfrak{Q}c(X)$. We have seen that the converse is also true if $X = \mathrm{Spec}(M)$ is affine and M is almost finitely generated (see Theorem 7.5.10). That is, any point of the topos $\mathfrak{Q}c(X)$ is isomorphic to the stalk functor at a point of X. This is part iv) of Theorem 8.0.1 from the introduction.

We wish to generalise this result to the non-affine case. We will show that this is true for a large class of monoid schemes, namely quasi-projective monoid schemes.

The graded monoids in this section are all assumed to be commutative and positively graded.

8.3.1 Monoid scheme Proj(M_*)

Let M_* be a positively graded monoid. We can construct the monoid scheme $\mathrm{Proj}(M_*)$ just like in classical algebraic geometry [18]. We recall the definition:

Denote by $V(M_+)$ the set of all prime ideals $\mathfrak{p}$ of $\mathrm{Tot}(M_*)$ for which $M_+ \subseteq \mathfrak{p}$. The map

$$\mathrm{Spec}(M_0) \to V(M_+), \quad \mathfrak{p}_0 \mapsto \mathfrak{p}_0 \coprod M_+$$

is a homeomorphism since $M_+ = \coprod_{d>0} M_d$ is a prime ideal of $\mathsf{Tot}(M_*) = \coprod_{d\geq 0} M_d$ and the union of prime ideals in a commutative monoid is again a prime ideal. We define $\mathsf{Proj}(M_*)$ as a topological space by

$$\mathsf{Proj}(M_*) := \{\mathfrak{p} \in \mathsf{Spec}(\mathsf{Tot}(M_*)) \mid M_+ \smallsetminus \mathfrak{p} \neq \varnothing\} = \mathsf{Spec}(\mathsf{Tot}(M_*)) \smallsetminus V(M_+).$$

Here, $M_+ \smallsetminus \mathfrak{p}$ is short for $M_+ \smallsetminus (M_+ \cap \mathfrak{p})$. Hence, $\mathsf{Proj}(M_*)$ is an open subset of $\mathsf{Spec}(\mathsf{Tot}(M_*))$. Take an element $f \in \mathsf{Tot}(M_*)$. Then $D(f) \subseteq \mathsf{Proj}(M_*)$ if and only if $f \in M_+$. The collection of open sets $\{D(f)\}_{f\in M_+}$ is an open cover of $\mathsf{Proj}(M_*)$.

We can regard $\mathsf{Proj}(M_*)$ as a monoid subscheme of $\mathsf{Spec}(\mathsf{Tot}(M_*))$. In many cases, this agrees with the punctures spectrum introduced in Subsection 5.6.1. But this spaces also admits another monoid scheme structure. We will need to introduce some additional notation in order to describe it.

Let $f \in M_d$, $d \geq 1$ and denote by $\mathsf{Tot}(M_*)_{(f)}$ the subset of degree zero elements of $\mathsf{Tot}(M_*)_f$. That is

$$\mathsf{Tot}(M_*)_{(f)} := \left\{ \frac{a}{f^k} \in \mathsf{Tot}(M_*)_f \mid a \in M_{kd} \right\}.$$

According to [18], there exists a unique monoid scheme structure on $\mathsf{Proj}(M_*)$, for which the structure sheaf $\mathcal{O}_X$, $X = \mathsf{Proj}(M_*)$ is given by

$$\Gamma(D(f), \mathcal{O}_X) = \mathsf{Tot}(M_*)_{(f)}, \quad f \in M_+.$$

The stalk at $\mathfrak{p} \in \mathsf{Proj}(M_*)$ is $\mathsf{Tot}(M_*)_{(\mathfrak{p})}$ and the restriction of the structural sheaf on $D(f)$, $f \in M_+$ is isomorphic to the affine monoid scheme $\mathsf{Spec}(M_{*(f)})$.

8.3.2 The functor $\iota^* : \mathscr{S}_{\mathrm{gr}M_*} \to \mathfrak{Qc}(\mathsf{Proj}(M_*))$

Our next aim is to relate the category $\mathfrak{Qc}(\mathsf{Proj}(M_*))$ of quasi-coherent sheaves over $X = \mathsf{Proj}(M_*)$ to the category $\mathscr{S}_{\mathrm{gr}M_*}$ of graded M_*-sets.

Assume A_* is a graded M_*-set. There exist a unique sheaf on $\mathsf{Proj}(M_*)$, denoted by $\iota^*(A_*)$, such that

$$\Gamma(D(f), \iota^*(A_*)) = \mathsf{Tot}(A_*)_{(f)}$$

for each $f \in M_d$, $d \geq 1$. It is clear from the description that $\iota^*(A_*)$ is a quasi-coherent sheaf. This yields the functor

$$\iota^* : \mathscr{S}_{\mathrm{gr}M_*} \to \mathfrak{Qc}(\mathsf{Proj}(M_*)).$$

The functor $A_* \mapsto \mathsf{Tot}(A_*)_{(f)}$ respects colimits and finite limits. Hence, ι^* also preserves all colimits and finite limits.

Lemma 8.3.1. *Let $\alpha_* : A_* \to B_*$ be a morphism of graded M_*-sets. The following conditions are equivalent:*

(i) The morphism $\iota^*(\alpha_*) : \iota(A_*) \to \iota^*(B_*)$ is a monomorphism.

ii) Let $f \in M_d$, $d \geq 1$ and $x, y \in A_k$ such that $\alpha_k(x) = \alpha_k(y)$. There exists $m \in \mathbb{N}$ such that $xf^m = yf^m$.

Proof. The family $\{D(f)\}_{f\in M_d,\ d\geq 1}$, forms an open cover. As such, $\iota^*(\alpha_*)$ is a monomorphism if and only if the restriction of $\iota^*(\alpha_*)$ on $D(f)$ is a monomorphism for each such f. The latter condition is equivalent to injectivity of the induced map $A_{*(f)} \to B_{*(f)}$, which happens if and only if ii) holds. $\mathcal{Q.E.D}$

Lemma 8.3.2. *Let $\alpha_* : A_* \to B_*$ be a morphism of graded M_*-sets. The following conditions are equivalent:*

(i) The morphism $\iota^*(\alpha_*) : \iota^*(A_*) \to \iota^*(B_*)$ is an epimorphism.

ii) Let $f \in M_d$, $d \geq 1$ and $z \in B_k$. There exists $m \in \mathbb{N}$ such that $zf^m \in$ Im(α_{mdk}).

Proof. The family $\{D(f)\}_{f\in M_d,\ d\geq 1}$, forms an open cover of Proj(M_*). Hence, $\iota^*(\alpha_*)$ is an epimorphism if and only if the restriction of $\iota^*(\alpha_*)$ on $D(f)$ is an epimorphism for each $f \in M_d, d \geq 1$. The sheaves involved here are quasi-coherent and $D(f)$ is affine. In this case, this condition is equivalent to the surjectivity of the canonical map $A_{*(f)} \to B_{*(f)}$. The latter holds if and only if part ii) of this Lemma holds. $\mathcal{Q.E.D}$

8.3.3 Some finiteness conditions

From this section onwards, we will assume that the positively graded monoid M_* satisfies the following properties:

(i) M_0 is a almost finitely generated monoid.

(ii) M_1 is a finitely generated M_0-set.

(iii) Tot(M_*) is generated by M_0 and M_1 as a monoid.

Lemma 8.3.3. *Let $f_1, \cdots, f_s \in M_1$ generate M_1 as an M_0-set. The open subsets $D(f_1), \cdots, D(f_s)$ cover* Proj(M_*).

Proof. Assume they do not. There exists a prime ideal $\mathfrak{p} \in$ Proj(M_*) such that $f_1, \cdots, f_s \in \mathfrak{p}$. We can find an element $f = f_0 f_1^{n_1} \cdots f_s^{n_s} \in M_+$ such that $f \notin \mathfrak{p}$ since $\mathfrak{p} \in$ Proj(M_*). Here, $f_0 \in M_0$, $n_i \geq 0$, $i \in \{1, \cdots, s\}$. We see that $n_i > 0$ for at least one $i > 0$ since otherwise $f \in M_+$, which contradicts $f \in M_+$. It follows that $f \in (f_i) \subseteq \mathfrak{p}$, which contradicts our assumptions on f. As usual, (f_i) denotes the principal ideal generated by f_i. $\mathcal{Q.E.D}$

We will need the shifting functors on $\mathfrak{Q}c$(Proj(M_*)), which are defined as follows: Denote by $\mathcal{O}_X(n)$, $n \in \mathbb{Z}$, the sheaf $\iota^*(M_*(n))$, where $X =$ Proj(M_*). Let $\mathcal{F}$ be a quasi-coherent sheaf. We set

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

Lemma 8.3.4. *Let $X =$ Proj(M_*). The sheaf $\mathcal{O}_X(n)$ is a line bundle for any $n \in \mathbb{Z}$.*

Proof. We regard Tot$(M_*)(n)_{(f)}$ as a Tot$(M_*)_{(f)}$-set. Let $f \in M_1$. The restriction of $\mathcal{O}_X(n)$ on $D(f)$ is isomorphic to the quasi-coherent sheaf on Spec(Tot$(M_*))_{(f)}$, associated to Tot$(M_*)(n)_{(f)}$. Since f is invertible in Tot$(M_*)_{(f)}$, Tot$(M_*)(n)_{(f)}$ is free of rank one over Tot$(M_*)_{(f)}$ with generator f^n. $\mathcal{Q.E.D}$

8.3.4 The geometric morphism $\iota : \mathfrak{Q}c(\mathsf{Proj}(M_*)) \to \mathscr{S}_{\mathrm{gr}M_*}$

Recall that M_* satisfies the finiteness conditions listed at the beginning of Section 8.3.3. We have already mentioned that the functor

$$\iota^* : \mathscr{S}_{\mathrm{gr}M_*} \to \mathfrak{Q}c(\mathsf{Proj}(M_*))$$

respects finite limits and all colimits. It has an adjoint i_*, which is given as follows: Take a quasi-coherent sheaf $\mathcal{F}$ on $X = \mathsf{Proj}(M_*)$ and define the graded set $\Gamma_*(\mathcal{F})$ by

$$\Gamma_n(\mathcal{F}) = \Gamma(X, \mathcal{F}(n)),\ n \in \mathbb{Z}.$$

We will define an action of M_* on $\Gamma_*(\mathcal{F})$. Take $m \in M_k$. Since $(M_*(k))_0 = M_k$, the element $\frac{m}{1}$ can be considered as an element of $(M_*(k))_{*(f)}$ for any $f \in M_1$. This is the same as $\Gamma(D(f), \iota((M_*)(k))$. Such elements satisfy the gluing condition and hence, define a global element in $\Gamma(X, \mathcal{O}_X(k))$. It will still be denoted by m. Take an element $x \in \Gamma_i(\mathcal{F})$. By Lemma 8.1.1, we have

$$\mathsf{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \simeq \Gamma(X, \mathcal{G})$$

for any sheaf of $\mathcal{O}_X$-sets $\mathcal{G}$. It follows that m and x define morphisms of coherent sheaves $\mathcal{O}_X \to \mathcal{O}_X(k)$ and $\mathcal{O}_X \to \mathcal{F}(i)$ respectively. We take the tensor product to obtain

$$\mathcal{O}_X = \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \to \mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{F}(i) = \mathcal{F}(k+i).$$

This is an element in $\Gamma_{k+i}(\mathcal{F})$, which we define to be mx. This gives us an action of M_* on $\Gamma_*(\mathcal{F})$.

The assignment $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ defines the functor $\iota_* : \mathfrak{Q}c(\mathsf{Proj}(M_*)) \to \mathscr{S}_{\mathrm{gr}M_*}$.

Lemma 8.3.5. *The functor ι^* is the left adjoint of ι_*.*

Proof. Let A_* be a graded M_*-set and $\mathcal{F}$ be a quasi-coherent sheaf on $X = \mathsf{Proj}(M_*)$. We need to show that there is a bijective correspondence between morphisms $\alpha : A_* \to \Gamma_*(\mathcal{F})$ of graded M_*-sets and the morphisms of quasi-coherent sheaves of sets $\beta : \iota^*(A_*) \to \mathcal{F}$.

To define β corresponding to α, it suffices to define it on the affine subscheme $D(f)$ for each $f \in M_1$, and verify that these morphisms are compatible on the intersections. The category of quasi-coherent sheaves over $D(f)$ is equivalent to the category of $M_{*(f)}$-sets. Hence, we need to construct a morphism of $M_{*(f)}$-sets: $A_{*(f)} \to \Gamma(D(f), \mathcal{F})$.

Take an element $\frac{a}{f^k}$ of $A_{*(f)}$, where $a \in A_k$. Thus, $\alpha(a) \in \Gamma_k(\mathcal{F})$ is a morphism $\mathcal{O}_X \to \mathcal{F}(k)$. On the other hand, $\frac{1}{f^k}$ can be interpreted as a morphism $\mathcal{O}_X \to \mathcal{O}_X(-k)$ over $D(f)$. Hence, $\frac{1}{f^k} \otimes a$ defines a morphism $\mathcal{O}_X \to \mathcal{F}$, which is an element of $\Gamma(D(f), \mathcal{F})$.

To show that it is the restriction of a morphism $\beta(\frac{a}{f^k})$ on $D(f)$, one needs to check that they are compatible on intersection. To this end take $g \in M_1$. The image of $\frac{ag^k}{(fg)^k}$ is

$$\alpha(ag^k) \otimes \frac{1}{f^k g^k} = \alpha(a) \otimes \frac{g^k}{f^k g^k} = \frac{\alpha(a)}{f^k}.$$

These local data are therefore compatible and we get $\beta : \iota^*(A_*) \to \mathcal{F}$. Thus, we have defined a map $\alpha \mapsto \beta$. It is an exercise to see that this is a bijection. $\mathcal{Q.E.D}$

We have constructed the geometric morphism $\iota := (\iota^*, \iota_*) : \mathfrak{Qc}(\mathsf{Proj}(M_*)) \to \mathscr{S}_{\mathsf{gr}M_*}$.

Proposition 8.3.6. *The inverse image of the geometric morphism $\iota : \mathscr{S}_{\mathsf{gr}M_*} \to \mathfrak{Qc}(\mathsf{Proj}(M_*))$ is a localisation of topoi.*

Proof. It suffices to show that the canonical map $\beta : \iota^*(\Gamma_*(\mathcal{F})) \to \mathcal{F}$ is an isomorphism for any quasi-coherent sheaf $\mathcal{F}$ over Proj(M_*). This is equivalent to saying that there is an isomorphism

$$\Gamma_*(D(f), \mathcal{F})_{(f)} \simeq \Gamma(D(f), \mathcal{F})$$

for any $f \in M_1$. Consider f as an element of $\Gamma(X, \mathcal{O}_X(1))$. The desired isomorphism is a direct consequence of Lemma 8.1.7, where the subset X_f is exactly $D(f)$. $\mathcal{Q.E.D}$

8.3.5 Points of the topos $\mathfrak{Qc}(\mathsf{Proj}(M_*))$

We will assume that our (positively) graded monoids satisfy the finiteness conditions listed in Section 8.3.3.

We would like to investigate the points of the topos $\mathfrak{Qc}(\mathsf{Proj}(M_*))$. Recall that we have a topos point

$$\mathsf{Stalk}(x) : \mathfrak{Qc}(\mathsf{Proj}(M_*)) \to \mathscr{S}$$

for any $x \in X = \mathsf{Proj}(M_*)$, given by $\mathcal{F} \mapsto \mathcal{F}_x$.

We will show that any point of the topos $\mathfrak{Qc}(\mathsf{Proj}(M_*))$ is of this form. We will need two auxiliary results for this.

Denote by Σ the following class of morphisms in $\mathscr{S}_{\mathsf{gr}M_*}$: A morphism $\alpha : X_* \to Y_*$ of graded M_*-sets belongs to Σ, if $\iota^*(\alpha) : \iota^*(X_*) \to \iota^*(Y_*)$ is an isomorphism.

Lemma 8.3.7. *Let $\mathfrak{p}$ be a prime ideal of* Tot(M_*). *If the functor $A_* \mapsto (A_*(i))_{(\mathfrak{p})}$ takes morphisms from Σ to isomorphisms, then $\mathfrak{p} \in \mathsf{Proj}(M_*)$.*

Proof. Without loss of generality, we may assume that $i = 0$. Observe that the natural inclusion $M_+ \subseteq \mathsf{Tot}(M_*)$ belongs to Σ. Thanks to Lemmas 8.3.1 and 8.3.2, $(M_+)_{(\mathfrak{p})} \to M_{*(\mathfrak{p})}$ is an isomorphism. It follows that 1 is in the image of this map. We have $1 = \frac{m}{s}$, where $s \notin \mathfrak{p}$ and $m \in M_+$. Thus, $|s| = |m| > 0$ and $s \in M_+$. It follows that $M_+ \setminus \mathfrak{p} \neq \varnothing$ as $s \in M_+$ and $s \notin \mathfrak{p}$, so $\mathfrak{p} \in \mathsf{Proj}(M_*)$. $\mathcal{Q.E.D}$

Lemma 8.3.8. *Let $\mathfrak{p} \in \mathsf{Proj}(M_*)$. The functors $A_* \mapsto (A_*(i))_{(\mathfrak{p})}$ and $A_* \mapsto (A_*)_{(\mathfrak{p})}$ are isomorphic for all $i \in \mathbb{Z}$.*

Proof. By definition, $(A_*)_{(\mathfrak{p})}$ (resp. $(A_*(i))_{(\mathfrak{p})}$) is the degree zero (resp. degree i-th) part of $\mathsf{Tot}(A_*)_{\mathfrak{p}}$. It follows from our finiteness assumption that there exists an element $f \in M_1$ such that $f \notin \mathfrak{p}$. Hence, $l_f : \mathsf{Tot}(A_*)_{\mathfrak{p}} \to \mathsf{Tot}(A_*)_{\mathfrak{p}}$, $x \mapsto fx$ is an isomorphism which raises the degree by one. The restriction of the i-th iteration of this isomorphism on $(A_*)_{(\mathfrak{p})}$ yields the isomorphism in question. $\mathcal{Q.E.D}$

Now we are in the position to state and prove the following result:

Theorem 8.3.9. *Let M_* be a positively graded commutative monoid satisfying the finiteness conditions listed in Section 8.3.3 and let $P^* : \mathfrak{Q}c(\mathrm{Proj}(M_*)) \to \mathscr{S}$ be the inverse image of a topos point $\mathfrak{Q}c(\mathrm{Proj}(M_*))$. There exists an isomorphism of points $P^* \simeq \mathrm{Stalk}(x)$ for some element $x \in \mathrm{Proj}(M_*)$.*

Proof. By Lemma 6.1.9 and Proposition 8.3.6, the isomorphism classes of topos points $\mathfrak{Q}c(\mathrm{Proj}(M_*))$ correspond to the isomorphism classes of points $P^* : \mathscr{S}_{\mathrm{gr}M_*} \to \mathscr{S}$, for which $P^*(\alpha)$ is a bijection for any $\alpha \in \Sigma$. Here, Σ is the same as in Section 8.3.3.

On the other hand, Theorem 8.2.6 gives a description of all the points of the topos $\mathscr{S}_{\mathrm{gr}M_*}$. The result follows from Lemmas 8.3.7 and 8.3.8. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 8.3.10. *Let M_* be a positively graded commutative monoid satisfying the finiteness conditions listed in Section 8.3.3 and let U be an open monoid subscheme of $X = \mathrm{Proj}(M_*)$. Any $P^* : \mathfrak{Q}c(U) \to \mathscr{S}$ is isomorphic to a point of the form $\mathrm{Stalk}(x)$, $x \in U$.*

We call a monoid scheme X *quasi-projective* if it is isomorphic to an open monoid subscheme of $\mathrm{Proj}(M_*)$. Here, M_* is a non-negative graded monoid satisfying the finiteness conditions listed in Section 8.3.3.

Theorem 8.3.11. *Let X and Y be quasi-projective monoid schemes such that the categories $\mathfrak{Q}c(X)$ and $\mathfrak{Q}c(Y)$ are equivalent. Then X and Y are isomorphic.*

Proof. The monoid scheme structure of X is determined by the underlying ordered set and stalk functors $x \mapsto \mathcal{O}_{X,x}$, as X it is of finite type [18, Proposition 2.11], [49, Proposition 2.3]. It suffices to show that the category of quasi-coherent sheaves determines X as a poset and the stalk functors.

Let us consider $\mathrm{Pts}(\mathfrak{Q}\mathrm{c}(\mathrm{X}))$, the equivalence classes of topos points of $\mathfrak{Q}c(X)$. Recall that their equivalence classes admitted a poset structure, where $P \leq Q$ if and only if $P \in \Delta(Q)$. Here, $P, Q \in \mathrm{Pts}(\mathfrak{Q}\mathrm{c}(\mathrm{X}))$ are topos points. This was shown in Corollary 7.5.12 and discussed just after it. Further, this order agrees with the ordering of the underlying set of X. We have also an isomorphism of monoids $\mathcal{O}_{X,P} \simeq \mathrm{End}(P)$, and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

8.4 The case of affine monoid schemes

In this section, we aim to prove part ii) of Theorem 3 (the main theorem) for affine monoid schemes, i.e, when $X = \mathrm{Spec}(M)$. In this case, the topos $\mathfrak{Qc}(X)$ is the category Sets_M of M-sets.

8.4.1 Grothendieck topologies on a commutative monoid M and subsets of $\mathrm{Spec}(M)$

The goal of this subsection is to construct a Galois connection between the posets of Grothendieck topologies on a commutative monoid M and subsets of $\mathrm{Spec}(M)$.

We start with the following observation.

Proposition 8.4.1. *Let M be a commutative monoid and $\mathfrak{p}$ be prime ideal of M. We set $S_{\mathfrak{p}} = M \setminus \mathfrak{p}$. The following assertions holds:*

(i) The set

$$\mathfrak{F}_{\{\mathfrak{p}\}} := \{\mathfrak{a} \in \Omega^M \quad \text{and} \quad \mathfrak{a} \cap S_{\mathfrak{p}} \neq \varnothing\}$$

is a topology on M.

(ii) For an ideal $\mathfrak{a}$, one has

$$\left(\mathfrak{a} \in \mathfrak{F}_{\{\mathfrak{p}\}}\right) \Longleftrightarrow (\mathfrak{p} \notin V(\mathfrak{a}))\,.$$

(iii) For prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, one has $\mathfrak{p} \subseteq \mathfrak{q}$ if and only if $\mathfrak{F}_{\{\mathfrak{q}\}} \subseteq \mathfrak{F}_{\{\mathfrak{p}\}}$.

Proof. i) Since $1 \in S_{\mathfrak{p}} = M \cap S_{\mathfrak{p}}$, we obtain $M \in \mathfrak{F}_{\{\mathfrak{p}\}}$. Assume $\mathfrak{a} \in \mathfrak{F}_{\{\mathfrak{p}\}}$ and $m \in M$. By assumption, there exists an element $s \in S$ such that $s \in \mathfrak{a}$. Since $s \in (\mathfrak{a} : m)$, the axiom (T2) follows. In order to verify (T3), let $\mathfrak{B}$ be an ideal from $\mathfrak{F}_{\{\mathfrak{p}\}}$ and let $\mathfrak{a}$ be an ideal such that $(\mathfrak{a} : m) \in \mathfrak{F}_{\{\mathfrak{p}\}}$ for any $m \in \mathfrak{B}$. By assumption, there are $s, t \in S$ such that $s \in \mathfrak{B}$ and $t \in (\mathfrak{a} : s)$. Thus, $st \in \mathfrak{a}$. Since $st \in S$, we see that $\mathfrak{a} \cap S \neq \varnothing$ and axiom (T3) holds.

ii) Obvious.

iii) Assume $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{a} \in \mathfrak{F}_{\{\mathfrak{q}\}}$. Then

$$\varnothing \neq \mathfrak{a} \setminus \mathfrak{q} \subseteq \mathfrak{a} \setminus \mathfrak{p}.$$

Hence, $\mathfrak{a} \in \mathfrak{F}_{\{\mathfrak{p}\}}$. It follows that $\mathfrak{F}_{\{\mathfrak{q}\}} \subseteq \mathfrak{F}_{\{\mathfrak{p}\}}$.

Conversely, assume $\mathfrak{F}_{\{\mathfrak{q}\}} \subseteq \mathfrak{F}_{\{\mathfrak{p}\}}$. Take an element $m \in \mathfrak{p}$. For the principal ideal mM we have $mM \notin \mathfrak{F}_{\{\mathfrak{p}\}}$. It follows that $mM \notin \mathfrak{F}_{\{\mathfrak{q}\}}$. Hence $mM \subseteq \mathfrak{q}$. In particular $m \in \mathfrak{q}$. Thus $\mathfrak{p} \subseteq \mathfrak{q}$. $\mathcal{Q.E.D}$

Recall our discussion on topologies on a monoid in Section 7.1.1.4. Denote by $\mathfrak{P}(X)$ the set of all subsets of a set X. We construct two maps

$$\Upsilon : \mathfrak{P}(\mathrm{Spec}(M)) \to \mathrm{Top}(M) \quad \text{and} \quad \Xi : \mathrm{Top}(M) \to \mathfrak{P}(\mathrm{Spec}(M)),$$

given by

$$\Upsilon(\mathcal{P}) := \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{F}_{\{\mathfrak{p}\}} \quad \text{and} \quad \Xi(\mathfrak{F}) = \{\mathfrak{p} \in \mathrm{Spec}(M) | \mathfrak{p} \notin \mathfrak{F}\},$$

where $\mathcal{P} \subseteq \mathrm{Spec}(M)$ is a subset of the set of prime ideals of M and $\mathfrak{F}$ is a topology on M.

Proposition 8.4.2. *i) For an ideal $\mathfrak{a}$, one has*

$$(\mathfrak{a} \in \Upsilon(\mathcal{P})) \Longleftrightarrow (\mathfrak{p} \notin V(\mathfrak{a}) \quad \text{for all} \quad \mathfrak{p} \in \mathcal{P})$$

and hence,

$$\Upsilon(\mathcal{P}) = \{\mathfrak{a} \mid V(\mathfrak{a}) \cap \mathcal{P} = \varnothing\}.$$

ii) The functions Υ and Ξ form a Galois correspondence between the posets $\mathfrak{P}(\mathrm{Spec}(M))$ and $\mathrm{Top}(M)$.

iii) Let $\mathfrak{F}$ be a topology on M. Then

$$\Upsilon(\Xi(\mathfrak{F})) = \{\mathfrak{a} \in \Omega^M \mid V(\mathfrak{a}) \subseteq \mathfrak{F}\}.$$

‘

iv) Let $\mathcal{P}$ be a subset of $\mathrm{Spec}(M)$. Then

$$\Xi(\Upsilon(\mathcal{P})) = \{\mathfrak{q} \in \mathrm{Spec}(M) \mid \exists \mathfrak{p} \in \mathcal{P}, \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. i) follows from part ii) of Proposition 8.4.1.

ii) Clearly both maps reverse the ordering. Thus, we have to show that if $\mathfrak{F}$ is a topology on M and $\mathcal{P}$ is a subset of $\mathrm{Spec}(M)$, then

$$(\mathfrak{F} \subseteq \Upsilon(\mathcal{P})) \Longleftrightarrow (\mathcal{P} \subseteq \Xi(\mathfrak{F})).$$

In fact, $\mathfrak{F} \subseteq \Upsilon(\mathcal{P})$ holds if and only if for any ideal $\mathfrak{a} \in \mathfrak{F}$ and any prime ideal $\mathfrak{p} \in \mathcal{P}$, one has $\mathfrak{p} \notin V(\mathfrak{a})$. Since $\mathfrak{p} \in V(\mathfrak{p})$, it follows that $\mathfrak{p} \notin \mathfrak{F}$ and $\mathcal{P} \subseteq \Xi(\mathfrak{F})$.

On the other hand, assume $\mathcal{P} \subseteq \Xi(\mathfrak{F})$. For any $\mathfrak{p} \in \mathcal{P}$ one has $\mathfrak{p} \notin \mathfrak{F}$. Take an ideal $\mathfrak{a} \in \mathfrak{F}$. If $V(\mathfrak{a}) \cap \mathcal{P} \neq \varnothing$, it follows that there exists $\mathfrak{p} \in \mathcal{P}$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Hence, $\mathfrak{p} \in \mathfrak{F}$. This contradicts our assumption on $\mathcal{P}$. We have $V(\mathfrak{a}) \cap \mathcal{P} = \varnothing$ and $\mathfrak{a} \in \Upsilon(\mathcal{P})$, showing $\mathfrak{F} \subseteq \Upsilon(\mathcal{P})$ and ii) follows.

iii) For an ideal $\mathfrak{a}$, one has

$$(\mathfrak{a} \in \Upsilon(\Xi(\mathfrak{F}))) \Longleftrightarrow (\forall \mathfrak{p} \in \Xi(\mathfrak{F}) \Longrightarrow \ \mathfrak{p} \notin V(\mathfrak{a})) \Longleftrightarrow (\mathfrak{p} \notin \mathfrak{F} \Longrightarrow \mathfrak{p} \notin V(\mathfrak{a}))\,.$$

The last condition is the same as $V(\mathfrak{a}) \subseteq \mathfrak{F}$ and we are done.

iv) We have $\mathfrak{q} \in \Xi(\Upsilon(\mathcal{P}))$ if and only if $\mathfrak{q} \notin \Upsilon(\mathcal{P})$. By i), this happens if and only if there exists $\mathfrak{p} \in \mathcal{P}$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Example 8.4.3. *i) We have*

$$\Upsilon(\{\varnothing\}) = \{\text{all nonempty ideals of } M\} = \Omega^M \setminus \{\varnothing\},$$

while

$$\Upsilon(\varnothing) = \Omega^M = \{\text{all ideals of } M\}.$$

ii) If $\mathfrak{m}(M)$ is the ideal of all non invertible elements of M, which is the greatest proper ideal of M, then

$$\Upsilon(\{\mathfrak{m}(M)\}) = \{M\}$$

and

$$\Upsilon(\mathsf{Spec}(M)) = \{M\}.$$

Lemma 8.4.4. *Let $f \in M$ be an element. We have*

$$\Upsilon(D(f)) = \{\mathfrak{a} \in \Omega^M \mid f \in \mathfrak{a}\}.$$

Moreover, the localisation corresponding to the topology $\Upsilon(D(f))$ via iii) of Lemma 6.1.5 is the category of M_f-sets.

Proof. For the first assertion, observe that, by part i) of Proposition 8.4.2, we have $\mathfrak{a} \in \Upsilon(D(f))$ if and only if $V(\mathfrak{a}) \cap D(f) = \varnothing$. In other words, if for any prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$, it also contains f. Obviously, this condition holds if $f \in \mathfrak{f}$. The converse is also true because if $f \notin \mathfrak{a}$, the ideal, which is maximal among the ideals containing $\mathfrak{a}$ and not containing any power of f, is prime.

For the second part, we have to show that an M-set A is a sheaf with respect to this topology if and only if the map $l_f : A \to A$, given by $l_f(a) = fa$, is a bijection.

Assume A is a sheaf in this topology. This means that for any ideal $\mathfrak{a}$ such that $f \in \mathfrak{a}$, the restriction map

$$A = \mathsf{Hom}_M(M, A) \to \mathsf{Hom}_M(\mathfrak{a}, A)$$

is a bijection. By considering $\mathfrak{a} = fM$, injectivity of this map implies the injectivity of $l_m : A \to A,\ a \mapsto ma$. For the surjectivity of l_f, take any element $a \in A$. Let $m, n \in M$ be elements such that $fm = fn$. We have $fma = fna$ and hence, $ma = na$ since l_f is injective. This shows the existence of a well-defined morphism of M-sets $fM \to A$ for which $fm \mapsto ma$. By the bijectivity of the map $A = \mathsf{Hom}_M(M, A) \to \mathsf{Hom}_M(\mathfrak{a}, A)$, it follows that there exist $b \in A$ such that $fb = a$. Thus, l_f is a bijection.

Conversely, assume that $l_f : A \to A$ is bijective. This condition obviously implies the bijectivity of the restriction map $A = \mathsf{Hom}_M(M, A) \to \mathsf{Hom}_M(\mathfrak{a}, A)$. Let us take any ideal $\mathfrak{a}$ such that $f \in \mathfrak{a}$. To show the bijectivity of $A = \mathsf{Hom}_M(M, A) \to \mathsf{Hom}_M(\mathfrak{a}, A)$, observe that it fits in the diagram

$$A = \mathsf{Hom}_M(M, A) \to \mathsf{Hom}_M(\mathfrak{a}, A) \to \mathsf{Hom}_M(fM, A).$$

Since the composite map is a bijection, we only need to show injectivity of the second map. For this, take any morphism of M-sets $\alpha : \mathfrak{a} \to A$. We have $\alpha(fx) = f\alpha(x)$ for any $x \in \mathfrak{a}$. Thus

$$\alpha(x) = l_f^{-1}(\alpha(fx)),$$

showing that α is uniquely defined by its restriction on fM and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Recall that a topology $\mathfrak{F}$ (resp. a subset $\mathcal{P}$) is stable if and only if $\mathfrak{F} = \Upsilon\Xi(\mathfrak{F})$ (resp. $\mathcal{P} = \Xi\Upsilon(\mathcal{P})$). From the properties of Galois correspondences, one obtains a bijective correspondence between stable topologies on M and stable subsets of Spec(M). Our next goal is to find a convenient way of characterising stable subsets and stable topologies.

Proposition 8.4.5. *Let M be a commutative monoid. A subset $\mathcal{P} \subseteq$ Spec(M) is stable if and only if it is an open subset in the order topology of* Spec(M).

Proof. This follows from part iv) of Proposition 8.4.2. $\mathcal{Q.E.D}$

Proposition 8.4.6. *Let $\mathfrak{F}$ be a topology on a commutative monoid M. Then $\mathfrak{F}$ is stable if and only if for any ideal $\mathfrak{a} \notin \mathfrak{F}$, there exists $\mathfrak{p} \in V(\mathfrak{a})$, such that $\mathfrak{p} \notin \mathfrak{F}$.*

Proof. The 'if' part is a consequence of Proposition 8.4.2. For the other side, it suffices to show that $\Upsilon\Xi(\mathfrak{F}) \subseteq \mathfrak{F}$. Take an ideal $\mathfrak{an} \in \Upsilon\Xi(\mathfrak{F})$. By Proposition 8.4.2. we have $V(\mathfrak{a}) \notin \mathfrak{F}$. So, there is a prime $\mathfrak{p} \in V(\mathfrak{a})$ such that $\mathfrak{p} \notin \mathfrak{F}$. $\mathcal{Q.E.D}$

Corollary 8.4.7. *For any commutative monoid M, there exist a bijection between the stable topologies of M and open subsets of* Spec(M) *in the order topology.*

Proof. This is a formal consequence of Galois correspondence and Proposition 8.4.5. $\mathcal{Q.E.D}$

Lemma 8.4.8. *Let $\mathfrak{F}$ be a topology on a commutative monoid M.*

(i) Let $\mathfrak{c}$ be an ideal of M. If $\mathfrak{a}$ is an ideal which is maximal with respect to the property that $\mathfrak{a} \notin \mathfrak{F}$ and $\mathfrak{c} \subseteq \mathfrak{a}$, then $\mathfrak{a}$ is a prime ideal.

(ii) Assume that for any ideal $\mathfrak{B} \in \mathfrak{F}$, there exists a finitely generated ideal $\mathfrak{a}$ such that $\mathfrak{a} \subseteq \mathfrak{B}$ and $\mathfrak{a} \in \mathfrak{F}$. For any ideal $\mathfrak{c} \notin \mathfrak{F}$, there exist a prime ideal $\mathfrak{p} \in V(\mathfrak{c})$, such that $\mathfrak{p} \notin \mathfrak{F}$.

Proof. (i) Take elements $a, b \notin \mathfrak{a}$ and assume $ab \in \mathfrak{a}$. Consider the ideals $\mathfrak{a}' = \mathfrak{a} \cup aM$ and $\mathfrak{B}' = \mathfrak{a} \cup bM \in \mathfrak{F}$. By our assumption, $\mathfrak{a}', \mathfrak{B}' \in \mathfrak{F}$. We have $\mathfrak{a}'\mathfrak{B}' \subseteq \mathfrak{a}$, which contradicts the fact that $\mathfrak{a}'\mathfrak{B}' \in \mathfrak{F}$. The latter holds due to Lemma (7.1.3), implying that $\mathfrak{a}$ is prime.

(ii) Consider the set $\mathcal{A}$ of all ideals $\mathfrak{a}$ such that $\mathfrak{c} \subseteq \mathfrak{a}$ and $\mathfrak{a} \notin \mathfrak{F}$. Clearly, $\mathcal{A}$ admits an ordering by inclusion and $\mathfrak{c} \in \mathcal{A}$. We will use Zorn's lemma.

Take any chain $(\mathfrak{a}_i)_{i\in I}$ in $\mathcal{A}$. Then $\mathfrak{B} = \bigcup_i \mathfrak{a}_i$ is an ideal since any union of ideals of a monoid is an ideal. We claim that $\mathfrak{B} \notin \mathfrak{F}$. Assume $\mathfrak{B} \in \mathfrak{F}$. We can find a finitely generated ideal $\mathfrak{B}' \in \mathfrak{F}$ such that $\mathfrak{B}' \subseteq \mathfrak{B} = \bigcup_i \mathfrak{a}_i$. Since $\mathfrak{B}'$ is finite generated, $\mathfrak{B}' \in \mathfrak{a}_i$ for some i. This contradicts the fact that $\mathfrak{a}_i \in \mathcal{A}$, proving the claim. Now we can use Zorn's lemma to conclude that $\mathcal{A}$ has a maximal element $\mathfrak{p}$, which is prime thanks to (i) and $\mathfrak{c} \subseteq \mathfrak{p}$. $\mathcal{Q.E.D}$

Lemma 8.4.9. *Let M be an s-Noetherian monoid. Any topology $\mathfrak{F}$ on M is stable and hence, there exists a subset $\mathcal{P} \subseteq$ Spec(M) such that $\mathfrak{F} = \mathfrak{F}_{\mathcal{P}}$.*

Proof. By Proposition (8.4.6), we have to show that for any ideal $\mathfrak{cn} \in \mathfrak{F}$, there exist a prime $\mathfrak{p} \in V(\mathfrak{c})$ such that $\mathfrak{pn} \in \mathfrak{F}$. This follows from the second part of Lemma (8.4.8). $\mathcal{Q.E.D}$

This fact, together with Corollary 8.4.7, implies the following:

Corollary 8.4.10. *Let M be an s-Noetherian monoid. There exists a bijection between the topologies of M and open subsets of* Spec(M) *in the order topology.*

We are now in a position to prove part ii) of Theorem 3 in the affine case.

Corollary 8.4.11. *Let $M/M^\times$ be a finitely generated monoid and $\mathscr{T}$ the category of M-sets. There is an isomorphism*

$$\mathfrak{Loc}(\mathscr{T}) \cong \mathfrak{Off}(\text{Spec}(M)).$$

Proof. We know (see Lemma 6.1.5, part iii)) that $\mathfrak{Loc}(\mathscr{T}) \cong \text{Top}(M)$ (this is valid for all M). Hence, the result follows from Lemma 4.1.5 and Corollary 8.4.10. $\mathcal{Q.E.D}$

8.5 The general case

8.5.1 Preliminaries on monoid Schemes

Recall that for any commutative monoid M and any M-set A, there exist a unique sheaf $\tilde{A}$ of sets on Spec(M) such that

$$\Gamma(D(f), \tilde{A}) = A_f.$$

The stalk of $\tilde{A}$ at the point $\mathfrak{p}$ is the localisation $A_\mathfrak{p}$ of A by the submonoid $M \setminus \mathfrak{p}$.

As we argued in 8.1.5, the category of quasi-coherent sheaves over an s-Noetherian scheme is a locally s-Noetherian topos. The next result describes the subobject classifier of this topos. To state it, we fix some notations. Let $U = \text{Spec}(M)$ be an s-Noetherian affine monoid scheme. We denote by Ω^U the quasi-coherent sheaf corresponding to the M-set Ω^M. Moreover, if V is an open affine monoid subscheme, then it is open in the order topology. Since it is also affine, V has a greatest element, say $\mathfrak{p}$. Then $V = \text{Spec}(M_\mathfrak{p})$. We can thus define an isomorphism of sheaves

$$\alpha_{U,V} : Res^U_V(\Omega^U) \to \Omega^V,$$

corresponding to the canonical isomorphism $S^{-1}\Omega^M \to \Omega^{S^{-1}M}$, constructed in Proposition 7.2.9. Here, $S = M \setminus \mathfrak{p}$ and Res^U_V is the restriction $\text{Sh}(U) \to \text{Sh}(V)$.

Theorem 8.5.1. *Let X be an s-Noetherian monoid scheme. There exists a unique quasi-coherent sheaf Ω^X, together with isomorphisms of quasi-coherent sheaves*

$$\psi_U : Res^X_U(\Omega^X) \to \Omega^U,$$

where U is an affine open monoid subscheme, such that if V is also an affine open subscheme and $W = U \cap V$, the diagram

$$\begin{array}{ccc} Res^X_W(\Omega^X) & \xrightarrow{Res^U_W(\psi_U)} & Res^U_W(\Omega^U) \\ {\scriptstyle Res^V_W(\psi_V)}\downarrow & & \downarrow{\scriptstyle \alpha_{U,W}} \\ Res^V_W(\Omega^V) & \xrightarrow[\alpha_{V,W}]{} & \Omega^W \end{array}$$

commutes. The sheaf Ω^X is the subobject classifier of the topos $\mathfrak{Q}c(X)$.

Remark 8.5.2. *In the statement of this theorem, we implicitly use the fact, that the intersection of two open affine subschemes of a separated monoid scheme is again an open affine subscheme [18].*

Proof. The collection of sheaves Ω^U and isomorphisms

$$\phi_{U,V} = \alpha^{-1}_{V,W} \circ \alpha_{U,W} : \mathsf{Res}^U_W(\Omega^U) \to \mathsf{Res}^V_W(\Omega^V)$$

satisfy the requirements of [28, Exercise II, 1.22] thanks to part v) of Lemma 7.1.4, and as such, glue as sheaves. Thus, the existence and uniqueness of Ω^X follows. Take quasi-coherent sheaves $\mathcal{A}$ and $\mathcal{B}$ and assume that $\mathcal{A} \subseteq \mathcal{B}$. We know that Ω^M is a subobject classifier in the topos of M-sets. Hence, locally, there exist a unique morphism of sheaves : $\xi_{\mathcal{B}} : \mathcal{A} \to \Omega^X$, as in the definition of the subobject classifier. Uniqueness implies that these local maps agree on the intersection of affine open subsets. This shows that they are restrictions of a uniquely defined morphism of sheaves $\mathcal{A} \to \Omega^X$ and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

8.5.2 The centre of $\mathfrak{Qc}(X)$

Lemma 8.5.3. *Let X be an s-Noetherian monoid scheme. Then*

$$\mathcal{Z}(\mathfrak{Qc}(X)) \cong \Gamma(X, \mathcal{O}_X).$$

Proof. We first consider the case when $X = \mathsf{Spec}(M)$ is affine. In this case, the result is classical and it follows basically from the Yoneda lemma since for any M-set A, we have a functorial isomorphism

$$A \cong \mathsf{Hom}_M(M, A).$$

This shows that M represents the identity functor on the category of M-sets. For a general monoid scheme X, observe that if U and V are open monoid subschemes of X, the category $\mathfrak{Qc}(U \cup V)$ is equivalent to the gluing of $\mathfrak{Qc}(U)$ and $\mathfrak{Qc}(V)$ along $\mathfrak{Qc}(U \cap V)$. This fact, well-known in classical algebraic geometry (see [20, Proposition 2, p.441]), follows directly from the gluing of sheaves discussed in [28, Ch.2. Exercise 1.22]. Thus, we can use Lemma 6.2.9 to finish our proof. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

8.5.3 The proof Theorem 8.0.1

The aim of this subsection is to prove Theorem 8.0.1, which is the main theorem of this chapter. We will need one preliminary lemma before that though. To state it, let us fix some terminology.

If $\iota : W_1 \subseteq W_2$ is the inclusion of open monoid subschemes, then $\iota^* : \mathfrak{Off}(W_2) \to \mathfrak{Off}(W_1)$ is a map given by $L \mapsto L \cap W_1$, $L \in \mathfrak{Off}(W_2)$. We also have a similar map

$$\iota^* : \mathfrak{Loc}(\mathfrak{Qc}(W_2)) \to \mathfrak{Loc}(\mathfrak{Qc}(W_1)),$$

which sends a localisation $\mathscr{D}$ of $\mathfrak{Qc}(W_2)$ to the intersection $\mathscr{D} \cap \mathfrak{Qc}(W_1)$. Here, we consider $\mathfrak{Qc}(W_1)$ as a localisation subcategory of $\mathfrak{Qc}(W_2)$ (see [54, 4.3.2]) and use the fact that the

intersection of localisations in a topos is again a localisation [7, Theorem 6.8 and Example 6.9 iv)].

We have a map $\Phi_X : \mathfrak{Off}(X) \to \mathfrak{Loc}(\mathfrak{Qc}(X))$, given by $\Phi_X(U) = \mathfrak{Qc}(U)$. By construction, Φ is compatible with ι, meaning that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{Off}(W_2) & \xrightarrow{\Phi_{W_2}} & \mathfrak{Loc}(\mathfrak{Qc}(W)) \\ {\scriptstyle \iota^*}\downarrow & & \downarrow{\scriptstyle \iota^*} \\ \mathfrak{Off}(W_1) & \xrightarrow[\Phi_{W_1}]{} & \mathfrak{Loc}(\mathfrak{Qc}(W_1)). \end{array}$$

Lemma 8.5.4. *Let X be a monoid scheme of finite type. Assume U_1 and U_2 are open monoid subschemes of X and $U_{12} = U_1 \cap U_2$, $V = U_1 \cup U_2$. Then both*

$$\begin{array}{ccc} \mathfrak{Off}(V) & \xrightarrow{\iota_1^*} & \mathfrak{Off}(U_1) \\ \downarrow{\scriptstyle \iota_2^*} & & \downarrow{\scriptstyle \iota_3^*} \\ \mathfrak{Off}(U_2) & \xrightarrow[\iota_4^*]{} & \mathfrak{Off}(U_{12}) \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{Loc}(\mathfrak{Qc}(V)) & \xrightarrow{\iota_1^*} & \mathfrak{Loc}(\mathfrak{Qc}(U_1)) \\ \downarrow{\scriptstyle \iota_2^*} & & \downarrow{\scriptstyle \iota_3^*} \\ \mathfrak{Loc}(\mathfrak{Qc}(U_2)) & \xrightarrow[\iota_4^*]{} & \mathfrak{Loc}(\mathfrak{Qc}(U_{12})) \end{array}$$

are pullback diagrams. Here, $\iota_1, \iota_2, \iota_3, \iota_4$ are the appropriate inclusions.

Proof. The statement related to the first diagram follows from the definition.

For the second diagram, observe that $\mathfrak{Loc}(\mathfrak{Qc}(V))$ is in a one-to-one correspondence with the Lawvere-Tierney topologies on the topos $\mathfrak{Qc}(V)$ (see iii) and iv) of Lemma 6.1.5). Thus, we can and we will work with Lawvere-Tierney topologies.

Any such topology j on $\mathfrak{Qc}(V)$ is a morphism of sheaves $\Omega^V \to \Omega^V$ with appropriate properties. The restriction of Ω^V on U_1 (resp. U_2, U_{12}) is Ω^{U_1} (resp. Ω^{U_2}, $\Omega^{U_{12}}$). The result follows from the fact that for any two morphisms $\Omega^{U_1} \to \Omega^{U_1}$ and $\Omega^{U_2} \to \Omega^{U_2}$, which restrict to the same morphisms $\Omega^{U_{12}} \to \Omega^{U_{12}}$, there exists a unique morphism $\Omega^V \to \Omega^V$, which restrict to the given morphisms. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

We are now in a position to prove Theorem 8.0.1, the main theorem of this chapter.

Proof of Theorem 8.0.1. i) and iv) As already sated in the introduction, parts i) and iv) were already proven in [54].

ii) Observe that by Lemma 8.5.4, it suffices to restricts ourself to the affine case. This is a consequence of Corollary 8.4.11 and Lemma 8.4.4.

iii) This follows from Lemma 8.5.3.

v) Let $p = (p_*, p^*)$ be a point of the topos $\mathscr{T}$ and $\mathscr{D}$ a localisation of $\mathscr{T}$. Denote the canonical inclusion $\mathscr{D} \subset \mathscr{T}$ by ι and the corresponding localisation functor by $\rho : \mathscr{T} \to \mathscr{D}$. The pair of morphisms (ι, ρ) is a geometric morphism $\mathscr{D} \to \mathscr{T}$. By our definition, $p \pitchfork \mathscr{D}$ if and only if $p_* : \mathrm{Sets} \to \mathscr{T}$ factors through $\mathscr{D}$. If this happens, the induced functor $\mathrm{Sets} \to \mathscr{D}$ will be the direct image part of a geometric morphism $\mathrm{Sets} \to \mathscr{D}$ and hence, give rise to a topos point of $\mathscr{D}$.

Equivalently, this happens if and only if $p^* : \mathscr{T} \to \mathrm{Sets}$ factors through the functor $\rho : \mathscr{T} \to \mathscr{D}$. That is to say, there exists a functor $p_1 : \mathscr{D} \to \mathrm{Sets}$ such that $p^* = p_1 \circ \rho$.

Let $\mathfrak{Qc}(X)$ be our topos, $x \in X$ a point and U an open subset of X. In this case, the functor $\rho : \mathfrak{Qc}(X) \to \mathfrak{Qc}(U)$ is just the restriction functor. If $x \in U$, then the stalk functor $\Pi(x) : \mathfrak{Qc}(X) \to \mathrm{Sets}$ factors through $\mathfrak{Qc}(U) \to \mathrm{Sets}$. Hence, $\Pi(x) \pitchfork \mathfrak{Qc}(U)$. Conversely, if $\Pi(x) \pitchfork \mathfrak{Qc}(U)$, then $\Pi(x) = p_1 \circ \rho$, where p_1 is the inverse image functor of a topos point of $\mathfrak{Qc}(U)$. As shown in [54], it corresponds to the stalk at some point $y \in U$. The equality $\Pi(x) = p_1 \circ \rho$ shows that for any quasi-coherent sheaf F defined on X, we have $F_x = F_y$. Thus $x = y$ and hence $x \in U$. $\mathcal{Q.E.D}$

Chapter 9

Elliptic Algebras

9.1 An exercise of Silverman-Tate

Except for the name, the following definition goes back to [64, p34] Exercise 1.11.

Definition 9.1.1. *An elliptic group is a set S equipped with a binary operation $*$ satisfying the properties*

- EG1) $x * y = y * x$.
- EG2) $x * (x * y) = y$.
- EG3) $x * (y * (z * w)) = w * (y * (z * x))$.

Example 9.1.2. *Let $\mathcal{E}$ be an elliptic curve defined over a field K. Take two points $P, Q \in \mathcal{E}(K)$. The line $\mathcal{L}_{P,Q}$ passing through P and Q intersects $\mathcal{E}$ on a third (not necessarily distinct) point, denoted by $P * Q$. Thanks to [64, Exercises 1.10], one obtains an elliptic group in this way.*

A morphism of elliptic groups $f : S \to T$ is a map such that $f(x * y) = f(x) * f(y)$. The category of elliptic groups is denoted by $\mathsf{Ell}_{\mathsf{GR}}$.

9.1.1 From $\mathsf{Ell}_{\mathsf{GR}}$ to Ab

The following is an exercise in [64, Exercises 1.11].

Proposition 9.1.3. *Let S be a nonempty elliptic group.*

i) Fix an element c in S and define

$$x +_c y := c * (x * y).$$

The pair $(S, +_c)$ is an abelian group with c as the zero element. The inverse of an element x is $x * (c * c)$, thus

$$-_c x = x * (c * c).$$

ii) If d is another point, the map $f : S \to S$, given by

$$f(x) = c * (x * d),$$

is a group isomorphism $(S, +_c) \to (S, +_d)$.

Proof. i) The commutativity of $*$ implies the commutativity of $+_c$. By EG2, we have

$$c +_d y = c * (c * y) = y.$$

Thus, c is the zero element of $+_c$. We also have

$$x +_c (x * (c * c)) = c * (x * (x * (c * c))) = c * (c * c) = c.$$

Hence, $x * (c * c)$ is the inverse of x. So far, we did not use condition EG3. We need it to prove the associativity of $+_c$. In fact we have

$$x +_c (y +_c z) = c * (x * (c * (y * z))).$$

On the other hand

$$(x +_c y) +_c z = c * ((c * (x * y)) * z).$$

So, for associativity, it suffice to observe that

$$x * (c * (y * z)) = z * (c * (y * x)) = (c * (x * y)) * z,$$

for which we use EG3 and then EG1.

ii) We have to show that

$$f(x +_c y) = f(x) +_d f(y).$$

To this end, observe that $f(x) = x +_c d$. Thus

$$f(x +_c y) = x +_c y +_c d.$$

To proceed further let us show the following cancellation law:

$$(x +_c d) +_d z = x +_c z.$$

We have

$$\begin{aligned}(x +_c d) +_d z =& d * ((c * (x * d)) * z)\\ =& d * (z * ((x * d) * c))\\ =& c * (z * ((x * d) * d)).\end{aligned}$$

Here we used EG3. The axioms EG1 and EG2 yield

$$(x * d) * d = d * (d * x) = x.$$

Thus we obtain $(x +_c d) +_d z = c * (z * x) = z +_c x$, which is the cancellation law. Based on this, we compute

$$f(x) +_d f(y) = (x +_c d) +_d (y +_c d) = x +_c y +_c d.$$

Comparing our computations for $f(x) +_d f(y)$ and $f(x +_c y)$ above, we see that they are equal. $\mathcal{Q.E.D}$

Thus, we can talk about the group associated to an elliptic group. In what follows we write $x -_c y$ instead of $x +_c (-_c y)$, which equals to $c * (x * (y * (c * c)))$.

Example 9.1.4. *Let us consider the elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$, defined over $\mathcal{F}_7$ by the equations*

$$x^3 + 2y^3 = 3$$

and

$$y^2 = x^3 + 2$$

respectively. In both examples there are 9 points:

$$\mathcal{E}_1(\mathcal{F}_7) = \{(1,1),(2,1),(4,1),(1,2),(2,2),(4,2),(1,4),(2,4),(4,4)\}$$

and

$$\mathcal{E}_2(\mathcal{F}_7) = \{\infty,(0,3),(0,4),(3,1),(3,6),(5,1),(5,6),(6,1),(6,6)\},$$

where $\infty = (0:1:0)$ in the projective notation. Let us choose $\mathcal{O} = (1,1) \in \mathcal{E}_1(\mathcal{F}_7)$ and $\infty \in \mathcal{E}_2(\mathcal{F}_7)$. The corresponding abelian groups are isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ but as elliptic groups they are not isomorphic. In fact, in the second case we have $\infty * \infty = \infty$, while in the first case, there is not a single P satisfying $P * P = P$. Such points are called flex points and we will discuss more about them in Section 9.4.

We will come back to this example in Example 9.4.6. We will use the following easy fact several times.

Lemma 9.1.5. *Let $f : S \to S'$ be a morphism of elliptic groups. The map f is a group homomorphism $(S, +_c) \to (S', +_{f(c)})$, for any element $c \in S$.*

9.2 Relations with abelian groups

9.2.1 General remarks

According to Proposition 9.1.3, we can associate an abelian group to an elliptic group, by singling out an element. Our first result in this section shows that any abelian group can be obtained in such a manner. This is based on a simple construction which creates an elliptic group, starting with an abelian group with a chosen element. Moreover, any elliptic group can be obtained in this way. So, elliptic groups and abelian groups are closely related. As we will show, the isomorphism classes of abelian groups and elliptic groups are, however, not the

same. More precisely, nonisomorphic elliptic groups could have isomorphic associated abelian groups and the aim of this section is to understand this phenomena more closely.

Before we proceed further, let us observe that the theory of elliptic groups is an example of universal algebras and therefore, all known notions and results are applicable. In particular, we can talk not only about morphisms of elliptic groups, but also about elliptic subgroups, free elliptic groups, congruences and so on.

It is a consequence of the general facts from the theory of universal algebras, that the category $\mathsf{Ell}_{\mathsf{GR}}$ posses all limits and colimits and also, the fact that limits can be computed "componentwise". In particular, if S and T are elliptic groups, then $S \times T$ is also an elliptic group with the operation being defined by

$$(s,t)*(s',t') := (s * s', t * t').$$

The forgetful functor $\mathsf{Ell}_{\mathsf{GR}} \to Sets$ has the left adjoint. The value of this functor on a set S is known as the free elliptic group generated by S. We will give an explicit construction of these objects in Section 9.7.

9.2.2 Reformulation of axioms of elliptic groups

Let S be an elliptic group. For an element $a \in S$, denote by ℓ_a the map $S \to S$, given by $x \mapsto a * x$. Axiom EG2 can be equivalently stated as

$$\ell_a^2 = \mathsf{Id}.$$

This can again be restated as:

$$\ell_a \quad \text{is a bijection and} \quad \ell_a^{-1} = \ell_a.$$

Axiom EG1 can be formulated as follows: for all $a, b \in S$ one has

$$\ell_a(b) = \ell_b(a).$$

It follows that the family of maps ℓ_a are totally distinct, meaning that if ℓ_a and ℓ_b have the same value on at least one point, then they are equal. In fact, if $\ell_a(x) = \ell_b(x)$ for some x, then $\ell_x(a) = \ell_x(b)$ and since ℓ_x is a bijection, we have $a = b$.

9.2.3 Elliptic groups from abelian groups

We start by constructing elliptical groups from abelian groups.

Lemma 9.2.1. *i) Let A be an additively written abelian group. Fix an element a and define*

$$x *_a y := a - x - y.$$

Then $(A, *_a)$ is an elliptic group, denoted by ${}_aA$.

ii) If $f : A \to B$ is a group homomorphism and $a \in A$, then f is also a morphism of elliptic groups $(A, *_a) \to (B, *_{f(a)})$.

iii) If c is an element of A, the abelian group $({}_aA, +_c)$, associated to the elliptic group ${}_aA$, is isomorphic to A. In fact, this group coincides with A if $c = 0$.

Proof. i) Commutativity of $*_a$ is obvious. Next, we have

$$x *_a (x *_a y) = a - x - (a - x - y) = a - x - a + x + y = y.$$

Finally, both $x * (y * (z * w))$ and $w * (y * (z * x)))$ are equal to $a + y - x - z - w$.

ii) This is straightforward to check.

iii) For any elements $x, y \in A$, we have

$$x +_c y = c *_a (x *_a y) = a - c - (a - x - y) = x + y - c.$$

We see that for $c = 0$, the group law coincides with the original addition. For general c, the map $f : A \to A$, given by $f(x) = x + c$, is an isomorphism of abelian groups $A \to (A, +_c)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 9.2.2. *Any abelian group is a group associated to an elliptic group.*

Proof. This follows from part iii) of Lemma 9.2.1. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

The next result shows that, up to isomorphisms, any elliptic group can be obtained as in Lemma 9.2.1. This is well-known, see [63].

Theorem 9.2.3. *Let S be a nonempty elliptic group and $o \in S$. Denote by A the abelian group $(S, +_o)$. For $c = o * o$, the elliptic group ${}_cA$ is the same as S.*

Proof. To simplify the notation, we write + instead of $+_o$. For an element a, define the map

$$\lambda_a : S \to S$$

by $\lambda_a(x) = a + x = o * (a * x)$. Thus, we have

$$\lambda_a = \ell_o \circ \ell_a,$$

which implies $\ell_a = \ell_o \circ \lambda_a$. Since $\ell_a^2 = \mathrm{Id}$, we obtain

$$\ell_o \lambda_a \ell_o \lambda_a = \mathrm{Id}$$

or

$$\lambda_a \ell_o \lambda_a = \ell_o.$$

It follows that for all $x \in S$, we have

$$a + \ell_o(a + x) = \ell_o(x).$$

Denote

$$c = \ell_o(o) = o * o.$$

Taking $x = c$, we obtain

$$a + \ell_o(a + c) = \ell_o(c) = o * (o * o) = o.$$

Since o is the zero element for $+$, we see that

$$\ell_a(a + c) = -a.$$

Since a was an arbitrary element we obtain

$$\ell_o(x) = c - x.$$

Hence

$$\ell_a(x) = \ell_o\lambda_a(x) = \ell_0(a + x) = c - a - x.$$

Thus

$$a * x = c - a - x,$$

proving that the operations in S and ${}_cA$ are the same. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

9.3 Morphisms of elliptic groups

If $S, T \in \mathsf{Ell}_{\mathsf{GR}}$, the set of morphism from S to T is denoted by $\mathsf{Hom}_{\mathsf{El}}(S, T)$. We use the notation Hom for the set of group homomorphisms.

9.3.1 Morphism of elliptic groups and affine maps of abelain groups

Let A and B be abelian groups. Recall that a map $f : A \to B$ is called *affine* if the map $f_1 : A \to B$ is a group homomorphism, where $f_1(x) = f(x) - f(0)$. Clearly any affine map $f : A \to B$ is of the form $f = f_0 + f_1$, where f_0 is a constant map and f_1 is a group homomorphism. We say that f_0 is the *constant* part of f and f_1 is the *linear part* of f.

Proposition 9.3.1. *Let A and B be abelian groups and $a \in A$, $b \in B$. Any morphism of elliptic groups $f : {}_aA \to {}_bB$ is affine as a map $A \to B$ and hence, of the form*

$$f(x) = f_0 + f_1(x), \text{ for all } x \in A,$$

where $f_0 \in B$ and $f_1 : A \to B$ is a group homomorphism. Moreover, these quantities satisfy the compatibility condition:

$$3f_0 + f_1(a) = b. \tag{9.1}$$

Conversely, if $f_1 : A \to B$ is group homomorphism and $f_0 \in B$ is an element for which the compatibility condition (9.1) holds, then

$$f(x) = f_0 + f_1(x)$$

defines a morphism of elliptic groups $f : {}_aA \to {}_bB$.

Proof. A map $f : A \to B$ is a morphism ${}_aA \to {}_bB$ if

$$f(x *_a y) = f(x) *_b f(y)$$

holds. In our circumstances, this means

$$f(a - x - y) = b - f(x) - f(y). \tag{9.2}$$

Denote by f_0 the value of f at zero, that is $f_0 := f(0)$. We also set $f_1(x) = f(x) - f_0$. Then $f(x) = f_0 + f_1(x)$ and $f_1(0) = 0$. Based on this, we can rewrite condition (9.2) as

$$3f_0 + f_1(a - x - y) = b - f_1(x) - f_1(y).$$

By putting $x = y = 0$, we obtain

$$3f_0 + f_1(a) = b.$$

We want to show that f_1 is a group homomorphism, which would imply that f is affine. Replacing b by $3f_0 + f_1(a)$, we obtain

$$f_1(a - x - y) = f_1(a) - f_1(x) - f_1(y). \tag{9.3}$$

Putting $y = 0$ yields $f_1(a - x) = f_1(a) - f_1(x)$ for all x. Finally, replacing x with $x + y$ and comparing it with equation (9.3) proves that f is a group homomorphism.

Conversely, let $f = f_0 + f_1$ be an affine map with $3f_0 + f_1(a) = b$. We have

$$f(x *_a y) = f(a - x - y) = f_0 + f_1(a - x - y) = f_0 + f_1(a) - f(x) - f(y).$$

On the other hand

$$f(x) *_b f(y) = b - f(x) - f(y) = b - 2f_0 - f_1(x) - f_1(y) = f_0 + f_1(a) - f_1(x) - f_1(y).$$

Comparing these expressions, we see that $f(x *_a y) = (x) *_b f(y)$ and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

9.3.2 Two consequences

As we shall soon see, it might happen that there is no morphism between elliptic groups. The following gives a criteria when this happens. It also gives a criteria when ${}_aA$ and ${}_bB$ are isomorphic.

Corollary 9.3.2. *Let A and B be abelian groups and $a \in A$, $b \in B$. Then*

$$\mathrm{Hom}_{\mathsf{El}}({}_aA, {}_bB) \neq \varnothing \quad \textit{if and only if} \quad b \in 3B + \mathcal{H}(a),$$

where $\mathcal{H}(a) \subseteq B$ is the set of all elements of the form $f(a)$, with f running through the set of all group homomorphisms $A \to B$. Moreover, ${}_aA$ and ${}_bB$ are isomorphic if and only if

$$b \in 3B + \mathcal{H}(a).$$

Here, $\mathcal{H}(a)$ is the set of all elements of the form $f(a)$, where f runs through the set of all group isomorphisms $A \to B$.

Proof. Assume there exist a morphism $f : {}_aA \to {}_bB$. We have seen that $f = f_0 + f_1$, where f_0 is a constant map and f_1 is a group homomorphism. We have also seen that $3f_0 + f_1(a) = b$. Thus, $b \in 3B + \mathcal{H}(a)$. Conversely, assume $b = 3f_0 + f_1(a)$ for some $f_1 \in \mathrm{Hom}(A, B)$. We put $f = f_0 + f_1$. Then f is a morphism ${}_aA \to {}_bB$, thanks to Proposition 9.3.1. If one can choose f_1 to be an isomorphism of abelian groups, then $f = f_0 + f_1$ will also be bijective and hence, an isomorphism of elliptic groups. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 9.3.3. *Assume the multiplication by 3 is a surjective endomorphism of an abelian group A. Then for any $a \in A$, the elliptic groups ${}_aA$ and ${}_0A$ are isomorphic. In particular, ${}_aA$ and ${}_0A$ are isomorphic if and only if $a \in 3A$.*

Proof. This is a direct consequence of Corollary 9.3.2, as $3A = A$ and $a = \mathrm{Id}(a) \in \mathcal{H}(a)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

9.4 Elliptic groups with a flex point

An element o of an elliptic group S is called a *flex* point if $o * o = o$. For example, 0 is a flex point of ${}_0A$, where A is an abelian group A. It is clear that any homomorphism of elliptic groups sends flex points to flex points.

The category $\mathrm{Ell}_{\mathrm{GR}}$ has a terminal object $\mathbf{0}$. As a set, it has the single element o, which is a flex point. For any elliptic group S, the unique morphism $S_! : S \to \mathbf{0}$ has a section if and only if S has a flex point.

Lemma 9.4.1. *Let S be an elliptic group and $c \in S$. Denote by $+_c$ the corresponding group structure. Then c is a flex point of S if and only if the inverse of any element $x \in S$ is $x * c$.*

Proof. The inverse of x is $x*(c*c)$ by Proposition 1.2 i), which implies the *if* part. Conversely, assume $x * (c * c) = x * c$ for all x. It follows that $\ell_c = \ell_{c*c}$ and hence, $c * c = c$, thanks to Section 9.2.2. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 9.4.2. *Let A be an abelian group and $a \in A$. Then ${}_aA$ has a flex point if and only if $a \in 3A$. If this condition holds, the set of all flex points is a torsor over the group*

$$\mathrm{Ann}_3(A) := \{t \in A \mid 3t = 0\}.$$

Proof. Let o be a flex point of ${}_aA$. We have

$$o = o * o = a - o - o,$$

hence a is a flex point if and only if $a = 3o \in 3A$. If $t \in Ann_3(A)$, then $3t = 0$. Hence

$$(o + t) * (o + t) = a - (o + t) - (o + t) = a - 2o - 2t = 3o + 3t - 2o - 2t = o + t.$$

This shows that $o + t$ is flex. Conversely, assume o' is also a flex point. Denote $t' = o' - o \in A$. We have $o' = o + t'$. Thus, $a = 3o' = 3o + 3t' = a + 3t'$ and $3t' = 0$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Remark 9.4.3. *Note that in Chapter 2, we used the notation ${}_3A$ to denote the group $\{a \in A \mid 3a = 0$, see for example Corollary 2.4.5. However, as ${}_aA$ already has a different meaning in this chapter, we use the notation* Ann_3.

Lemma 9.4.4. *i) An elliptic group S is isomorphic to an elliptic group of the form ${}_0A$ if and only if S has a flex point.*

ii) Denote by $\mathsf{Ell}_{\mathsf{GR}}{}^{Flex}$ *the category of elliptic groups with chosen flex points and morphisms preserving those points. The functor*

$$\mathsf{Ab} \to \mathsf{Ell}_{\mathsf{GR}}{}^{Flex}, \quad A \mapsto {}_0A,$$

is an equivalence of categories, where $\mathbf{Ab}$ *is, as usual, the category of abelian groups.*

Proof. i) This follows from Corollary 9.3.3.

ii) Let S be an elliptic group with a fixed flex point o. By Theorem 2.3, ${}_0A$ and S are the same elliptic groups, where $A = (S, +_o)$ is the group corresponding to S. Hence, the above functor is essentially surjective. It is obviously faithful. It only remains to show that it is full. Let $f : S \to T$ be a morphism of elliptic groups. By assumption, we have a chosen flex $o \in S$ and $f(o)$ is the chosen flex in T. As we already said, $S = {}_0A, T =_0 B$, where $A = (S, +_o)$ and $B = (T, +_{f(o)})$. The map f, considered as a map $A \to B$, sends 0 to 0. As such, the constant part of f vanishes and thus, it is a homomorphism of abelian groups, proving the result. $\mathcal{Q.E.D}$

Lemma 9.4.5. *For any elliptic group S, there is an injective morphism of elliptic groups $\kappa : S \to T$, where T has a flex point.*

Proof. We can assume $S = {}_cA$ for an abelian group A. We can embed A into a divisible group (in fact, 3-divisible is enough) B. Then ${}_cB$ has a flex, thanks to Lemma 9.4.2, so we can take $T = {}_cA$. $\mathcal{Q.E.D}$

The above lemma can be compared with the well-known fact that any elliptic curve over an algebraically closed field has a flex point, see [31, Ch.2. Proposition 4.8]. However, over non-algebraically closed fields, elliptic curves might not have a flex point and thus, their corresponding elliptic groups will be not of the form ${}_0A$. For example, the flex points of the elliptic curve

$$x^3 + 2y^3 - 3z^3 = 0$$

are

$$(0, \sqrt[3]{3}, \sqrt[3]{2}), (\sqrt[3]{3}, 0, 1) \quad \text{and} \quad (\sqrt[3]{2}, -1, 0).$$

Hence, this elliptic curve has no rational flex, see also Example 9.1.4.

Example 9.4.6. *Let us once again consider the elliptic curve from Example 9.1.4. Since ∞ is a flex point of the elliptic group $\infty \in \mathcal{E}_2(\mathbb{F}_7)$, it follows that, as an elliptic group, it is isomorphic to ${}_0(\mathbb{Z}/3\mathbb{Z})^2$.*

If, on the other hand, we choose $\mathcal{O} = (1,1) \in \mathcal{E}_1(\mathbb{F}_7)$ *as the zero element, the points* $(1,2)$, $(2,4)$ *become the generators for the group*

$$(\mathcal{E}_1(\mathbb{F}_7), +_{\mathcal{O}}) \cong (\mathbb{Z}/3\mathbb{Z})^2.$$

Since $\mathcal{O} * \mathcal{O} = (2,4)$, *it follows from Theorem 9.2.3, that one has an isomorphisms of elliptic groups*

$$\mathcal{E}_1(\mathbb{F}_7) \cong {}_{(1,0)}(\mathbb{Z}/3\mathbb{Z})^2.$$

In particular, this shows that $\mathcal{E}$ *has no flex point, which is an alternative way of proving this.*

9.5 $\mathsf{Hom}_{\mathsf{El}}(S,T)$ as an elliptic group

Lemma 9.5.1. *Let* S *and* T *be elliptic groups. Assume* $f, g : S \to T$ *are homomorphisms of elliptic groups. Define* $f * g : S \to T$ *to be the map given by*

$$(f * g)(s) = f(s) * g(s).$$

Then $f * g$ *is also a homomorphism of elliptic groups.*

Proof. We can and we will assume that $S = {}_aA$ and $T = {}_bB$, for abelian groups A and B, with $a \in A$, $b \in B$. To check that $h = f * g$ is a morphism of elliptic groups, it is enough to show that $f * g$ is an affine map from the group A to B, satisfying the compatibility condition:

$$b = 3h_0 + h_1(a).$$

To show this assertion, write $f(s) = f_0 + f_1(s)$ and $g(s) = g_0 + g_1(s)$, where $f_0, g_0 \in B$ and $f_1, g_1 \in \mathsf{Hom}(A,B)$. It follows that

$$h(s) = f(s) * g(s) = b - f_0 - f_1(s) - g_0 - g_1(s).$$

Thus, h is affine with

$$h_0 = b - f_0 - g_0$$

and

$$h_1(s) = -(f_1 + g_1).$$

Now, we can compute

$$3h_0 + h_1(a) = 3b - 3f_0 - 3g_0 - f_1(a) - g_1(a) = 3b - (b - f_1(a)) - (b - g_1(a)) - f_1(a) - g_1(a) = b$$

and the Lemma follows. $\mathcal{Q.E.D}$

Corollary 9.5.2. *For any elliptic groups* S, T, *the set* $\mathsf{Hom}_{\mathsf{El}}(S,T)$ *of all homomorphisms of elliptic groups* $S \to T$ *is again an elliptic group.*

We will henceforth always equip the set $\mathsf{Hom}_{\mathsf{El}}(S,T)$ with this structure.

9.5.1 Determining Hom in $\mathsf{Ell}_{\mathsf{GR}}$

Lemma 9.5.3. *Let A and B be abelian groups and $b \in B$. One has an isomorphism of elliptic groups*

$$\mathsf{Hom}_{\mathsf{El}}({}_0A, {}_bB) \cong \begin{cases} \varnothing, & \text{if } b \notin 3B \\ {}_0\mathsf{Hom}(\mathbb{Z}/3\mathbb{Z} \oplus A, B), & \text{if } b \in 3B. \end{cases}$$

The second isomorphism is natural with respect to A.

Proof. The first part follows directly from Corollary 9.3.2. Assume $b = 3f_0$ for some $f_0 \in B$. We observe that

$$\mathsf{Hom}(\mathbb{Z}/3\mathbb{Z} \oplus A, B) \cong \mathsf{Hom}(\mathbb{Z}/3\mathbb{Z}, B) \oplus \mathsf{Hom}(A, B) \cong \mathsf{Ann}_3(B) \oplus \mathsf{Hom}(A, B),$$

where $\mathsf{Ann}_3(B) = \{t \in B \mid 3t = 0\}$. Define the map

$$\phi : \mathsf{Ann}_3(B) \oplus \mathsf{Hom}(A, B) \to \mathsf{Maps}(A, B)$$

by

$$\phi(t, h)(a) := f_0 + t + h(a),$$

where $t \in B$ is element such that $3t = 0$ and $h \in \mathsf{Hom}(A, B)$. Obviously, $\phi(t, h) : A \to B$ is an affine map with constant term $f_0 + t$. Since

$$3(f_0 + t) + h(0) = 3f_0 = b,$$

the compatibility condition 9.1 holds and thus, $\phi(t, h) \in \mathsf{Hom}_{\mathsf{El}}({}_0A, {}_bB)$. Hence, ϕ can be considered as a map

$$\phi : \mathsf{Ann}_3(B) \oplus \mathsf{Hom}(A, B) \to \mathsf{Hom}_{\mathsf{El}}({}_0A, {}_bB).$$

We have

$$\begin{aligned} \phi(-t - t', -h - h')(a) &= f_0 - t - t' - h(a) - h'(a) \\ &= b - (f_0 + t + h(a)) - (f_0 + t' + h'(a)) \\ &= \phi(t, h)(a) * \phi(t', h')(a). \end{aligned}$$

Thus, $\phi : {}_0\mathsf{Hom}(\mathbb{Z}/3\mathbb{Z} \oplus A, B) \to \mathsf{Hom}_{\mathsf{El}}({}_0A, {}_bB)$ is a morphism of elliptic groups. It suffices to show that it is a bijection. Take any morphism $g : {}_0A \to {}_bB$. By Proposition 9.3.1, it has a form $g = g_0 + g_1$, where $3g_0 = b$ and $g_1 : A \to B$. By putting $h = g$ and $t = f_0 - g_0$, we see that $g \mapsto (t, h)$ is the inverse of the map ϕ. $\mathcal{Q.E.D}$

Lemma 9.5.4. *Let B be an abelian group and $b \in B$.*

i) One has an isomorphism of elliptic groups

$$\mathsf{Hom}_{\mathsf{El}}({}_1\mathbb{Z}, {}_bB) \cong {}_bB.$$

ii) Let $n = 3^k$. There exists an isomorphism of elliptic groups

$$\mathsf{Hom}_{\mathsf{El}}({}_1(\mathbb{Z}/n\mathbb{Z}),\, {}_bB) \cong \begin{cases} \varnothing, & \text{if } 3^k b \notin 3^{k+1}B \\ {}_bB, & \text{if } 3^k b \in 3^{k+1}B. \end{cases}$$

Proof. i) Any affine map $f : \mathbb{Z} \to B$ has the form $f(x) = f_0 + xc$, where $f_0, c \in B$. The compatibility condition (9.1) reads

$$3f_0 + c = b,$$

as $a = 1$ in our case. See Proposition 9.3.1. This shows that f_0 can be taken arbitrary and then $c = b - 3f_0$ is uniquely determined. Thus, $f \mapsto f_0$ is an isomorphism.

ii) Any affine map $f : \mathbb{Z}/n \to B$ has the form $f(x) = f_0 + xc$, where $f_0, c \in B$ and $nc = 0$. Since $c = b - 3f_0$, as argued above, the condition $nc = 0$ implies $3^k b = 3^{k+1} f_0$, and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

9.6 Congruences of elliptic groups

We consider congruences in elliptic groups next. According to the theory of universal algebras, an equivalence relation $\sim$ on an elliptic group S is a *congruence* relation if and only if $x \sim y$ and $u \sim v$ implies $x * u \sim y * v$. In this case, the quotient set $S/\sim$ has a unique elliptic group structure for which the canonical map $S \to S/\sim$ is a homomorphism.

Lemma 9.6.1. *i) Let S be a nonempty elliptic group and $\sim$ be a congruence on S. For any $c \in S$, the set*

$$K_c = \{x \in S \mid x \sim c\}$$

is a subgroup of the group $(S, +_c)$.

ii) Conversely, if K is a subgroup of $(S, +_c)$, then

$$(x \sim y) \Longleftrightarrow (x +_c (-_c y) \in K)$$

is a congruence on S, where $-_c y = y * (c * c)$ is the inverse of y in the group $(S, +_c)$.

Proof. i) Assume $x \sim y$ and $u \sim v$. We have

$$x +_c u = c * (x * u) \cong c * (y * v) = y +_c v.$$

Thus, $\sim$ is also a congruence in the group $(S, +_c)$ and hence, the result follows from the well-know correspondence between subgroups and congruences of abelian groups.

ii) The same correspondence shows that $\sim$ is a congruence in an abelian group $(S, +_c)$. It remains to show that if $x \sim y$ and $u \sim v$, then $x * u \sim y * v$. Thanks to Theorem 9.2.3, we can assume that $S = {}_aA$. In this case, we have

$$x +_c y = c * (x * y) = a - c - (a - x - y) = x + y - c.$$

Since K is a subgroup under $+_c$, it now follows that K satisfies the following properties: a) $c \in K$, b) if $x, y \in K$, then $x + y - c \in K$ and c) if $x \in K$, then $-_c x = 2c - x \in K$. By definition, $x \sim y$ if and only if

$$K \ni x -_c y = x +_c (2c - y) = x - y + c.$$

Similarly, $u - v + c \in K$. Now we have

$$y * v - x * u + c = a - y - v - a + x + u + c = (x - y + c) + (u - v + c) - c \in K,$$

thanks to the property b). $\mathcal{Q.E.D}$

Corollary 9.6.2. *The constructions in Lemma 9.6.1 establish a one-to-one correspondence between congruences on S and subgroups of $(S, +_c)$. We also have $K_c \simeq K_{c'}$ for all $c, c' \in S$.*

9.7 Free elliptic groups

9.7.1 The description

9.7.1.1 The finitely generated case

The first part of Lemma 9.5.4 shows that ${}_1\mathbb{Z}$ is a free elliptic group with one generator 0. We now describe other free objects. As usual, let $e_1, \cdots, e_n$ denote the standard basis in the free abelian group $\mathbb{Z}^n$:

$$e_1 = (1, \cdots, 0), \cdots, e_n = (0, \cdots, 1).$$

Theorem 9.7.1. *The elliptic group ${}_{e_1}\mathbb{Z}^n$ is a free object in* $\mathsf{Ell}_{\mathsf{GR}}$*, on the n-element set $\{0, e_2, \cdots, e_n\}$. In other words, $f \mapsto (f(0), f(e_2), \cdots, f(e_n))$ induces an isomorphism of elliptic groups:*

$$\mathsf{Hom}_{\mathsf{El}}({}_{e_1}\mathbb{Z}^n, {}_aA) \cong {}_{(a,\cdots,a)}A^n.$$

Proof. Recall that the elliptic group structure on ${}_{e_1}\mathbb{Z}^n$ is given by

$$(x_1, \cdots, x_n) * (y_1, \cdots, y_n) = (1 - x_1 - y_1, -x_2 - y_2, \cdots, -x_n - y_n).$$

We have to show that for any elliptic group S and elements $s_1, \cdots, s_n \in S$, there exists a unique morphism of elliptic groups $f : {}_{e_1}\mathbb{Z}^n \to S$ such that $f(0) = s_1$, and $f(e_i) = s_i$ for all $2 \le i \le n$. According to Theorem 9.2.3, we can assume $S = {}_aA$ for an abelian group A and $a \in A$.

Uniqueness. Assume $f : {}_{e_1}\mathbb{Z}^n \to {}_aA$ is such a morphism. According to Proposition 9.3.1, we have $f = f_0 + f_1$, where f_0 is the constant function with value $f_0 \in A$ and $f_1 : \mathbb{Z}^n \to A$ is a group homomorphism. Since $f_0 = f(0) = s_1$, is follows that f_0 is uniquely defined. On the other hand, f_1 is completely determined by the values $f_1(e_j)$ for $1 \le j \le n$, as f_1 is a group homomorphism. Since $0 *_{e_1} 0 = e_1$, we see that $f(e_1) = f(0) *_a f(0) = c - 2s_1$. On the other hand, for any $2 \le i \le n$, we have $f_1(e_i) = f(e_i) - f(0) = s_i - s_1$ and the uniqueness of f is proven.

Existence. As $f(e_1) = f_1(e_1) - f_0(e_1)$, it follows that

$$\begin{aligned} f_1(e_1) &= f(e_1) - f_0(e_1) \\ &= c - 2s_1 - s_1, \end{aligned}$$

see our argument above. This suggests that we consider

$$f(x_1, \cdots, x_n) = s_1 + x_1(a - 3s_1) + \sum_{i=2}^{n} x_i(s_i - s_1).$$

This formula defines a map $\mathbb{Z}^n \to A$. We first check that this map takes the expected values at $0, e_2, \cdots, e_n$. In fact, we have

$$f(0, \cdots, 0) = s_1$$

and for any $2 \leq i \leq n$, we have

$$f(e_i) = s_1 + 1(s_i - s_1) = s_i.$$

It thus only remains to be checked that this map is a morphism of elliptic groups. We have

$$\begin{aligned} f((x_1, \cdots, x_n) *_{e_1} (y_1, \cdots, y_n)) &= f(1 - x_1 - y_1, -x_2 - y_2, \cdots, -x_n - y_n) \\ &= s_1 + (1 - x_1 - y_1)(a - 3s_1) + \sum_{i=2}^{n}(-x_i - y_i)(s_i - s_1) \\ &= (a - 2s_1) + (-x_1 - y_1)(a - 3s_1) + \sum_{i=2}^{n}(-x_i - y_i)(s_i - s_1). \end{aligned}$$

On the other hand

$$\begin{aligned} f(x_1, \cdots, x_n) *_a f(y_1, \cdots, y_n) &= a - f(x_1, \cdots, x_n) - f(y_1, \cdots, y_n) \\ &= (a - 2s_1) - (x_1 + y_1)(a - 3s_1) - \sum_{i=2}^{n}(x_i + y_i)(s_i - s_1). \end{aligned}$$

Comparing these expressions we see that f is a morphism of elliptic groups, implying the result. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

9.7.1.2 The general case

We have constructed the free elliptic group over a finite nonempty set. The free elliptic group on the empty set is just the empty set. The construction of a free elliptic group generated by an infinite set is quit similar to the finite case: If X is any nonempty set, we let A to be the free abelian group generated by X. Choose an element $a \in X$ and consider the elliptic group ${}_aA$. It is a free elliptic group on the set $0 \cup X'$, where $X' = X \setminus a$.

9.7.2 Finitely generated elliptic groups

Let S be an elliptic group. We say that a subset $X \subseteq S$ generates S if S is the minimal elliptic subgroup containing X. An elliptic group S is called finitely generated if there is a finite subset $X \subseteq S$ which generates S. Equivalently, if there exists a surjective morphism of elliptic groups $F \twoheadrightarrow S$, where F is a finitely generated free elliptic group.

Lemma 9.7.2. *i) If S is a nonempty and finitely generated elliptic group, the associated group is a finitely generated abelian group.*

ii) If A is a finitely generated abelian group, then for any $c \in A$, the elliptic group ${}_cA$ is a finitely generated elliptic group.

Proof. i) Since the associated group $(S, +_c)$ is independent of the chosen point $c \in S$, it suffices to show that $(S, +_c)$ is a finitely generated abelian group for at least one c. By assumption, there is a surjective morphism of elliptic groups ${}_{e_1}\mathbb{Z}^n \xrightarrow{f} S$, which gives rise to a surjective homomorphism of abelian groups $(\mathbb{Z}^n, +) \to (S, +_{f(0)})$. As such, $(S, +_{f(0)})$ is a finitely generated abelian group and part i) follows.

ii) Assume $a_1, \cdots, a_k$ are generators of an abelian group A. Consider the elliptic group ${}_{e_1}\mathbb{Z}^{k+1}$, which is the free elliptic group with generators $0, e_2, \cdots, e_{k+1}$. Thus, there is a homomorphism of elliptic groups ${}_{e_1}\mathbb{Z}^{k+1} \xrightarrow{f} {}_cA$, which satisfies $f(0) = 0$ and $f(e_2) = a_1, \cdots, f(e_{k+1}) = a_k$. The same f can be seen as a homomorphism of abelian groups $\mathbb{Z}^{k+1} = ({}_{e_1}\mathbb{Z}^{k+1}, +_0) \to ({}_cA, +_0) = A$. Since f is a group homomorphism whose image contains all the generators of the group A, it is surjective. It follows that S is a finitely generated elliptic group. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

9.7.3 A funny example

We now turn to ${}_0\mathbb{Z}$. Recall that it is equipped with the operation $*_0$, defined by $x *_0 y = -x - y$. Observe that the subsets $3\mathbb{Z}$, $1 + 3\mathbb{Z}$ and $2 + 3\mathbb{Z}$ are closed under $*_0$. So, they are elliptic subgroups of ${}_0\mathbb{Z}$, giving a partition by elliptic subgroups

$$ {}_0\mathbb{Z} = (3\mathbb{Z}) \biguplus (1 + 3\mathbb{Z}) \biguplus (2 + 3\mathbb{Z}). $$

Here, $\biguplus$ denotes the disjoint union, i.e. the coproduct in the category of sets. We use the above notation as we will shortly need $\coprod$ for the coproduct.

The elliptic groups $1 + 3\mathbb{Z}$ and $2 + 3\mathbb{Z}$ are isomorphic to the elliptic group ${}_1\mathbb{Z}$. Hence, they are free with a single generator each. The corresponding isomorphisms

$$ f : {}_1\mathbb{Z} \to 1 + 3\mathbb{Z}, \quad g : {}_1\mathbb{Z} \to 2 + 3\mathbb{Z} $$

are given by $f(x) = 3x + 1$ and $g(x) = 3x - 1$. Thus, two out of the three elliptic subgroups in the partition of ${}_0\mathbb{Z}$ are free with one generator. The third one $3\mathbb{Z}$ is isomorphic to ${}_0\mathbb{Z}$, by sending $\mathbb{Z} \ni x \mapsto 3x \in 3\mathbb{Z}$. Hence, we can repeat this and obtain another partition

$$ {}_0\mathbb{Z} = (9\mathbb{Z}) \biguplus (1 + 3\mathbb{Z}) \biguplus (2 + 3\mathbb{Z}) \biguplus (3 + 9\mathbb{Z}) \biguplus (6 + 9\mathbb{Z}), $$

where now four members are free with one generators and the fifth is isomorphic to the whole object. This can, of course, be repeated indefinitely.

9.8 Indecomposable finitely generated elliptic groups

Let $\mathfrak{C}$ be a category with products. An object A is called *indecomposable* if for any isomorphism $A \xrightarrow{(f_1,f_2)} A_1 \times A_2$, either A_1 or A_2 is the terminal object. It is well-known that in the category of finitely-generated abelian groups, the indecomposable objects are exactly $\mathbb{Z}$ and $\mathbb{Z}/p^k\mathbb{Z}$, where p is prime and $k \geq 1$. Moreover, any finitely generated abelian group is isomorphic to the product of indecomposable ones and such a decomposition is unique up to the order of the factors. We can translate these classical results to elliptic groups.

Lemma 9.8.1. *Let A and B be groups. For any elements $a \in A$ and $b \in B$, the elliptic groups ${}_{(a,b)}(A \times B)$ and ${}_aA \times {}_bB$ are isomorphic.*

Proof. Both have the same underlying set. The operation in ${}_{(a,b)}(A \oplus B)$ is given by

$$(x,y) *_{(a,b)} (x',y') = (a,b) - (x,y) - (x',y') = (a - x - x', b - y - y') = (x *_a x', y *_b y').$$

The last expression is the result of the operation in ${}_aA \times {}_bB$ and hence, the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 9.8.2. *There are exactly two elliptic groups up to isomorphisms, ${}_1\mathbb{Z}$ and ${}_0\mathbb{Z}$, whose associated groups are infinite cyclic groups.*

Proof. Since any group isomorphism $\mathbb{Z} \to \mathbb{Z}$ is given by either $x \mapsto x$ or $x \mapsto -x$, we see that in the notation of Corollary 9.3.2, we have $\mathcal{H}(a) = \{a, -a\}$. Hence, for integers m, n, the elliptic groups ${}_m\mathbb{Z}$ and ${}_n\mathbb{Z}$ are isomorphic if and only if $m \equiv n$ mod(3) or $m \equiv -n$ mod(3), thanks to Corollary 9.3.2. It follows that ${}_m\mathbb{Z}$ is isomorphic to either ${}_1\mathbb{Z}$ or ${}_0\mathbb{Z}$ and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 9.8.3. *Let $p \neq 3$. Up to isomorphisms, there is only one elliptic group ${}_0\mathbb{Z}/p^k\mathbb{Z}$ whose associated group is the cyclic group $\mathbb{Z}/p^k\mathbb{Z}$.*

Proof. Since 3 is invertible, we can apply Corollary 9.3.3. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Lemma 9.8.4. *Up to isomorphisms, there are exactly two elliptic groups ${}_1\mathbb{Z}/3^k\mathbb{Z}$ and ${}_0\mathbb{Z}/3^k\mathbb{Z}$ whose associated group is the cyclic group $\mathbb{Z}/3^k\mathbb{Z}$.*

Proof. Take $a \in \mathbb{Z}/3^k\mathbb{Z} = A$. If $a \in 3A$, then ${}_aA$ is isomorphic to ${}_0A$, thanks to Corollary 9.3.2. If $a \notin 3A$, then a is invertible in $A = \mathbb{Z}/3^k\mathbb{Z}$. So, $1 = ab$ for some $b \in A$. Thus, $1 = \mu_b(1)$, where $\mu_b : A \to A$ is the multiplication by b, which is an isomorphism of abelian groups Thus, in the notations of Corollary 9.3.2, we have $1 \in \mathcal{H}(a)$. According to the same corollary, ${}_aA$ and ${}_1A$ are isomorphic. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

The following fact was proven in [63].

Theorem 9.8.5. *Any non-empty and non-terminal finitely generated elliptic group is isomorphic to the finite product of indecomposable elliptic groups. Moreover, the following is a full list of pairwise nonisomorphic, finitely generated, indecomposable elliptic groups*

$$ {}_0\mathbb{Z},\ {}_1\mathbb{Z},\ {}_0\mathbb{Z}/p^k\mathbb{Z},\ {}_0\mathbb{Z}/3^k\mathbb{Z},\ {}_1\mathbb{Z}/3^k, $$

where $p \neq 3$ is a prime and $k \geq 1$ is an integer.

Proof. Take a finitely generated elliptic group S. We know that the corresponding group is a finitely generated abelian group. Without loss of generality, we can assume that $S = {}_aA$, where A is a finitely generated abelian group. Based on Lemma 9.8.1, we can conclude that S is a finite product of elliptic groups of the form ${}_bB$, where B is an indecomposable abelian group. Clearly, any such elliptic group will also be indecomposable as an elliptic group, because any decomposition of elliptic groups induces a decomposition of the corresponding groups. The rest just summarises previous lemmas. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Remark 9.8.6. *Unlike abelian groups, the decomposition in Theorem 9.8.5 is not unique. We have the following examples:*

Example 9.8.7. *i) There is an isomorphism*

$$ f : ({}_1\mathbb{Z}) \times ({}_1\mathbb{Z}) = {}_{(1,1)}(\mathbb{Z} \oplus \mathbb{Z}) \to {}_{(1,0)}(\mathbb{Z} \oplus \mathbb{Z}) = ({}_1\mathbb{Z}) \times ({}_0\mathbb{Z}), $$

given by $f(x, y) = (x, 2x + y - 1)$.

ii) For any integer m, the same map induces an isomorphism

$$ ({}_1\mathbb{Z}) \times ({}_1\mathbb{Z}/3^m\mathbb{Z}) = {}_{(1,1)}(\mathbb{Z} \oplus \mathbb{Z}/3^m\mathbb{Z}) \to {}_{(1,0)}(\mathbb{Z} \oplus \mathbb{Z}/3^m\mathbb{Z}) = ({}_1\mathbb{Z}) \times ({}_0\mathbb{Z}/3^m\mathbb{Z}). $$

ii) For any integers $k \geq m$, the same map induces an isomorphism

$$ ({}_1\mathbb{Z}/3^k\mathbb{Z}) \times ({}_1\mathbb{Z}/3^m\mathbb{Z}) = {}_{(1,1)}(\mathbb{Z}/3^k\mathbb{Z} \oplus \mathbb{Z}/3^m\mathbb{Z}) \to {}_{(1,0)}(\mathbb{Z}/3^k\mathbb{Z} \oplus \mathbb{Z}/3^m\mathbb{Z}) = ({}_1\mathbb{Z}/3^k\mathbb{Z}) \times ({}_0\mathbb{Z}/3^m\mathbb{Z}). $$

The proofs can be done directly by hand and will be omited.

Corollary 9.8.8. *Any finitely generated elliptic group is of one of the following forms:*

$$ {}_0A,\ {}_1\mathbb{Z} \times {}_0A \text{ or } {}_1\mathbb{Z}/3^k\mathbb{Z} \times {}_0A, $$

where A is a finitely generated abelian group.

Proof. According to Theorem 9.8.5, any such object is a product of the above listed elliptic groups. Moreover, two copies of elliptic groups of the form ${}_1Z$, with $Z = \mathbb{Z}$ or $Z = \mathbb{Z}/3^k\mathbb{Z}$, can be replaced by a single one, according to Example 9.8.7 and we are done. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

As we have seen, any elliptic group is embeddable into an elliptic group with a flex. We aim to prove a more precise statement for finitely generated elliptic groups.

Lemma 9.8.9. *For any finitely generated elliptic group S, there is an injective morphism of elliptic groups $\kappa : S \to T$, where T has a flex point and is also finitely generated.*

Proof. Since S is finitely generated, we can use Theorem 9.8.5 to reduice our assertion to the cases $S = {}_1\mathbb{Z}$ and $S = {}_1\mathbb{Z}/3^k\mathbb{Z}$, $k \geq 1$, as the others have a felx point. In the first one, we have an injective morphism of elliptic groups $\kappa : {}_1\mathbb{Z} \to {}_0\mathbb{Z}$, where $\kappa(n) = 1 - 3n$. In the second case, we take $\kappa : {}_1\mathbb{Z}/3^k\mathbb{Z} \to {}_0\mathbb{Z}/3^{k+1}\mathbb{Z}$, where $\kappa(x) = 1 - 3x$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

9.9 Coproduct

The goal of this section is to study the coproduct in the category of elliptic groups. In particular, our results give very explicit descriptions of the coproduct of finitely generated elliptic groups.

As usual, $\oplus$ denotes the direct sum of abelian groups, which is simultaneously the product and the coproduct in the category of abelian groups. For coproducts in the category of elliptic groups, we will use the symbol $\coprod$, while $\times$ will denote the product of elliptic groups. We further let $[a]_n$ (or simply $[a]$) be the class of an integer a in $\mathbb{Z}/n\mathbb{Z}$.

As a reminder, a morphism $f : {}_aA \to {}_cC$ of elliptic groups is a map for which the map $f_1 : A \to C$, given by $f_1(x) = f(x) - f(0)$, is a group homomorphism and the following compatibility condition holds:

$$3f(0) + f_1(a) = c.$$

The last condition is equivalent to $2f(0) + f(a) = c$.

Proposition 9.9.1. *Let A and B be abelian groups. There is an isomorphism of elliptic groups:*

$${}_0A \coprod {}_0B \cong {}_0(A \oplus B \oplus \mathbb{Z}/3\mathbb{Z}).$$

More precisely, the diagram of elliptic groups

$$i : {}_0A \to _0(A \oplus B \oplus \mathbb{Z}/3\mathbb{Z}) \leftarrow _0B : j,$$

given by $i(x) = (x, 0, 0)$, $j(y) = (0, y, [1])$, is a coproduct diagram of elliptic groups. That is, for any elliptic group E and for any morphisms of elliptic groups $f : {}_0A \to E$, $g : {}_0B \to E$, there exists a unique morphism of elliptic groups

$$h : {}_0(A \oplus B \oplus \mathbb{Z}/3\mathbb{Z}) \to E$$

such that $f = h \circ i$ and $g = h \circ j$.

Proof. Since $A \mapsto {}_0A$ is a functor $\mathbf{Ab} \to \mathbf{Ell}_{\mathbf{Gr}}$, the map i is a morphism of elliptic groups. Since $[1]$ is of order 3, one sees that the compatibility condition holds for j and thus, j is also a morphism. Take $E = {}_cC$ for an abelian group C and an element $c \in C$. For f and g the compatibility conditions

$$3f(0) = c = 3g(0)$$

holds. Define the map $\tilde{h} : A \oplus B \oplus \mathbb{Z} \to C$ by

$$\tilde{h}(x, y, [n]) = f(x) - nf(0) + g(y) + (n-1)g(0).$$

Since

$$\tilde{h}(x,y,[n+3]) - \tilde{h}(x,y,[n]) = -3f(0) + 3g(0) = -c + c = 0,$$

we see that $\tilde{h}$ yields a well-defined map

$$h : A \oplus B \oplus \mathbb{Z}/3\mathbb{Z} \to C.$$

Next,

$$3h(0,0,[0]) = 3(f(0) + g(0) - g(0)) = 3f(0) = c.$$

So, h is a morphism of elliptic groups ${}_0(A \oplus B \oplus \mathbb{Z}/3\mathbb{Z}) \to {}_cC$. We have

$$h \circ i(x) = h(x,0,[0]) = f(x) + g(0) - g(0) = f(x)$$

and

$$h \circ j(y) = h(0,y,[1]) = f(0) - f(0) + g(y) = g(y).$$

We have checked all the conditions except for the uniqueness.

To this end, assume $p : {}_0(A \oplus B \oplus \mathbb{Z}/3\mathbb{Z}) \to {}_cC$ is also a morphism such that $p \circ i = f$ and d$p \circ j = g$. We have

$$p(0,0,0) = pi(0) = f(0).$$

So, h and p have the same constant terms. Hence $q = h - p$ is a group homomorphism. Clearly $q \circ i = 0$. So $q(x,0,0) = 0$ for all x. We also have

$$q \circ j(y) = h \circ j(y) - p \circ j(y) = g(y) - g(y) = 0.$$

It follows that $q(0,y,1) = 0$ for all y. First, by putting $y = 0$, we obtain that $q(0,0,[1]) = 0$ and hence, $q(0,0,[n]) = 0$ for all n. Using the fact that q is a group homomorphism, we obtain

$$q(0,y,0) = q((0,y,1) - (0,0,1)) = q(0,y,[1]) - q(0,0,[1]) = 0.$$

Thus, the restriction of the group homomorphism q on each summand is zero and therefore q itself is zero. This finishes the proof. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 9.9.2. *As a special case, we have* $\mathbf{1} \coprod \mathbf{1} \simeq_0 (\mathbb{Z}/3\mathbb{Z})$.

Proposition 9.9.3. *Let A and B be abelian groups and a an element of A. One has an isomorphism of elliptic groups:*

$$({}_1\mathbb{Z} \times {}_0B) \coprod {}_aA \cong {}_{(0,0,a)}(\mathbb{Z} \oplus B \oplus A).$$

More precisely, the diagram of elliptic groups

$$i : {}_1\mathbb{Z} \times {}_0B \to {}_{(0,0,a)}(\mathbb{Z} \oplus B \oplus A) \leftarrow {}_aA : j,$$

given by $i(n,b) = (1-3n, b, na)$ and $j(x) = (0,0,x)$, is a coproduct diagram of elliptic groups. That is to say, for any elliptic group E and any morphisms $f : {}_1\mathbb{Z} \times {}_0B \to E$ and $g : {}_aA \to E$, there exists a unique morphism $h : {}_{(0,0,a)}(\mathbb{Z} \oplus B \oplus A) \to E$ such that $f = h \circ i$ and $g = h \circ j$.

Proof. We first have to show that i and j define morphisms of elliptic groups. Since they obviously are affine maps, we need to check that they satisfy the compatibility condition. We have $2i(0,0)+i(1,0) = 2(1,0,0)+(-2,0,a) = (0,0,a)$. Thus, i is a morphism. We also have $2j(0)+j(a) = (0,0,a)$ and hence, j is also a morphism.

Without loss of generality, we can assume that $E = {}_cC$. Our next aim is to show that if f and g are morphisms, there exist a unique morphism h with the above described properties.

For uniqueness, observe that any affine map $h : \mathbb{Z} \oplus B \oplus A \to C$ has the form

$$h(n,b,x) = u + nv + \alpha(x) + \beta(b),$$

where (n,b,x) is any element from $\mathbb{Z} \oplus B \oplus A$ and $u, v \in C$ and $\alpha : A \to C$, $\beta : B \to C$ are group homomorphisms. The condition $h \circ j = g$ gives us $g(x) = h(0,0,x) = u + \alpha(x)$ for all $x \in A$. This implies that $u = g(0)$ and $\alpha(x) = g(x) - g(0)$. Thus, u and α are uniquely defined. The condition $f = h \circ i$ gives

$$f(n,b) = h(1-3n, b, na) = u + (1-3n)v + \alpha(na) + \beta(b).$$

Putting $n = 0, b = 0$, we obtain $f(0,0) = u + v$. Thus, $v = f(0,0) - u = f(0,0) - g(0)$ is well-defined. We put $n = 0$ to obtain $f(0,b) = u + v + \beta(b)$. It follows that $\beta(b) = f(0,b) - f(0,0)$ and hence, β is also uniquely defined.

For existence, we put

$$h(n,b,x) = g(0) + n(f(0,0) - g(0)) + g(x) - g(0) + f(0,b) - f(0,0)$$

as the uniqueness proof suggests. By definition, h is affine. We have

$$2h(0,0,0) + h(0,0,a) = 2g(0) + g(0) + g(a) - g(0) = 2g(0) + g(a) = c,$$

since g is a morphism. It follows that h is a morphism

$$h : {}_{(0,0,a)}(\mathbb{Z} \oplus B \oplus A) \to_c C.$$

Let us compare $h \circ j$ and g. We have

$$h \circ j(x) = h(0,0,x) = g(0) + g(0) - g(0) = g(x),$$

as required. We write

$$\begin{aligned} h \circ i(n,b) &= h(1-3n, b, na) \\ &= g(0) + (1-3n)(f(0,0) - g(0)) + g(na) - g(0) + f(0,b) - f(0,0) \\ &= f(0,0) - 3nf(0,0) + 3ng(0) + n\alpha(a) + \beta(b) \\ &= f(0,0) - 3nf(0,0) + nc + \beta(b). \end{aligned}$$

It will be helpful to write $f(n,b) = f(0,0) + nw + \beta(b)$, where $w \in C$ and $\beta : B \to C$ is a group

homomorphism. The compatibility condition for f gives $c = 3f(0,0) + w$. Hence, we have

$$h \circ i(n,b) = f(0,0) + nw + \beta(b) = f(n,b)$$

and the proof is finished. $\mathcal{Q.E.D}$

Corollary 9.9.4. *As a special case, we have* ${}_1\mathbb{Z} \coprod \mathbf{1} \simeq_0 \mathbb{Z}$.

Proposition 9.9.5. *Let A and B be abelian groups. There is an isomorphism of elliptic groups*

$$({}_1\mathbb{Z}/3^k\mathbb{Z} \times {}_0B) \coprod {}_0A \cong {}_0(\mathbb{Z}/3^{k+1}\mathbb{Z} \oplus B \oplus A).$$

More precisely, the diagram

$$i : {}_1\mathbb{Z}/3^k\mathbb{Z} \times {}_0B \to {}_0(\mathbb{Z}/3^{k+1}\mathbb{Z} \oplus B \oplus A) \leftarrow {}_0A : j,$$

given by $i([a]_{3^k}) = ([1-3a]_{3^{k+1}}, 0)$, $j(x) = (0,0,x)$, is a coproduct diagram of elliptic groups. That is, for any elliptic group E and for any morphisms of elliptic groups $f : {}_1\mathbb{Z}/3^k\mathbb{Z} \times {}_0B \to E$, $g : {}_0A \to E$, there exists a unique morphism of elliptic groups

$$h : {}_0(\mathbb{Z}/3^{k+1}\mathbb{Z} \oplus A) \to E$$

such that $f = h \circ i$ and $g = h \circ j$.

The proof of is quite similar to the previous case and we omit it.

Proposition 9.9.6. *Let A and B be abelian groups and $\ell \leq k$ be natural numbers. We have an isomorphism of elliptic groups*

$$({}_1\mathbb{Z}/3^k\mathbb{Z} \times {}_0A) \coprod ({}_1\mathbb{Z}/3^\ell\mathbb{Z} \times {}_0B) \cong {}_{(0,1,0,0)}(\mathbb{Z}/3^{k+1}\mathbb{Z} \oplus \mathbb{Z}/3^\ell\mathbb{Z} \oplus A \oplus B).$$

More precisely, the diagram

$$i : {}_1\mathbb{Z}/3^k\mathbb{Z} \times {}_0A \to {}_{-}(0,1,0,0)(\mathbb{Z}/3^{k+1}\mathbb{Z} \oplus \mathbb{Z}/3^\ell\mathbb{Z} \oplus A \oplus B) \leftarrow {}_1\mathbb{Z}/3^\ell\mathbb{Z} \times {}_0B : j,$$

given by $i([n]_{3^k}, a) = ([1-3n]_{3^{k+1}}, [n]_{3^\ell}, a, 0)$, $j([m]_{3^\ell}, b) = (0, [m]_{3^\ell}, 0, b)$ is a coproduct diagram of elliptic groups. This in term means that for any elliptic group E and for any morphisms of elliptic groups $f : {}_1\mathbb{Z}/3^k\mathbb{Z} \times {}_0A \to E$, $g : {}_1\mathbb{Z}/3^\ell\mathbb{Z} \times {}_0B \to E$, there exists a unique morphism of elliptic groups

$$h : {}_{(0,1,0,0)}(\mathbb{Z}/3^{k+1}\mathbb{Z} \oplus \mathbb{Z}/3^\ell\mathbb{Z} \oplus A \oplus B) \to E$$

such that $f = h \circ i$ and $g = h \circ j$.

Proof. The map j is a group homomorphism and $2j(0,0) + j(1,0) = (0,1,0,0)$. Hence, the compatibility condition holds and j is a morphism of elliptic groups.

We next aim to show that i is well-defined. This requires us to prove that $i([n]_{3^k}, a) = i([n+3^k]_{3^k}, a)$. This holds because

$$([1-3n]_{3^{k+1}}, [n]_{3^\ell}, [a], 0) = [1-3(n+3^k)]_{3^{k+1}}, [n+3^k]_{3^\ell}, a, 0).$$

The condition $k \geq \ell$ is essential here. Since i is well-defined, it is affine and the compatibility condition $2i(0,0) + i(1,0) = 2(1,0,0,0) + (-2,1,0,0) = (0,1,0,0)$ holds. Thus, i is a morphism.

Without loss of generality, we can assume that $E = {}_cC$. Our next aim is to show that if f and g are morphisms, there exist a unique morphism h with the described properties.

For uniqueness, we observe that any affine map $h : \mathbb{Z} \oplus \mathbb{Z} \oplus A \oplus B \to C$ has the form

$$h(n,m,a,b) = u + nv + mw + \alpha(a) + \beta(b),$$

where $u, v, w \in C$ and $\alpha : A \to C$, $\beta : B \to C$ are group homomorphisms. The condition $h \circ j = g$ yields

$$u = h(0,0,0,0) = h \circ j(0,0) = g(0,0).$$

Quite similarly, we obtain

$$u + w = g(1,0), \quad u + \beta(b) = g(0,b).$$

In the same way, the condition $h \circ i = f$ gives us

$$h(1,0,0,0) = u + v, \quad h(1,0,0,0) = u + v + \alpha(a).$$

As u, v, α, β are unique, h too is uniquely defined.

For the existence, we first observe that since f is affine, it can be written as

$$f([n]_{3^k}, a) = p + nq + \alpha(a),$$

where $p, q \in C$ and $\alpha : A \to C$ is a group homomorphism. The compatibility condition says that $3p + q = c$. Hence

$$f([n]_{3^k}, a) = p + n(c - 3p) + \alpha(a).$$

Since the LHS depends only on the residue class of n, the same must also be true for the RHS. Thus

$$3^k c = 3^{k+1} p.$$

Quite similarly, for g we have

$$g([m], b) = r + m(c - 3r) + \beta(b)$$

and

$$3^\ell c = 3^{\ell+1} r.$$

Here, $\beta : B \to C$ is a group homomorphism. Since $k \geq \ell$, we obtain

$$3^{k+1}(r - p) = 0.$$

Define h by

$$h([n]_{3^{k+1}}, [m]_{3^\ell}, a, b) = r + n(p - r) + m(c - 3r) + \alpha(a) + \beta(b).$$

The last equality implies that h is a well-defined affine map

$$\mathbb{Z}/3^{k+1}\mathbb{Z} \oplus \mathbb{Z}/3^{\ell}\mathbb{Z} \oplus A \oplus B \to C.$$

We have

$$2h([0],[0],0,0) + h([0],[1],0,0) = 2r + r + c - 3r = c.$$

This shows that the compatibility condition also holds for h. We have

$$hj([m],b) = h([0],[m],0,b) = r + m(c-3r) + \beta(b) = g([m],b)$$

and

$$\begin{aligned} hi([n],a) &= h([1-3n],[n],a,0) \\ &= r + (1-3n)(p-r) + n(c-3r) + \alpha(a) \\ &= p - 3np + nc + \alpha(a) \\ &= p - 3np + n(3p+q) + \alpha(a) \\ &= p + nq + \alpha(a) = f([n],a). \end{aligned}$$

Thus $hi = f$ as required. $\mathcal{Q.E.D}$

By Corollary 9.8.8, we have considered every case.

9.10 The group of automorphisms of an elliptic group

Let S be an elliptic group. We let $\mathsf{Aut}_{\mathsf{E}}(S)$ be the group of automorphism of S in the category $\mathsf{Ell}_{\mathsf{GR}}$. Denote by $\mathsf{Aut}(A)$ the group of automorphisms of the group A.

Recall that any morphism $f : {}_cA \to {}_cA$ is of the form $f = f_0 + f_1$, where $f_0 : A \to A$ is a constant map with value f_0 and $f_1 : A \to A$ is a group homomorphism such that

$$3f_0 + f_1(c) = c.$$

Proposition 9.10.1. *Let A be an abelian group and $c \in A$. For an elliptic group $S = {}_cA$, one has an exact sequence*

$$0 \to \mathsf{Ann}_3(A) \xrightarrow{\beta} \mathsf{Aut}_{\mathsf{E}}({}_cA) \xrightarrow{\alpha} \mathsf{Aut}(A) \xrightarrow{\eta} A/3A,$$

where $\mathsf{Ann}_3(A) = \{a \in A \mid 3a = 0\}$ and the maps α, β, η are defined by

$$\beta(a) = a + 1_A,\ \alpha(f_0 + f_1) = f_1,\ \eta(g) = c - g(c) + 3A.$$

Here, $g \in \mathsf{Aut}(A)$, $a \in A$, $3a = 0$ and $f = f_0 + f_1$ is an automorphism of ${}_cA$.

Note that η is in genral not a group homomorphisms, only a pointed map.

Proof. An affine map $f = f_0 + f_1$ is an isomorphism if and only if f_1 is an isomorphism. Hence, the map α is well defined. We have already observed that

$$3f_0 + f_1(c) = c.$$

Since the kernel of α corresponds to the case $f_1 = 1_A$, we have $3f_0 = 0$ and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 9.10.2. *i) If $S = {}_0A$, the group $\mathsf{Aut}_{\mathsf{E}}(S)$ is isomorphic to the semidirect product $\mathsf{Ann}_3(A) \rtimes \mathsf{Aut}(A)$, where the group $\mathsf{Aut}(A)$ acts on $\mathsf{Ann}_3(A)$ by $(g, a) \mapsto g(a)$.*

ii) If S is a free elliptic group with n generators, then $\mathsf{Aut}_{\mathsf{E}}(S)$ is isomorphic to the subgroup of $GL_n(\mathbb{Z})$, consisting of invertible integral matrices (a_{ij}) such that $a_{11} \equiv 1 \mod 3$ and $a_{i1} \equiv 0 \mod 3$ for all $2 \leq i \leq n$.

Proof. i) In this case, $\eta = 0$ and $g \mapsto 0 + g$ gives a splitting of α.

ii) In this case $A = \mathbb{Z}^n$. Hence, $\mathsf{Ann}_3(A) = 0$, $\mathsf{Aut}(A) = GL_n(\mathbb{Z})$ and $\eta(a_{ij}) = (1 - a_{11}, a_{21}, \cdots, a_{n1}) + 3A$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Let S be an elliptic group and G be a group. The group homomorphisms $G \to \mathsf{Aut}_{\mathsf{E}}(S)$ can be considered as the representations of G in an elliptic group S. Such objects arise in the following situation: Let $\mathcal{E}$ be an elliptic curve defined over a field K and assume $K \subset L$ is a finite Galois extension with Galois group G. We obtain a natural representation of G in $\mathcal{E}(L)$. We might come back to this situation in the future.

9.11 Elliptic tensor product

Let R, S and T be elliptic groups. A map $f : R \times S \to T$ is called a *bihomomorphism* of elliptic groups if for any $r \in R$ and $s \in S$, the maps $S \to T, x \mapsto f(r, x)$ and $R \to T, y \mapsto f(y, s)$ are homomorphisms. Denote by $\mathsf{Bimor}(R \times S, T)$ the set of such maps. It is an elliptic group by pointwise operation and one has the following isomorphism

$$\Phi : \mathsf{Hom}_{\mathsf{El}}(R, \mathsf{Hom}_{\mathsf{El}}(S, T)) \cong \mathsf{Bimor}(R \times S, T).$$

Just like in the classical case, it is given by

$$\Phi(h)(r, s) := h(r)(s),$$

where $h : R \to \mathsf{Hom}_{\mathsf{El}}(S, T)$ is homomorphism of elliptic groups.

Let R and S be elliptic groups. As for abelian groups, there is an universal bihomomorphism $u : R \times S \to R \circledast S$, which satisfies the following property: for any bihomomorphsm $f : R \times S \to T$, there exist a unique homomorphism $\bar{f} : R \circledast S \to T$ such that $f = \bar{f} \circ u$. In particular, we have

$$\mathsf{Hom}_{\mathsf{El}}(R, \mathsf{Hom}_{\mathsf{El}}(S, T)) \cong \mathsf{Hom}_{\mathsf{El}}(R \circledast S, T)).$$

It follows from the properties of adjoint functors, that the functor $R \circledast - : X \mapsto R \circledast X$, which is the left adjoint of $\mathsf{Hom}_{\mathsf{El}}(R, -)$, preserves all colimits, in particular coproducts.

Lemma 9.11.1. *For any abelian groups A and B, one has*

(i) ${}_1\mathbb{Z} \circledast A \cong A$,

(ii) ${}_0A \circledast {}_0B \cong {}_0(\mathbb{Z}/3\mathbb{Z} \oplus A/3A \oplus B/3B \oplus A \otimes B)$.

Proof. i) Is fairly straighforward.
ii) We have

$$\begin{aligned}\mathsf{Hom}_{\mathsf{El}}({}_0A \circledast {}_0B, C) &= \mathsf{Hom}_{\mathsf{El}}({}_0A, \mathsf{Hom}_{\mathsf{El}}({}_0B, {}_cC)) \\ &= \begin{cases} \varnothing & \text{if} c \notin 3C \\ 0 & \text{if} c \in 3C. \end{cases}\end{aligned}$$

Let us consider the case $c \in 3C$. Then

$$\mathsf{Hom}_{\mathsf{El}}({}_0A, {}_0\,\mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}/3\mathbb{Z} \oplus B, C)) = \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}/3\mathbb{Z} \oplus A, \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}/3\mathbb{Z} \oplus B, C))$$

by Lemma 9.5.3. This furter equals

$$\begin{aligned}&= {}_0\,\mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}/3\mathbb{Z} \oplus A, \mathsf{Hom}_{\mathsf{El}}(\mathbb{Z}/3\mathbb{Z}, C) + \mathsf{Hom}_{\mathsf{El}}(B, C)) \\ &= {}_0(\mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}/3\mathbb{Z}, \mathsf{Hom}_{\mathsf{El}}(\mathbb{Z}/3\mathbb{Z}, C)) + \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}/3\mathbb{Z}, \mathsf{Hom}_{\mathsf{El}}(B, C)) \\ &\quad + \mathsf{Hom}_{\mathsf{Ab}}(A, \mathsf{Hom}_{\mathsf{El}}(\mathbb{Z}/3\mathbb{Z}, C)) + \mathsf{Hom}_{\mathsf{Ab}}(A, \mathsf{Hom}_{\mathsf{El}}(B, C))) \\ &= {}_0(\mathsf{Hom}_{\mathsf{El}}(\mathbb{Z}/3\mathbb{Z}, C) + \mathsf{Hom}_{\mathsf{El}}(B/3B, C) + \mathsf{Hom}_{\mathsf{El}}(A/3A, C) + \mathsf{Hom}_{\mathsf{El}}(A \oplus B, C)) \\ &= {}_0\mathsf{Hom}_{\mathsf{El}}({}_0(\mathbb{Z}/3\mathbb{Z} \oplus A/3A \oplus B/3B \oplus A \otimes B, C)).\end{aligned}$$

Since this is true for any $C \in \mathsf{Ell}_{\mathsf{GR}}$, the desired result follows. $\mathcal{Q.E.D}$

9.12 Elliptic rings

Using the tensor product, we can define monoid objects in the category $(\mathsf{Ell}_{\mathsf{GR}}, \circledast)$ of elliptic groups. We now wish to study them in more detail.

Definition 9.12.1. *An associative elliptic ring is a quadruple $(A, \dotplus, \divideontimes, e)$, where $(A, \dotplus)$ is an elliptic group, $\divideontimes$ is a bilinear operation on A and $e \in A$, such that the following axioms hold*

(i) $(x \dotplus y) \divideontimes z = (x \divideontimes z) \dotplus (y \divideontimes z)$,

(ii) $x \divideontimes (y \dotplus z) = (x \divideontimes y) \dotplus (x \divideontimes z)$,

(iii) $x \divideontimes (y \divideontimes z) = (x \divideontimes y) \divideontimes z$,

(iv) $x \divideontimes e = x = e \divideontimes x$.

If additionally $x \divideontimes y = y \divideontimes x$, then we say that $(A, \divideontimes, e)$ is a commutative elliptic ring.

For example, as we will show shortly, $(\mathbb{Z}, \dotplus, \divideontimes, 0)$ is a commutative elliptic ring, where $x \dotplus y = 1 - x - y$ and $x \divideontimes y = x + y - 3xy$. In fact, it is the initial object in the category of elliptic rings.

9.12.1 Examples of elliptic rings

i) The endomorphisms $(\mathsf{Hom}_{\mathsf{El}}(S,S), \dotplus, \circ, \mathsf{Id})$ of an elliptic group S form an elliptic ring under the composition $\divideontimes = \circ$, while $\dotplus$ is defined by

$$(f \dotplus g)(s) = f(s) \dotplus g(s),$$

for any $f, g \in \mathsf{Hom}_{\mathsf{El}}(S,S)$.

As an example, take $S = {}_1\mathbb{Z}$. Recall that any endomorphism of S is of the form $f_a(x) = a + (1-3a)x$, for a uniquely defined $a \in \mathbb{Z}$ (see Lemma 9.5.4, part 1). One easily checks that

$$f_a \dotplus f_b = f_{1-a-b}, \quad f_a \divideontimes f_b = f_a \circ f_b = f_{a+b-3ab},$$

from which we see that $x \circ y = x + y - 3xy$ defines an elliptic ring structure on ${}_1\mathbb{Z}$, whose elliptic unit is 0.

Quite similarly, any endomorphism of ${}_0\mathbb{Z}$ is of the form $g_a(x) = ax$, for a uniquely defined $a \in \mathbb{Z}$ (see Lemma 9.5.4). It is, once again, readily checked that

$$g_a \dotplus g_b = g_{-a-b}, \quad g_a \divideontimes g_b = g_a \circ g_b = g_{ab}.$$

As such, $x \circ y = xy$ defines an elliptic ring structure on ${}_0\mathbb{Z}$, whose elliptic unit is 1.

More generally, we have the following facts.

Lemma 9.12.2. *i) Let R be a ring with unit. Define*

$$r \dotplus s := 1 - r - s \quad \textit{and} \quad r \divideontimes s := r + s - 3rs.$$

Then $(R, \dotplus, \divideontimes, 0)$ is an elliptic ring, denoted by $\mathsf{Ell}_1(R)$.

ii) Let R be a ring with unit. Define

$$r \dotplus s := -r - s \quad \textit{and} \quad r \divideontimes s := rs.$$

Then $(R, \dotplus, \divideontimes, 1)$ is an elliptic ring, denoted by $\mathsf{Ell}_0(R)$.

The proofs are left as exercises to the reader. We have just shown that $\mathsf{Hom}_{\mathsf{El}}({}_1\mathbb{Z}, {}_1\mathbb{Z})$ and $\mathsf{Hom}_{\mathsf{El}}({}_0\mathbb{Z}, {}_0\mathbb{Z})$ are isomorphic to $\mathsf{Ell}_1(\mathbb{Z})$ and $\mathsf{Ell}_0(\mathbb{Z})$, respectively.

To compute $\mathsf{Hom}_{\mathsf{El}}({}_{e_1}\mathbb{Z}^n, {}_{e_1}\mathbb{Z}^n)$, we first fix some notations. As usual, $A = (a_{ij})$ denotes a matrix and $v = (v_1, \cdots, v_n)^t$ denotes a column vector (here t stands for the transpose). Set $\mathsf{Aff}_n(\mathbb{Z})$ to be $\mathbb{Z}^n \times \mathsf{Mat}_n(\mathbb{Z})$ as a set, and define the multiplication by

$$(v, A) \divideontimes (w, B) = (v + Av, AB).$$

It also has an elliptic group structure, where $\dotplus$ is defined as in the product of $(n+1)$-copies of $e_1\mathbb{Z}^n$. For example, for $n = 2$ we have

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) \dotplus \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) = \left(\begin{pmatrix} 1-u_1-v_1 \\ -u_2-v_2 \end{pmatrix}, \begin{pmatrix} 1-a_{11}-b_{11} & 1-a_{12}-b_{12} \\ -a_{21}-b_{12} & -a_{22}-b_{22} \end{pmatrix}\right).$$

An *elliptic matrix* is a pair $(u, A) \in \mathrm{Aff}_n(\mathbb{Z})$, for which

$$a_{11} = 1 - 3u_1, \text{ and } a_{k1} = -3u_k, \ 2 \leq k \leq n.$$

This comes from Corollary 9.10.2 part ii), as the endomorphisms behave exactly like the automorphisms.

One can show that the set Ell.Mat$(\mathbb{Z})$ of all elliptic matrices forms an elliptic ring with respect to the above defined $+\!\!+$ and $\ast\!\!\ast$. This elliptic ring is isomorphic to $\mathsf{Hom}_{\mathsf{El}}({}_{e_1}\mathbb{Z}^n\,{}_{e_1}\mathbb{Z}^n)$. This straightforward exercise is left to the reader.

9.12.2 Arithmaetical properties of the elliptic ring $\mathrm{Ell}_1(\mathbb{Z})$

Recall that $\mathrm{Ell}_1(\mathbb{Z})$ is an elliptic ring whose elements are integers and operations are defined by

$$a +\!\!+ b = 1 - a - b, \quad a \ast\!\!\ast b = a + b - 3ab.$$

The multiplicative unit element is $e = 0$. The elliptic ring $\mathrm{Ell}_1(\mathbb{Z})$ is in fact the initial object of the category of elliptic rings. One can compare it to the initial object in the category of rings.

In this section, we aim to show that the fundamental theorem of arithmetic holds in $\mathrm{Ell}_1(\mathbb{Z})$ as well. For this, we investigate some arithmetical properties of $\mathrm{Ell}_1(\mathbb{Z})$, but first some terminology.

If a and b be are non-zero integers, we say that b is $\ast\!\!\ast$-divisible by a if $b = a \ast\!\!\ast c$ for an integer c. A non-zero integer a is called $\ast\!\!\ast$-*prime*, if it has no non-trivial decomposition. That is, if $a = b \ast\!\!\ast c$ implies $b = 0$ or $c = 0$.

Here are a few positive $\ast\!\!\ast$-primes: $1, 2, 4, 6, 8, 10, 14, \cdots$, while $-2, -4, -6, -10, \cdots$ are negative $\ast\!\!\ast$-primes. One can note that $-1, 12$ and -8 are not in the list.

The multiplicative monoid of $\mathrm{Ell}_1(\mathbb{Z})$ is obviously cancellative. As we will see, it is, in fact, a free commutative monoid. More precisely, the following is true:

Theorem 9.12.3. *i) An integer a is $\ast\!\!\ast$-prime if and only if $p = |3a - 1|$ is a prime number in the classical sense.*

ii) Any nonzero integer can be decomposed as $a = q_1 \ast\!\!\ast \cdots \ast\!\!\ast q_k$, where each q_i is $\ast\!\!\ast$-prime and such a decomposition is unique (up to reordering).

The proof, given below, is based on the properties of the map $\Sigma : \mathbb{Z} \to \mathbb{Z}$, defined by $\Sigma(a) = 1 - 3a$. One easily checks that it is injective and

$$\Sigma(a \ast\!\!\ast b) = \Sigma(a)\Sigma(b).$$

Moreover, the image of this map is exactly the coset $1 + 3\mathbb{Z}$. In other words, it is the set of integers which are congruent to 1 modulo 3. Further, it should be noted that we have no non-trivial units, meaning decompositions can only differ up to reordering, not units.

Lemma 9.12.4. *If $\Sigma(a) = mn$, then either $m = \Sigma(b)$, $n = \Sigma(c)$ or $m = -\Sigma(b)$, $n = -\Sigma(c)$ for some $b, c \in \mathbb{Z}$.*

Proof. Since $mn = \Sigma(a) \equiv 1 \mod 3$, it follows that either $m \equiv n \equiv 1 \mod 3$ or $m \equiv n \equiv 2 \mod 3$. In the first case, m and n are in the image of Σ. In the second one, $-m$ and $-n$ are in the image of Σ. $\mathcal{Q.E.D}$

Proof of Theorem 9.12.3. i) Assume $p = |\Sigma(a)|$ is prime and $a = b \divideontimes c$. Then $p = |\Sigma(b)| \cdot |\Sigma(c)|$. Thus, $\Sigma(b) = \pm 1$ or $\Sigma(c) = \pm 1$. So, $b = 0$ or $c = 0$ as $\Sigma(b), \Sigma(c)$ can not take -1 as a value, since both are $1 \mod 3$, and it follows that a is $\divideontimes$-prime.

Conversely, assume a is $\divideontimes$-prime. If $\Sigma(a) = mn$, then either $m = \Sigma(b), n = \Sigma(c)$ or $m = -\Sigma(b), n = -\Sigma(c)$, thanks to Lemma 9.12.4. Hence, $\Sigma(a) = \Sigma(b \divideontimes c)$ in both cases. Since Σ is injective, we obtain $a = b \divideontimes c$, which yields $b = 0$ or $c = 0$. Thus, $m = \pm 1$ or $n = \pm 1$. Hence p is prime.

ii) For the existence, take any non-zero number a and consider the prime decomposition of $\Sigma(a)$

$$\Sigma(a) = up_1 \cdots p_k q_1 \cdots q_n,$$

where $u = \pm 1$, $p_1, \cdots, p_k$ are usual primes which are congruent to 1 modulo 3, while $q_1, \cdots, q_n$ are usual primes which are congruent to 2 modulo 3. Such a decomposition exists and is unique since 3 does not divide $\Sigma(a)$. By Lemma 9.12.4, we have

$$p_i = \Sigma(b_i), q_j = -\Sigma(c_j),$$

since by the definition of Σ, for any $x \in \mathbb{Z}$, $\Sigma(x) \equiv 1 \mod 3$. As $p_i \equiv 1 \mod 3$ and $q_j \equiv -1 \mod 3$, the above is clear. For the same reason, n has to be even, as otherwise we would have $\Sigma(a) \equiv -1 \mod 3$. It follows

$$\Sigma(a) = \Sigma(b_1 \divideontimes \cdots \divideontimes b_k \divideontimes c_1 \divideontimes \cdots \divideontimes c_n),$$

which, together with the injectivity of Σ, gives us $a = b_1 \divideontimes \cdots \divideontimes b_k \divideontimes c_1 \divideontimes \cdots \divideontimes c_n$. This is a $\divideontimes$-prime decomposition, thanks to part i).

For the uniqueness, observe that if $t_1 \divideontimes \cdots \divideontimes t_k = s_1 \divideontimes \cdots \divideontimes s_n$ are two decompositions by $\divideontimes$-primes integers, we obtain

$$\Sigma(t_1) \cdots \Sigma(t_k) = \Sigma(s_1) \cdots \Sigma(s_n).$$

This implies $|\Sigma(t_1)| \cdots |\Sigma(t_k)| = |\Sigma(s_1)| \cdots |\Sigma(s_n)|$. By part i), each term is prime. It follows that $k = n$ and after reindexing $|\Sigma(t_i)| = |\Sigma(s_i)|$. This implies $\Sigma(t_i) = \pm \Sigma(s_i)$. By considering this equality modulo 3, we see that the sign must be $+1$ and the uniqueness follows from the injectivity of Σ. $\mathcal{Q.E.D}$

As a corollary of the theorem, we see that the only odd number which is $\divideontimes$-prime is 1 and an even number $2k$ is $\divideontimes$-prime if and only if $|6k - 1|$ is prime. Based on this, we can give an "elementary proof" of the fact that there are infinitely many primes of the form $6t - 1$ (a particular case of Dirchlet's famous theorem). By our theorem, this is equivalent to the fact that there are infinitely many $\divideontimes$-primes. This fact can be proven in exactly the same way as Euclid did for usual primes, we simply replace the expression $p_1 \cdots p_k + 1$ in that argument by $p_1 \divideontimes \cdots \divideontimes p_k \dotplus 0$ and use Lemma 9.12.5 and the fact that the equation $x \divideontimes y = 0$ has only the

trivial solution $x = y = 0$.

An other oddity we may observe is that $\mathbb{Z}$ becomes a free (commutative) monoid with countable generators under the operation $\divideontimes$. Of course, with the normal multiplication $\times$, $\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{N}^{(m)}$, where $\mathbb{N}^{(m)}$ is the natural numbers under multiplication, which is a free as a monoid.

Lemma 9.12.5. *Assume a is a non-zero integer. If b and $b + c$ are $\divideontimes$-divisible by a, then c is $\divideontimes$-divisible by a.*

Proof. If $b = a \divideontimes x$ and $b + c = a \divideontimes y$, we obtain

$$c = b + (b + c) = (a \divideontimes x) + (a \divideontimes y) = a \divideontimes (x + y),$$

hence the result. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

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