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Geometry of Monoids

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1. Introduction

After Alexander Grothendieck, schemes have been fundamental in algebraic geometry. Schemes are obtained by gluing together affine schemes, which themselves are obtained by introducing a topology (the Zariski Topology) on the set of prime ideals $Spec(R)$, of a commutative ring R. In 1994, Kato [16] had the idea to do the same for monoids, and introduced the monoid schemes, thereby generalizing the notion of toric varieties. To be more specific, for any commutative monoid M, he constructed the affine monoidal scheme by considering the set of all prime ideals $Kspec(M)$, with the same canonical topology as for rings, and the structure sheaf given by localisation. Then, mimicking Grothendieck's definition, he glued affine monoid schemes to get the global objects known as monoid schemes. 10 years later, Anton Deitmar saw it as a model for algebraic geometry over $\mathbb{F}_1$ which had a strong response by many mathematicians. Several authors, including Connes and Consani [4], [5], López Peña and Lorscheid [17], Cortiñas, Haesemeyer, Walker and Weibel [6] and others extended on his work and added much of their own. The monoid schemes play an important role, not only in the theory of toric varieties and $\mathbb{F}_1$-geometry, but also in p-adic Hodge theory [12] and logarithmic geometry [19].

The basic idea here was that monoids are like rings, when forgetting the notion of addition and everything (prime ideals, submonoids, quotient monoids etc.) was defined with this philosophy in mind. However, nice as the prime ideals are, they are not flawless, as they are not general enough. In rings, ideals are in a one to one relation with quotient-rings, where as in monoids, this relation fails. To solve this problem, the notion of prime ideals was to be replaced with the notion of prime congruences (which already appears in [13]). While the topology on the set of prime congruences was still given in a nice and intuitive way (with some minor problems, such as that the main opens sets are not closed with respect to intersection, but this problem can be solved without too much complications), the sheaf turned out to be the main problem. In rings, if one had $f = g$, then one could consider $f - g = 0$, and hence localise with $f - g$. But in monoids, as the minus operation is no longer defined, localising become a big problem. To solve this particular dilemma, different authors suggested different solutions like blueprints and sesquiads, splitting the field into many different directions. Based on the results of my project, we introduce yet a new approach, which differs from the others.

My idea is that we might have problems associating "functions" to open sets, but we are able to associate the "dual of the space of functions" to every open set, in this way we don't get a sheaf but a co-sheaf. The reason this construction makes sense is that, as will be shown in this project, the Kato-spectrum $Kspec$, which of course has a contravariant character (i.e. given $f : M \to N$, it induces $f_* : Kspec(N) \to Kspec(M)$), is equivalent to M^{sl}, the semilattice reduction described in Def. 4.8, which has a covariant character.

The contravariant character was of course known to everyone, but the covariant one seems to be new (see Corollary 4.18). It is a direct consequence of one of the

main theorems (see Section 4.3) that I prove, which also simplifies the $Kspec$ as a topological space, as well as its calculation.

More to the point, for a monoid M, one considers the monoid M^{sl} which is the quotient monoid of M by the congruence $\sim$ generated by $m^2 \sim m$, $\forall m \in M$. The assignment $M \mapsto M^{sl}$ obviously gives rise to a covariant functor as for any homomorphism $f : M \to N$ one has the induced homomorphism $f_*^{sl} : M^{sl} \mapsto N^{sl}$, or shortly f_*, such that the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M^{sl} & \xrightarrow[f_*]{} & N^{sl} \end{array}$$

commutes Since the left vertical map is surjective, f_* is defined uniquely. In the monoid M^{sl} the identity $x^2 = x$ holds for every $x \in M^{sl}$, because of this it can be equally well considered as a semilattice, where $x \leq y$ if nd only if $xy = y$ (see Section 4.2). We will prove that if M is finitely generated, then f_* considered as a map of posets has a right adjoint $f^! : N^{sl} \to M^{sl}$.

Our interest of M^{sl} comes from the fact that for any monoid M, the canonical map $M \to M^{sl}$ induces the homeomorphism (see Theorem 4.11)

$$Kspec(M^{sl}) \to Kspec(M)$$

and moreover there is an injective map

$$\alpha_M : M^{sl} \to Kspec(M),$$

such that for any homomorphism of monoids $f : M \to N$ with finitely generated M the diagram

$$\begin{array}{ccc} N^{sl} & \xrightarrow{f^!} & M^{sl} \\ {\scriptstyle \alpha_N}\downarrow & & \downarrow{\scriptstyle \alpha_M} \\ Kspec(N) & \xrightarrow{f^*} & Kspec(M) \end{array}$$

commutes. This is because, for any finitely generated monoid M the map α_M is a bijection (see Theorem 4.16 and Corollary 4.18). And since in essence, elements in M^{sl} turn out to be nothing more than complements of prime ideals, the only difference between M^{sl} and $Spec(M)$ is that the ordering is reversed. Hence what we will effectively show is that defining a scheme on open sets of $Spec(M)$ is the same as defining it on closed sets. The only difference will be that one will be contravariant and the other covvariant. But everything, including gluing and morphisms will be done basically equally. Using this observation, I will generalise M^{sl} instead of $Spec(M)$, i.e. define a congruence cosheaf, that restricted will agree with the sheaf on M^{sl}.

Let $CSpec(M)$ be the set of all prime congruences of M. For any elements a and b, we let $\mathfrak{D}(a,b)$ be the following subset of $CSpec(M)$

$$\mathfrak{D}(a,b) = \{K \in CSpec(M) |\ a \not\sim_K b\ \}.$$

We equip the set $CSpec(M)$ with the smallest topology where the subsets $\mathfrak{D}(a,b)$ are open sets.

To have an idea about what sort of objects we are talking about, consider the simple example $M = \mathbb{N}$, the additive monoid of natural numbers. In this case

$$Kspec(\mathbb{N}) = \{\emptyset, (1)\}$$

has two prime ideals $\emptyset$ and $(1) = \{1, 2, \cdots\}$ see Example 2.18 below. The point (1) is a closed point while the point $\emptyset$ is the generic point of $Kspec(\mathbb{N})$.

In contrast, $CSpec(\mathbb{N})$ is a much more interesting space. For each $n \neq 1$ there is a distinct prime congruence $K(n)$, generated by $n \sim_{K(n)} 0$ and the prime congruence Z, where $2 \sim_Z 1$, corresponding to the congruence, where $1 \sim \infty$. Hence

$$CSpec(\mathbb{N}) = \{Z, K(0), K(2), K(3), \cdots\}$$

The point $K(0)$ is the generic point of $Kspec(\mathbb{N})$, while closed points are exactly the points

$$K(2), K(3), K(5), K(7), \cdots, K(p), \cdots, Z$$

corresponding to prime numbers and $'infinity'$. What is even more interesting is that the closure of the point $K(n)$ is the set

$$\overline{K(n)} = \{K(m) |\ m \text{ divides } n \text{ and } m \neq 1\}$$

As we have seen, even after forgetting 'multiplication' in $\mathbb{N}$ we still get $Spec(\mathbb{Z})$ the classical spectrum of the ring of integers as a subspace (in fact, as the set of maximal congruences, where we say $K \subset L$ if $a \sim_K b \Rightarrow a \sim_L b\ \forall a, b \in M$) of $CSpec(\mathbb{N})$. This example demonstrates the complexity and importance of $CSpec(M)$.

For cancellative monoids, (though I believe, after some refinement, this can be extended for general monoids as well,) there is a monotonic map

$$\mu : M^{sl} \to CSpec(M)$$

associating to a given element $[a]$ of M^{sl} (which is a subset of M) the congruence where $x \sim y$ if and only if both $x, y \in [a]$, and then $x \sim 1, y \sim 1$.

But since in M^{sl} open sets (which are in essence closed sets in $Spec(M)$) are in a one to one relation with co-ideals, we also get a map

$$\nu : Open(M^{sl}) \to Open(Spectrum(M))$$

where for every principal open set $\langle [a] \rangle$ we have

$$\nu(\langle [a] \rangle) = \bigcup_{f \in [a]} \mathfrak{D}(f, 1)$$

I also prove that if M is a cancellative monoid, then there is a cosheaf $\mathfrak{X}_M$ on $CSpec(M)$ such that

$$\mathfrak{X}_M(\mathfrak{D}(f,g)) = M[\frac{f}{g}, \frac{g}{f}]$$

This might seen artificial, but as we will see the restriction of $\mathfrak{X}_M$ on M^{sl} has the following remarkable property: For any point $[a] \in M^{sl}$ the monoid $\mathfrak{X}_M(\nu(\langle [a] \rangle)$ coincides with $M_{[a]}$, the monoid associated to $\mathfrak{D}([a])$ by the sheaf of M^{sl} and hence the is canonically isomorphic of the stalk of the structure sheaf $\mathfrak{O}_{Kspec(M)}$ at the point $\alpha_M([a]) \in Kspec(M)$.

2. Basic facts about Monoids

2.1. **Introduction.** This chapter is meant to give the basic definitions and fix notations that we will use later. Here I will explicitly define what a monoid, morphism of monoids, submonoid, ideal, prime ideal, the Grothendieck group (which is just a localised monoid), congruence, kernel (whose definition differs from that of rings and groups, and captures both), product and coproduct is. We will also fix notions like integral, cancellative, cyclic and free. Finally, we will have a few examples to clarify things. The material of this section is of course well-known and good reference for this is [14].

2.2. **Monoids, submonoids and ideals.**

Definition 2.1. *A* **monoid** M *is a set equipped with a binary operation (usually written multiplicatively) such that the following conditions hold*

- *the product is associative*

$$(xy)z = x(yz)$$

- *we have a unit, i.e. an element* $1 \in M$ *such that*

$$x1 = x = 1x$$

Here $x, y, z \in M$*. Moreover,* M *is* **commutative** *if additionally one has*

- $xy = yx$

Hence monoids are groups without necessarily having inverses. Note that for my project, I will restrict myself to the commutative case.

Definition 2.2. *Let* M, N *be monoids. Then by a* **monoid homomorphism** *we mean a map* $\varphi : M \longrightarrow N$ *such that*

- $\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2), \ \ \forall m_1, m_2 \in M$
- $\varphi(1_M) = 1_N$

Remark 2.3. *We define the empty product to be the identity.*

Definition 2.4. *A* **submonoid** *of* M *is a subset* N *of* M *which is closed under the inherited operation. Thus* N *is a submonoid if*

- $1 \in N$

- *If $x, y \in N$, then $xy \in N$.*

Lemma 2.5. *Let M, N be monoids and let $\varphi : M \longrightarrow N$ be a monoid homomorphism. Then the image* $\mathbf{Im}(\varphi)$ *is a submonoid of N.*

Proof. Since $\varphi(1) = 1$ we see that $1 \in Im(\varphi)$. Similarly, $\varphi(a)\varphi(b) = \varphi(ab) \in Im(\varphi)$ and $Im(\varphi)$ is a submonoid of N.

□

As the intersection of submonoids is a submonoid, we define the **submonoid generated by** a subset X, denoted by $\langle X \rangle$ to be the intersection of all submonoids containing it, and consists of all monomials of the generators and their powers. If X is just one element x, then $\langle x \rangle$ is just the set of all positive powers of x and the unit. A monoid M is **finitely generated** if there is a finite subset $X \subset M$ such that $\langle X \rangle = M$.

Definition 2.6. *Given $X = \{x_1, \cdots, x_k\}$, define $\langle X \rangle$ to be the set*

$$\{x_1 \cdots x_k | n_i \in \mathbb{N}\},$$

with multiplication defined as

$$(x_1^{m_1} \cdots x_l^{m_k}) \times (x_1^{n_1} \cdots x_l^{n_k}) = x_1^{m_1+n_1} \cdots x_l^{m_k+n_k}.$$

Then we say that $\langle X \rangle$ is a **free commutative monoid** *or simply $\langle X \rangle$ is* **free**.

A free monoid with one generator is isomorphic to $\mathbb{N}^+$. A free commutative monoid with two generators is isomorphic to $(\mathbb{N}^+)^2$ etc.

Definition 2.7. *An* **ideal** *I is a subset of M such that $IM \subset I$.*

Note that an ideal can be empty. This is in essence the 'zero-ideal' in ring theory. It is clear that the union and intersection of ideals is again an ideal. Since the intersection of ideals is again an ideal, we can consider the **ideals generated** by subsets of M, being the intersection of all ideals containing them. If the generating set is one element a, we call it a **principal ideal**, and write it as (a), if its finite we write it as $(a_1, ..., a_n)$. Since the union of ideals is again an ideal,

$$(a_1, ..., a_n) = (a_1) \cup ... \cup (a_n).$$

Example 2.8. *By a* **cyclic monoid** *$C_{n,k|0 \leq k \leq n-1}$, I mean a monoid with one generator x such that $x^n = x^k$. Clearly they are all non-isomorphic as $C_{n,k}$ has $k+1$ ideals including the empty ideal, where*

$$(x^i) = \begin{cases} C_{n,k} \setminus \{1, ..., x^{i-1}\} & i \leq k \\ (x^k) & i \geq k. \end{cases}$$

In this case we have n non-isomorphic monoids as k can be any of $\{0, .., n-1\}$.

Note that both n, k can be infinite, then we just get a free moniod.

Definition 2.9. *The* **product** *IJ of two ideals I and J is the set $\{x_i y_j\}_{|\forall x_i \in I, y_j \in J}$ which is again an ideal and clearly contained in the intersection.*

If I and J are finitely generated by $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$ respectively, then $IJ = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (e_i f_j)$ the union of principal ideals obtained by multiplying all the generators. In particular, if both are finitely generated, then their product is again finitely generated.

2.3. **Monoids and groups.** The idea behind the geometry of monoids is that monoids are to groups, what rings are to fields. This comparison is of course not perfect, but fundamental. Hence studying the relation between groups and monoids is important. One side is trivial, every group is a monoid. The other side is a bit more tricky. Of course monoids are in general not groups, but there are two important constructions of groups from monoids.

There first one is to consider $U(M)$, the set of invertible elements in M. This clearly forms a group and is in some sense the 'biggest' group contained in M. This has some interesting subgroups such as the set of elements of finite orders, i.e. elements satisfying $x^n = 1$ for some finite n. Some subgroups of this are obtained by considering the set of all elements satisfying $x^n = 1$ for some fixed n.

Subgroups of monoids are important because they define quotient monoids just like in group theory. In more detail, if $G \subset M$ is a subgroup of M, then it induces cosets, being to be the subsets of the form $mG,\ m \in M$. As for abelian groups cosets define a partition of M, corresponding to an equivalence relation where $a \sim b$ if $a = gb$ for some $g \in G$. The set of cosets form a monoid as in the case of abelian groups.

The second construction is similar to that of the field of fractions from rings. This is also known as the **Grothendieck** group of a commutative monoid. Given any monoid M we can consider any submonoid S and consider $S^{-1}M$, the **localisation** at S, where elements of $S^{-1}M$ are given as equivalence classes of pairs (a, b), $a \in S, b \in M$. The pair (a, b) is said to be equivalent to (c, d), $c \in S, d \in M$ if there exists an element $x \in S$ such that $adx = bcx$. If $S = M$ we get a group, which is denoted by M^{gr}. But clearly there is a little complication, if $0 \in S$, then $S^{-1}M$ becomes just one element, namely 0. While this is all right in some cases (later in scheme theory, for example) this is not quite what we want. So if we make a further restriction that S is cancellative and has no zero, the homomorphism $\varphi : M \longrightarrow S^{-1}M$ will be injective.

2.4. **Prime ideals.**

Definition 2.10. *A* **prime ideal** *is a* **proper ideal** I *(i.e.* $I \neq M$*) such that whenever* $ab \in I$*, and* $a \notin I$ *then* $b \in I$*.*

There are two other equivalent ways to define a prime ideal, but first we need a few more definitions.

Definition 2.11. *Let* M *be a monoid and* I *an ideal, then* M/I *is the* **Rees factorisation** [14] *of* M *with* I*, where the ideal is contracted to one point, namely zero (if the monoid has no zero, we add one). In other words,* a *and* b *will become*

equivalent in the quotient monoid, if and only if they are both in I. Then they will both become equivalent to zero.

Integral domains from commutative algebra have two related, but different analogues in monoid theory, namely integral and cancellative monoids.

Definition 2.12. *We say a monoid with zero 0 is* **integral** *if $ab = 0, \Rightarrow a = 0$ or $b = 0$.*

Definition 2.13. *We say a monoid is* **cancellative** *if whenever*

$$ab = ac \text{ and } a \neq 0 \Rightarrow b = c.$$

We can now give equivalent definitions of prime ideals.

Proposition 2.14. *A* **proper ideal** *I is prime if and only if M/I is integral.*

Proof. Let I be a prime ideal. Assume $x, y \in M$ are elements such that the class xy in M/I is zero. Then $xy \in I$. Hence $x \in I$ or $y \in I$ and M/I is integral. Conversely, assume M/I is integral. If $xy \in I$, then the class of xy in M/I is zero. Thus the class of x is zero, or the class of y is zero, so $x \in I$ or $y \in I$ and I is prime. □

Note that we can not get integral with prime ideals (for that we will need the later defined prime congruence). Consider $M = \langle x_1x_2x_3x_4 \rangle / (x_2x_3 = x_3x_4)$. Then (x_1) is a prime ideal, but $M/(x_1)$ is not cancellative, (x_1) does not affect x_2, x_3 or x_4.

Definition 2.15. *A subset $S \subset M$ is called a* **face** *if it is a submonoid and whenever $ab \in S$ then $a \in S$ and $b \in S$.*

Proposition 2.16. *A subset $I \subset M$ is prime if and only if $S = M \setminus I$ is a face.*

Proof. First assume I is a prime ideal. Since I is a proper ideal we have $1 \notin I$. Hence $1 \in S$. Take $a, b \in S$, then clearly $ab \notin I$ since I is prime. Hence S is a submonoid of M. Assume $ab \in S$, since I is an ideal we have $a \in S$ and $b \in S$. Thus S is a face. Conversely, assume S is a face. First note that I is proper, since the unit is in every submonoid. Now take $ab \in I$. Clearly either a or $b \in I$, since otherwise both a and b would be in S and since by assumption it is a submonoid, it follows that $ab \in S$, which contradicts to the fact that $ab \in I$. It remains to show that I is an ideal. Take $a \in I$ and $b \in M$. Since S is face we have $ab \notin S$ and hence $ab \in I$. □

Example 2.17. *If G is a group then the empty set is the only prime ideal of G. In fact, assume I is a nonempty ideal of G and $g \in I$. Then $1 \in gg^{-1} \in I$ and $I = M$. Hence I is not proper.*

Example 2.18. *We claim that the monoid of natural numbers $\mathbb{N}$ has only two prime ideals $\emptyset$ and $(1) = \{1, 2, 3, \cdots\}$. The fact that these are primes is trivial to check. Assume I is a nonempty prime ideal. Since I is proper $0 \notin I$. Assume n*

is the minimal element of I. I have to prove that $n = 1$, if not then $n = m + 1$ for $m > 0$. Since $n \in I$ and $1 \notin I$ it follows that $m \in I$, this contradicts to the minimality of m.

2.5. **Basic algebra.**

Definition 2.19. *The* **Kernel** *of φ is defined to be an equivalence relation, where $a \sim b$ if $\varphi(a) = \varphi(b)$, we write it as $Ker(\varphi)$.*

Definition 2.20. *We say that an equivalence relation is a* **congruence relation** *if whenever $a \sim b$ and $c \sim d \Rightarrow ac \sim bd$.*

Notation: I will usually write $\overline{a}$ or $[a]$ for the congruence class of an element a and if there is ambiguity, I will write $a \sim_\varphi b$ to specify which congruence I mean.

Proposition 2.21. *The kernel is a congruence relation on M.*

Proof. Say $a \sim b$ and $c \sim d$. Then $\varphi(a) = \varphi(b)$ and $\varphi(c) = \varphi(d) \Rightarrow \varphi(ac) = \varphi(bd)$. □

Definition 2.22. *Let K be a congruence relation on M. Then the* **quotient monoid** *M/K is the set of equivalence classes with the following multiplication*

$$\overline{a}\overline{b} = \overline{ab}$$

where $\overline{a}$ is the equivalence class containing $a \in M$.

Note that M/I, where I is an ideal, is a special case of this if we add a zero to the monoid M, as the Rees factorisation induces a congruence relation where $a \sim 0$ if $a \in I$.

Proposition 2.23. *M/K is a monoid and we have a homomorphism $\varphi : M \longrightarrow M/K$ with $m \mapsto \overline{m}$, where $\overline{m}$ is the congruence class of m.*

Proof: Since K is a congruence, it is clear that if

$$\overline{m_1}, \overline{m_2} \in M/I, \Rightarrow \overline{m_1 m_2} \in M/K.$$

Furthermore clearly $\overline{1}$ is the unit of M/K.

Now we are ready to staet and prove the (first) isomorphism Theorem.

Theorem 2.24. First Isomorphism Theorem *: Let $\varphi : M \longrightarrow N$ be a homomorphism. Then there exists an isomorphism*

$$\phi : M/Ker(\varphi) \longrightarrow Im(\varphi).$$

Proof. Define the map $\phi(\overline{m}) = \varphi(m)$. Since $\overline{m} = \{x | \varphi(m) = \varphi(x)\}$ this map is clearly well defined. Also since by definition congruence relations respect operations, $\overline{mn}$ is again a congruence class so ϕ is a homomorphism as

$$\phi(\overline{mn}) = \varphi(\overline{mn}) = \varphi(\overline{m})\varphi(\overline{n}) = \phi(\overline{m})\phi(\overline{n}).$$

It is clearly surjective and if $\phi(\overline{m}) = \phi(\overline{n})$ then $\varphi(\overline{m}) = \varphi(\overline{n})$, but by definition of the kernel, $\overline{m} = \overline{n}$, so its injective, hence bijective. □

Proposition 2.25. *Let $\varphi : M \longrightarrow N$ and $\phi : N \longrightarrow O$ be homomorphisms. Then the composition $\phi \circ \varphi$ is a homomorphism.*

Proof.

$$\begin{aligned}(\phi \circ \varphi)(1) &= \phi(\varphi(1)) = \phi(1) = 1.\\ (\phi \circ \varphi)(m_1 m_2) &= \phi(\varphi(m_1 m_2)) = \phi(\varphi(m_1)\varphi(m_2))\\ &= \phi(\varphi(m_1))\phi(\varphi(m_2)) = (\phi \circ \varphi)(m_1)(\phi \circ \varphi)(m_2)\end{aligned}$$

□

Proposition 2.26. *If $\varphi, \psi : M \to N$ are homomorphisms, then the map*

$$\varphi\psi : M \to N \text{ defined by } (\varphi\psi)(m) = \varphi(m)\psi(m)$$

is also a homomorphism.

Proof.

$$\begin{aligned}(\phi\varphi)(1) &= \phi(1)\varphi(1) = 1\\ (\phi\varphi)(m_1 m_2) &= \phi(m_1 m_2)\varphi(m_1 m_2) = \phi(m_1)\phi(m_2)\varphi(m_1)\varphi(m_2)\\ &= \phi(m_1)\varphi(m_1)\phi(m_2)\varphi(m_2) = (\phi\varphi)(m_1 m_2)(\phi\varphi)(m_1 m_2)\end{aligned}$$

□

Theorem 2.27. *The set of all homomorphism from M to M, denoted* $\mathbf{End}(\mathbf{M})$ *is a (non-commutative) semi-ring.*

Proof. Let the operation of the monoid M be addition, then the addition of $End(M)$ will be $(f + g)(m) = f(m) + g(m)$ and the multiplication of $End(M)$ will be composition.

addition:

$$\begin{aligned}Associativity : ((f_1 + f_2) + f_3)(x) &= (f_1(x) + f_2(x)) + f_3(x) = f_1(x) + (f_2(x) + f_3(x))\\ &= (f_1 + (f_2 + f_3))(x)\\ Commutativity : \quad (f_1 + f_2)(x) &= f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_1 + f_2)(x)\end{aligned}$$

The zero is the trivial (zero) mapping

multiplication:

$$\begin{aligned}Associativity : ((f_1 \circ f_2) \circ f_3)(x) &= (f_1(f_2) \circ f_3)(x) = f_1(f_2(f_3(x)))\\ &= f_1 \circ f_2(f_3)(x) = (f_1 \circ (f_2 \circ f_3))(x)\end{aligned}$$

The unit is the identity mapping

distributivity:

$$\begin{aligned}(f_1 \circ (f_2 + f_3))(x) &= (f_1(f_2(x) + f_3(x)) = f_1(f_2(x)) + f_1(f_3(x)) \\ &= (f_1 \circ f_2)(x) + (f_1 \circ f_3)(x) \\ ((f_1 + f_2) \circ f_3)(x) &= (f_1 + f_2)(f_3(x)) = f_1(f_3(x)) + f_2(f_3(x)) \\ &= (f_1 \circ f_3)(x) + (f_2 \circ f_3)(x)\end{aligned}$$

The unit is the identity mapping

□

We have now shown that $R = End(M)$ is a semi-ring.

2.5.1. *Products and Coproducts.*

Definition 2.28. *If $\{M_i\}_{i\in I}$ is a family of monoids, define their* **product** $\prod_{i\in I} M_i$ *to be the cartesian product of sets $\{M_i\}_{i\in I}$, equipped with the following multiplication $(x_i)(y_i) = (x_i y_i)$, where $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are elements of $\prod_{i\in I} M_i$. For any $i \in I$ the* projection *mapping*

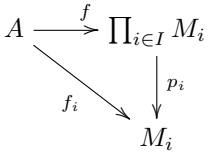
$$p_i : \prod_{i\in I} M_i \longrightarrow M_i$$

which sends $(x_i)_{i\in I}$ to x_i is a homomorphism.

Proposition 2.29. *Assume A is a monoid and one has homomorphisms*

$$f_i : A \longrightarrow M_i, i \in I.$$

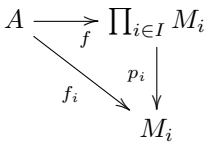
Then there is a unique homomorphism $f : A \to \prod_{i\in I} M_i$ for which all of the following diagrams commute:



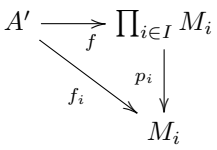
Proof. We define the homomorphism

$$f : A \longrightarrow \prod_{i\in I} M_i$$

by $f(a) = (f_i(a))$. It is straight forward to check that f is a homomorphism. By the construction the following diagram



commutes. In fact f is the only homomorphism with this property, since given another homomorphism $f' : A \longrightarrow \prod_{i\in I} M_i$ with the property that the diagram



commutes for all $i \in I$, it follows that the i'th coordinate of $f'(a)$ is $f_i(a)$. So

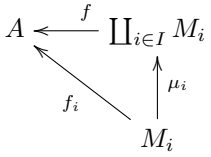
$$f'(a) = (f_i(a))_{i\in I} = f(a).$$

□

Definition 2.30. *If $\{M_i\}_{i\in I}$ is a family of commutative monoids, define their* **coproduct** *$\{\coprod_{i\in I} M_i\}_{i\in I}$ consisting of elements $(x_i)_{i\in I} \in \prod_{i\in I} M_i$ for which $x_i = 1$, for all but finitely many i's. For any $i \in I$ define $\mu_i : M_i \longrightarrow \coprod_{i\in I} M_i$ to be the homomorphism for which*

$$(\mu_i(x))_j = \begin{cases} 1 & \text{for } i \neq j \\ x & \text{for } i = j \end{cases} \quad \forall i \in I$$

Proposition 2.31. *If A is a commutative monoid and $\{f_i : M_i \longrightarrow A\}_{i\in I}$ is a family of homomorphism then there exists a unique homomorphism $f : \coprod_{i\in I} M_i \longrightarrow A$ for which the diagram*



commutes.

Proof.
Existence
For

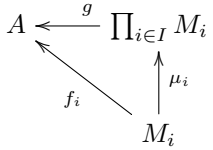
$$\begin{aligned} x &= (x_i)_{i\in I} \in \coprod_{i\in I} M_i, \\ \text{put} \quad f(x) &= \prod_{i\in I} f_i(x_i) \in A. \end{aligned} \tag{1}$$

Since $x_i = 1$ for all but finitely many i's, the product in (1) makes sense. To see that f is a homomorphism, take $y = (y_i) \in M$. Then $xy = (x_i y_i)_{i\in I}$ and we have

$$\begin{aligned} f(xy) = \prod_{i\in I} f_i(x_i y_i) &= \prod_{i\in I} f_i(x_i) f_i(y_i) \\ &= (\prod_{i\in I} f_i(x_i))(\prod_{i\in I} f_i(y_i)) \\ &= f(x)f(y) \end{aligned} \tag{2}$$

Hence f is a homomorphism. Observe that in (2) we used the fact that all M_i are commutative.

Uniqueness
Assume



commutes for all i's. Then for any $x = (x_i) \in \coprod_{i\in I} M_i$ one has $x = \prod_{i\in I} \mu_i(x_i)$. Again, this product makes sense as it is finite. So

$$g(x) = \prod_{i\in I} g\mu_i(x_i) = \prod_{i\in I} f_i(x) = f(x).$$

□

3. Preliminaries on posets, latices and $\mathfrak{P}$-topological spaces

3.1. **Introduction.** The most fundamental definition in algebraic geometry it the spectrum, i.e. the set of all prime ideals. So understanding is well is of course of great importance. In the next chapter, titled *The Kato-spectrum* I will show that the spectrum is in fact a lattice with respect to the union. This fundamental difference between prime ideals of monoids and prime ideals of rings, will simplify many things later on, but for us to be able to use this, will require some basic facts about lattices and partially ordered sets. So the aim of this chapter is to state and prove results that will be useful in the later chapters.

I will introduce the notions of posets, lattices, semilattices, semilattice morphism, adjoints, poset-topologies, ideals (of semilattices) and coideals.

Finally, I will define the notions of contravariant functors over posets (partially ordered sets) and sheaves over the poset-topologies and prove that they are equivalent.

3.2. Posets and adjoint morphisms of posets.

Definition 3.1. *A* **partial order** *is a binary relation $\leq$ over a set P which is reflexive, antisymmetric, and transitive, i.e. for all a, b, and c in P, we have*

- *$a \leq a$ (reflexivity)*
- *$a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry)*
- *$a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity)*

A partially ordered set $(P, \leq)$, or **poset***, is a set P with a partial order $\leq$.*

Definition 3.2. *We say an element a is the* **greatest element** *of P if for all x we have $x \leq a$. Similarly, we say an element a is the* **least element** *if for all x we have $a \leq x$. Note that both are unique if they exist. We sometimes use the notations*

$$\mathbf{Max}\{x \mid x \in P\} \quad \text{and} \quad \mathbf{Min}\{x \mid x \in P\}$$

for the greatest and least element of P respectively.

Definition 3.3. *For a poset P we let P^{op} be the* **opposite poset** *which is P as a set, but with the reverse ordering. In other words $x \leq y$ holds in P^{op} if and only if $y \leq x$ in P.*

The greatest element of P^{op}, if it exists, is the least element of P.

Definition 3.4. *Let X and Y be posets. Assume $f : X \to Y$ and $g : Y \to X$ are maps. We say that g is a* **left adjoint** *of f and f is a* **right adjoint** *of g, if*

$$(y \leq f(x)) \Leftrightarrow (g(y) \leq x)), \quad x \in X, y \in Y.$$

Definition 3.5. *We say the map f is* **monotonic** *if $x \leq y \Rightarrow f(x) \leq f(y)$.*

Proposition 3.6. *Let X and Y be posets.*

(1) A map $g : Y \to X$ has a right adjoint $f : X \to Y$ if and only if for all $x \in X$ the set $\{y \in Y | g(y) \leq x\}$ has a greatest element. If this is the case, then f is unique and

$$f(x) = \mathbf{Max}\{y \in Y | g(y) \leq x\}.$$

Also both f and g are monotonic maps.

(2) A map $f : X \to Y$ has a left adjoint $g : Y \to X$ if and only if for all $y \in Y$ the set $\{x \in X | y \leq f(x)\}$ has a least element. If this is the case, then g is unique and

$$g(y) = \mathbf{Min}\{x \in X | y \leq f(x)\}.$$

Also both f and g are monotonic maps.

Proof. First say that g has a right adjoint f. Then $f(x) \geq \mathbf{Max}\{y \in Y | g(y) \leq x\}$, since if there exists y satisfying $g(y) \leq x$, it follows by the definition that $f(x) \geq y$. By taking $y = f(x)$ and plugging it in the definition of the adjoint, we get that $g(f(x)) \leq x$, i.e. $f(x) \in \{y \in Y | g(y) \leq x\}$, so $f(x) = \mathbf{Max}\{y \in Y | g(y) \leq x\}$. This also shows that the adjoint is uniquely defined, if it exists.

On the other hand if the set $\{y \in Y | g(y) \leq x\}$ has the greatest element, then just define $f(x)$ to be the $\mathbf{Max}\{y \in Y | g(y) \leq x\}$.

To show that its monotone, take $x_2 \geq x_1$. As already shown. $x_1 \geq g(f(x_1))$ so it follows that $x_2 \geq g(f(x_2))$ and by definition of adjoints, $f(x_2) \geq f(x_1)$. The other one is proven in a similar way. □

Since an adjoint is unique, if it exists, we will from now on say **the** adjoint, and write $g = f^*$ for the left adjoint of f and $f = g^\dagger$ for the right adjoint of g.

3.3. **Semilattices, lattices and a duality of semilattices.**

Definition 3.7. *Let P be a poset and $a, b \in P$. Then the* **join***, if it exists, $a \vee b$ of $a, b \in L$ is the least element among the elements x such that $a \leq x$ and $b \leq x$. A* **join semi-lattice** *is a poset L with a least element such that for any two elements $a, b \in L$ the join $a \vee b$ exists. Let L and L' be semi-lattices. A map $f : L \to L'$ is a*

morphism of semi-lattices *if it respects the least element and for all* $a, b \in L$

$$f(a \vee b) = f(a) \vee f(b).$$

is satisfied.

In the same way one defines the **meet** $a \wedge b$ of elements a and b to be the greatest element among the elements $\{x\}$ such that $x \leq a$ and $x \leq b$. The meaning of **meet semi-lattice** and of **morphism of meet semi-lattices** should be clear.

Lemma 3.8. *A morphism of join (resp. meet) semi-lattices is monotonic.*

Proof. If $x \leq y$, then $x \vee y = y$. Hence $f(x) \leq f(x) \vee f(y) = f(x \vee y) = f(y)$. □

Definition 3.9. *A subset S of a join semi-lattice L is called a* **join sub semi-lattice** *provided S contains the least element and for any $a, b \in S$ we have $a \vee b \in S$.*

Definition 3.10. *A* **lattice** *is a poset L which is simultaneously a join and a meet semi-lattice.*

Lemma 3.11. *If L is a finite join semi-lattice, then L is a lattice.*

Proof. First we show that L contains the greatest element. In fact, if $L = \{x_1, \cdots, x_n\}$, then $x_1 \vee \cdots \vee x_n$ is the greatest element. Now for any $a, b \in L$ we consider the subset

$$Q_{a,b} := \{x \in L | x \leq a, \text{ and } x \leq b\}.$$

It is clear that the least element belongs to $Q_{a,b}$, so it is nonempty. It is also clear that $Q_{a,b}$ is a join sub semi-lattice of L. Thus $Q_{a,b}$ has the greatest element $a \wedge b$ and we are done. □

If $f : L \to L'$ is a morphism of finite join semi-lattices then as we have seen L and L' are lattices, but in general f is not a morphism of lattices. However, we have the following:

Proposition 3.12. *Let L and L' be lattices.*

(1) Let $f : L \to L'$ be a morphism of join semi-lattices. If L is finite, then f considered as a morphism of posets has a right adjoint $f^\dagger : L' \to L$ which is a morphism of meet semi-lattices.

(2) Let $f : L \to L'$ be a morphism of meet semi-lattices. If L' is finite, then f considered as a morphism of posets has a left adjoint $f^* : L' \to L$ which is a morphism of join semi-lattices.

Proof. Take an element $y \in L'$ and consider the set

$$Q_{f,y} = \{x \in L | fx \leq y\}$$

We claim that $Q_{f,y}$ is a join sub semi-lattice of L. First, the least element of L belongs to $Q_{f,y}$ and next, if $x_1, x_2 \in Q_{f,y}$, then $f(x_i) \leq y$ for $i = 1, 2$. Hence

$$f(x_1 \vee x_2) = f(x_1) \vee f(x_2) \leq y$$

and $x_1 \vee x_2 \in Q_{f,y}$. By Lemma 3.11 $Q_{f,y}$ contains the greatest element. Hence we can use Proposition 3.6 to deduce the existence of $f^\dagger$. Take now $y_1, y_2 \in L$. Then we have

$$\begin{aligned}(fx \leq y_1 \wedge y_2) &\Leftrightarrow (f(x) \leq y_1 \text{ and } f(x) \leq y_2)\\ &\Leftrightarrow \left(x \leq f^\dagger(y_1) \text{ and } x \leq f^\dagger(y_2)\right)\\ &\Leftrightarrow \left(x \leq f^\dagger(y_1) \wedge f^\dagger(y_2)\right).\end{aligned}$$

Hence

$$f^\dagger(y_1 \wedge y_2) = f^\dagger(y_1) \wedge f^\dagger(y_2)$$

and the first assertion is proved. A similar argument shows the second assertion. □

Corollary 3.13. *The category of finite join semi-lattices is contravariantly equivalent to the category of finite meet semi-lattices.*

Proof. By Lemma 3.11, we can assume that both categories in question have finite lattices as objects. The duality functor from the category of finite join semi-lattices to the category of finite meet semi-lattices is the identity on objects and sends a join semi-lattice morphism f into $f^\dagger$. The functor in the opposite direction is also identity on objects and sends a meet semi-lattice morphism g into g^*. □

3.4. **Ideals and coideals of posets.**

Definition 3.14. *A subset I of a poset X is said to be an* **ideal** *provided for any $x, y \in I$ such that $x \leq y$ and $x \in I$ we have $y \in I$.*

Dually, a subset I^c is called a **coideal** *provided for any $x, y \in I$ such that $x \leq y$ and $y \in I^c$ we also have $x \in I^c$. Note that both can be empty.*

We have the following obvious facts.

Lemma 3.15. *i) If $(I_\alpha)_{\alpha\in\Lambda}$ is a family of ideals of X, then $\bigcup_{\alpha\in\Lambda} I_\alpha$ and $\bigcap_{\alpha\in\Lambda} I_\alpha$ are also ideals.*

ii) If $(I^c_\alpha)_{\alpha\in\Lambda}$ are coideals of X then $\bigcup_{\alpha\in\Lambda} I^c_\alpha$ and $\bigcap_{\alpha\in\Lambda} I^c_\alpha c$ are also coideals.

Proof. Let us check that any intersection of coideals is a coideal. But this is clear, since if x belongs to the intersection, then x belongs to each coideal I^c_α, and by the definition of coideals, $\{y | y \leq x\}$ belongs to each coideal I^c_α, so it belongs to the intersection. The rest is similar. □

Definition 3.16. *For any $a \in X$, let*

$$(a) = \{x \in X | a \leq x\}.$$

Then (a) is an ideal called the **principal ideal** *generated by a. Observe that if $a \vee b$ exists in X, then*

$$(a) \cap (b) = (a \vee b)$$

Similarly, for any $a \in X$ the set

$$\langle a \rangle = \{x \in X | x \leq a\}$$

is a coideal, called the **principal coideal** *generated by a. Observe that if $a \wedge b$ exits in X, then*

$$\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle .$$

Note that $\langle a \rangle$ is in fact a sub semi-lattice.

We let $\mathbf{Ideal}(X)$ *denote the set of all ideals of X. Dually, we let* $\mathbf{Coideal}(X)$ *be the set of all coideals of X.*

Lemma 3.17. *(1) We have $(a) \subset (b)$ if and only if $b \leq a$.*

(2) We have $\langle b \rangle \subset \langle a \rangle$ if and only if $b \leq a$.

(3) We have $(a) = (b)$ if and only if $b = a$.

(4) We have $\langle b \rangle = \langle a \rangle$ if and only if $b = a$.

(5) The map $a \mapsto (a)$ yields an injective morphism of posets

$$X^{op} \to \mathbf{Ideal}(X).$$

Dually, the map $a \mapsto \langle a \rangle$ yields an injective morphism of posets

$$X \to \mathbf{Coideal}(X).$$

Proof. We will show the first assertion only. The rest are either dual or a direct consequence of this. Assume $(a) \subset (b)$. Since $a \in (a)$, we have $a \in (b)$. Hence $b \leq a$. Conversely, if $b \leq a$ and $x \in (a)$, then $a \leq x$. Thus $b \leq x$ and $x \in (a)$. □

3.5. T_0-topological spaces and posets.

Definition 3.18. *A topological space X is said to be a* T_0**-space** *if for any pair of distinct points of X at least one of them has an open neighbourhood not containing the other.*

In what follows, let $\overline{x}$ be the *closure* of a point x in X, which is by definition the intersection of all closed subsets containing x.

Lemma 3.19. *The following conditions are equivalent*

- $\overline{x} \subset \overline{y}$
- *any open subset containing x also contains y.*

Proof. Assume $\overline{x} \subset \overline{y}$ and $x \in U$, where U is open. Let $F = X \setminus U$. Then F is closed and $x \notin F$. If $y \notin U$, then $y \in F = X \setminus U$. Since F is closed $\overline{y} \subset F$. Thus $\overline{x} \subset F$, which contradicts to the assumption $x \notin F$. Conversely, assume any open subset containing x also contains y. Take any element $z \in \overline{x}$, if $z \notin \overline{y}$ there is an open set V such that $z \in V$ and $y \notin V$. Then $x \notin V$. Hence $z \notin \overline{x}$, which contradicts our choice. This shows that $z \in \overline{y}$. □

Based on this fact we show that in any T_0-space we have a poset structure as well.

Proposition 3.20. *Let X be a T_0 space and $x, y \in X$. Define $x \leq y$ provided $\overline{y} \subset \overline{x}$. Then $\leq$ is a partial order on X, called the* **underlying poset** *of the topological space X. Moreover, if $f : X \to Y$ is a continuous function of T_0-spaces then f is monotonic.*

Proof. It is clear that $\leq$ is a reflexive and transitive relation. Assume $x \leq y$ and $y \leq x$. In other words $\overline{x} = \overline{y}$. We claim that $x = y$. If not, then x or y has an open neighbourhood not containing the other element. Let us assume x has such a neighbourhood U. Then $y \in F = X \setminus U$ and $x \notin F$. So $\overline{y} \subset F$ and $\overline{x} \not\subset F$. This contradicts to the fact that $\bar{x} = \bar{y}$. This proves that $\leq$ is a partial order. Assume now Y is also a T_0-space, $f : X \to Y$ is a continuous map and $x_1 \leq x_2$ are points in X. We have to shoe that $\overline{f(x_2)} \subset \overline{f(x_1)}$. To this end take any open neighbourhood U of $f(x_2)$. Then $x_2 \in f^{-1}(U)$. By Lemma 3.19 $x_1 \in f^{-1}(U)$. Hence $f(x_1) \in U$. Again we can use Lemma 3.19 to get $\overline{f(x_2)} \subset \overline{f(x_1)}$. □

3.6. $\mathfrak{P}$-topological spaces and posets.

Definition 3.21. *We say that a topology is a* **$\mathfrak{P}$-topology** *(poset-topology), if*

- *the space is T_0,*
- *any intersection of open sets is open.*

Let X be a $\mathfrak{P}$-topological space and let $x \in X$ be a point. We let $B(x)$ be the intersection of all open subsets containing x. Then $B(x)$ is the minimal open set containing x.

Lemma 3.22. *Let X be a $\mathfrak{P}$-topological space. Then $x \leq y$ if and only if $B(x) \subseteq B(y)$.*

Proof. By Lemma 3.19 we have $x \leq y$ if and only if any open subset containing y also contains x. Since $B(y)$ is the smallest open subset containing y, this condition equivalent to $x \in B(y)$. □

Lemma 3.23. *Let P be a poset. The topology obtained by setting open sets to be the coideals of P, is a $\mathfrak{P}$-topology.*

Proof. By Lemma 3.15 any intersection and union of coideals is a coideal. Hence it defines a topology on P. To show that the space is T_0, we have to show that for $\forall x, y \in P, x \neq y$ there exists a coideal I_x^c such that $x \in I_x^c$ but $y \in I_x^c$ if we know that there is no coideal I_y^c containing y, that does not contain x.

So assuming that, it follows that $x \leq y$ as $\langle y \rangle = \{m | m \leq y, \ \ m \in P\}$ and $x \neq y$ as given. So if there is no coideal I_y^c containing y, that does not contain x, it follows that $x \in \langle y \rangle$. So if we also had $y \in \langle x \rangle$, then this would imply that $\langle x \rangle = \langle y \rangle$ and $x = y$ by Lemma 3.17. This finishes the proof. □

Theorem 3.24. *The category of posets with monotone maps is equivalent to the category of $\mathfrak{P}$-topological spaces with continuous maps.*

Proof. For a T_0-space X we denote the underlying poset by $O(X)$. For a poset P we denote by $T(X)$ the topological space described in Lemma 3.23.

Our firs claim is that for any $\mathfrak{P}$-topological space X one has

$$T(O(X)) = X$$

In other words, any open set in $\mathfrak{P}$-topology corresponds to a coideal in the underlying poset. Clearly every open set in $\mathfrak{P}$-topology is the union of the minimal opens

subsets of the form $B(x)$. Hence it suffices to show that $B(x) = \langle x \rangle$. Take an element $y \in B(x)$. Then $B(y) \subseteq B(x)$.Hence $y \leq x$ and therefore $y \in \langle x \rangle$. This shows that $B(x) \subset \langle x \rangle$. Take now an element $z \in \langle x \rangle$. Then $z \leq x$. Thus $B(z) \subseteq B(x)$ and $z \in B(x)$). Hence $B(x) = \langle x \rangle$. So every open set is a coideal and our first claim is proved.

Our second claim is that for any ordered set P one has

$$O(T(P)) = P$$

In fact, we have to prove that for any two elements $x, y \in P$ we have $x \leq y$ in P if and only if the $B(x) \subset B(y)$ in $T(P)$. But this is a consequence of the fact that $\langle x \rangle = B(x)$ which we already proved. □

It follows from Claim 1 and Claim 2 that we have nice correspondence on objects. The fact that this correspondence extends to morphisms is a consequence of the next lemma.

Lemma 3.25. *Monotone maps in posets and continuous maps of $\mathfrak{P}$-topological spaces are equivalent.*

Proof. By Lemma 3.20 we only need to show that if $f : X \longrightarrow Z$ is a monotone map i.e. $x \leq y \Rightarrow f(x) \leq f(y)$, then the preimage of a coideal is a coideal. It is enough to check that the preimage of a principal coideal is a coideal. Say $y \in f^{-1}(\langle x \rangle)$ and $z \leq y$. Then $f(z) \leq f(y)$ and $f(y) \in \langle x \rangle$. So $f(z) \in \langle x \rangle$ and $z \in f^{-1}(\langle x \rangle)$. This proves that $f^{-1}(\langle x \rangle)$ is a coideal. □

3.7. **Functors on posets and sheaves over $\mathfrak{P}$-topological spaces.**

Definition 3.26. *Let P be a poset. A* **functor** *from a poset P to monoids is given by the following data:*

- *$\forall x \in P$ a monoid M_x,*
- *$\forall x \leq y$ a homomorphism $f_{x,y} : M_x \longrightarrow M_y$ of monoids such that $f_{x,x} : M_x \longrightarrow M_x$ is the identity and if $x \leq y \leq z$, the following diagram*

$$\begin{array}{ccc} M_x & \xrightarrow{f_{x,z}} & M_z \\ & \searrow^{f_{x,y}} & \uparrow{\scriptstyle f_{y,z}} \\ & & M_y \end{array}$$

  commutes.

A contravariant functor defined on a poset P is simply a functor defined on the opposite poset P^{op}.

This is equivalent to consider the category with P as a set of objects and a unique morphism $x \to y$ if $x \leq y$ and no morphism otherwise. Then functors defined on this category are precisely functors defined on the poset P.

In a similar way one can talk on functors from posets to sets, rings, groups etc.

Definition 3.27. *Assume M and N are two functors on a poset P to monoids. A* **natural transformation** *or a* **morphism of functors** *$\varphi : M \to N$ is the following data:*

- *$\forall x \in P$ a homomorphism of monoids $\varphi_x : M_x \to N_x$*
- *$\forall x \leq y$ the following diagram*

$$\begin{array}{ccc} M_x & \xrightarrow{\varphi_x} & N_x \\ {\scriptstyle f_{x,y}}\downarrow & & \downarrow{\scriptstyle f_{x,y}} \\ M_y & \xrightarrow{\varphi_y} & N_y \end{array}$$

commutes.

Now we recall the notion of a sheaf and stalk of a sheaf.
Let X be a topological space. A **presheaf** (say of monoids) on X is simply a contravariant functor on the poset of open subspaces of X. If F is a presheaf and $V \subset U$ then we have the morphism of monoids $F(U) \to F(V)$. For an element $m \in F(U)$ the image of m under this morphism is written by $m|_V$ and is called the **restriction** of m on V.

A presheaf F is called **sheaf**, provided for any open subset U, any open covering $(U_i)_{i\in I}$ of U (meaning that $\bigcup_i U_i = U$) and any collection of elements $m_i \in F(U_i)$ such that m_i and m_j have the same restrictions on $U_i \cap U_j$ for all $i, j \in I$, there exists a unique element $m \in F(U)$ which restricts to m_i for all i.

For a sheaf F and a point $x \in X$, define a **stalk** F_x as follows. Elements of F_x are equivalence classes of pairs (U, m), where U is an open neighbourhood of x and $m \in F(U)$. Two pairs (U, m) and (V, n) are equivalent if there is an open neighbourhood W of x, such that $W \subset U \cap V$ and the restriction of m and n on W are the same.

Lemma 3.28. *Let X be a T_0-space. Then for any sheaf F the map $x \mapsto F_x$ gives rise to a contravariant functor on the poset X.*

Proof. Let $x \leq y$ be points in X. We have to define a homomorphism of monoids $F_y \to F_x$. Elements of F_y have representatives of the form (U, m), where U is an open neighbourhood of y and $m \in F(U)$. Since $x \leq y$, it follows that $x \in U$. Hence the pair (U, m) defines also an element in F_x and we get the map $F_y \to F_x$. □

Our next goal is to show that for a $\mathfrak{P}$-topological space X the notion of a sheaf and the the notion of a functor on the poset X are essentially the same. To this end we need a notion of an inverse limit for a functor defined on a poset.

Definition 3.29. *Let P be a poset and M a functor defined on P. The* **inverse limit** *$\varprojlim_{x\in P} M_x$ is the subset of $\prod_{x\in P} M_x$ consisting of elements $(m_x)_{x\in P}$ such that if $x \leq y$ then $m_y = f_{x,y}(m_x)$.*

Theorem 3.30. *Let X be a $\mathfrak{P}$-topological space. Then the map $F \to F_x$ defined in Lemma 3.28 gives rise to the equivalence of the category of sheaves on X and the category of contravariant functors on the poset X.*

Proof. The functor corresponding to a sheaf F was constructed in Lemma 3.28. Conversely, assume such a functor M is given. Thus we have M_x, for all $x \in X$ and morphisms $M_y \to M_x$ for all $x \leq y$. From this data we have to construct a sheaf F. So I have to define $F(U)$ for all open U. If $U = B(x)$, then I define $F(B(x))$ to be M_x. If $B(x) \subset B(y)$, then $x \leq y$ and the restriction map $F(B(y)) \to F(B(x))$ coincides with the homomorphism $M_y \to M_x$. Take now a general U and define

$$F(U) = \varprojlim_{x \in U^{op}} M_x$$

Argument given in [20], p.237 shows that in this way we really get a sheaf. □

4. The Kato-spectrum

4.1. **Introduction.** In this section I will define the Kato-spectrum explicitly and show that it is a lattice with respect to union. Furthermore, I will construct the sharpest possible reduction of the moniod, such that the spectrum does not change. Indeed after the reduction, the number of elements in the quotient monoid, denoted M^{sl} will be exactly the same as in the Kato-spectrum. In fact, we will show that these two objects are homeomorphic, after defining a topology on M^{sl}. Finally, I will have an example, showing how this theorem helps us calculate the spectrum of a given monoid with ease.

Recall that (see Definition (2.10)) an ideal $\mathfrak{p}$ of a monoid M is called prime if $\mathfrak{p}$ is a proper ideal (i.e. $1 \notin \mathfrak{p}$) and from $ab \in \mathfrak{p}$ it follows that $a \in \mathfrak{p}$, or $b \in \mathfrak{p}$.

For an element $a \in M$ we let $\mathfrak{D}(a)$ be the set of all prime ideals $\mathfrak{p}$ such that $a \notin \mathfrak{p}$. Then we have

Lemma 4.1. *If $a, b \in M$, then*

$$\mathfrak{D}(a) \cap \mathfrak{D}(b) = \mathfrak{D}(ab).$$

Proof. In fact if $\mathfrak{p} \in \mathfrak{D}(a) \cap \mathfrak{D}(b)$, then $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$. Hence $ab \notin \mathfrak{p}$. Conversely, if a or b is in $\mathfrak{p}$, then $ab \in \mathfrak{p}$, because $\mathfrak{p}$ is an ideal. □

Definition 4.2. *By the* **Kato-spectrum** *of a monoid M we mean the set of all prime ideals of M and write it as $Kspec(M)$. Its elements will sometimes be called* **points** *and written as $\mathfrak{p}$. It is equipped with the Zariski topology, where the sets*

$$\mathfrak{D}(a) = \{\mathfrak{p} \in Kspec(M) | a \notin \mathfrak{p}\}$$

and the empty set form a basis of open sets. Here a is an arbitrary element of M, see [16],[8],[9].

Lemma 4.1 shows that we really get a topology. Moreover we have

Lemma 4.3. *The space $Kspec(M)$ is a T_0-space (see Definition 3.18).*

Proof. If $\mathfrak{p} \not\subset \mathfrak{q}$, then there exist an element $a \in \mathfrak{p}$ such that $a \notin \mathfrak{q}$. Thus $\mathfrak{p} \notin D(a)$, while $\mathfrak{q} \in D(a)$.

□

We will prove in Corollary (4.13) below that for any finitely generated monoid M the space $Kspec(M)$ is in fact finite and hence a $\mathfrak{P}$-topological space.

Lemma 4.4. *(1) The empty set is a prime ideal, which is the smallest prime ideal.*

(2) The union of prime ideals is a prime ideal.

(3) The set of noninvertible elements of M is a prime ideal, which is the only maximal ideal.

(4) If $\mathfrak{a}$ is an ideal, then the set of the prime ideals contained in $\mathfrak{a}$ has a (unique) greatest element.

(5) The set $Kspec(M)$ with respect to union is a lattice, with the empty set as the least element, and the maximal ideal as the greatest element.

(6) The open sets $\mathfrak{D}(a)$ are sublattices of $Kspec(M)$.

Proof. The first two facts are obvious. Since the set of invertible elements is not only a submonoid of M but also a face, the third property follows. To see (4), one needs to consider the union of all prime ideals contained in $\mathfrak{a}$.
The join in $Kspec(M)$ is simply the union, while the meet is the maximal prime ideal contained in the intersection of prime ideals. To see the last one, observe that if $a \in \mathfrak{p}$, then a is also contained in every prime ideals containing $\mathfrak{p}$. So in other words, if $a \notin \mathfrak{p}$, then a can also not be in any prime ideal contained in $\mathfrak{p}$ and the set $\{\mathfrak{q}|\mathfrak{q} \subset \mathfrak{p}\}$ for a fixed $\mathfrak{p}$ forms a sublattice. □

The fact that $Kspec(M)$ is a lattice with respect to union is of fundamental importance for us. The following question arises naturally:

Question. Given a lattice L does there exist a monoid M such that $L = Kspec(M)$?.

We will see that the answer on this question is positive for every finite lattice L (see Corollary 4.17).

Since $Kspec(M)$ is again a monoid under the union of prime ideals, we can iterate the construction and consider $Kspec(Kspec(M)))$. Is there any relation between them? As we will later see, the answers to this two questions are very closely related (see Corollary 4.17 and Corollary 4.18).

The following describes the contravariant character of $Kspec$.

Lemma 4.5. *Let $f : M \to N$ be a homomorphism of monoids. If $\mathfrak{p}$ is a prime ideal of N, then*

$$f^{-1}(\mathfrak{p}) = \{m \in M | f(m) \in \mathfrak{p}\}$$

is a prime ideal of M.

Proof. Clearly $1 \notin f^{-1}(\mathfrak{p})$ as otherwise we would have $f(1) = 1 \in \mathfrak{p}$. If $ab \in f^{-1}(\mathfrak{p})$, then $f(a)f(b) = f(ab) \in \mathfrak{p}$. Hence $f(a) \in \mathfrak{p}$ or $f(b) \in \mathfrak{p}$. So $a \in f^{-1}(\mathfrak{p})$ or $b \in f^{-1}(\mathfrak{p})$ and $f^{-1}(\mathfrak{p})$ is a prime ideal. □

It follows that the Kato-spectrum is a contravariant functor, i.e. every homomorphism $f : M \longrightarrow N$, induces an 'inverse' map

$$\begin{array}{rcl} f^* : Kspec(N) & \longrightarrow & Kspec(M) \\ \mathfrak{p} & \mapsto & f^{-1}(\mathfrak{p}) = \{m \in M | f(m) \in \mathfrak{p}\} \end{array}$$

Our next lemma is needed in order to show that f^* is a continuous map with respect to the Zariski topology.

Lemma 4.6. *Let $f : M \to N$ be a homomorphism of monoids. Then for any element $a \in M$, one has*

$$(f^*)^{-1}(\mathfrak{D}(a)) = \mathfrak{D}(f(a))$$

Proof. We have $(\mathfrak{p} \in (f^*)^{-1}(\mathfrak{D}(a))) \Longleftrightarrow (f^{-1}(\mathfrak{p}) \in \mathfrak{D}(a)) \Longleftrightarrow (a \notin f^{-1}(\mathfrak{p})) \Longleftrightarrow (f(a) \notin \mathfrak{p}) \Longleftrightarrow (\mathfrak{p} \in \mathfrak{D}(f(a))$. □

Lemma 4.7. *Let $f : M \to N$ be a homomorphism of monoids. Then*

$$f^* : Kspec(N) \longrightarrow Kspec(M)$$

is a continuous map. If f additionally is surjective, then f^* is injective and $Kspec(N)$ is homeomorphic to $f^*(Kspec(N))$.

Proof. To see continuity, we have to show that the preimage of an open set is open. The general open set in $Kspec(M)$ is of the form $\bigcup_{i\in I} D(a_i)$ and hence it suffices to show that the preimage of $D(a)$ is open, but this follows from Lemma 4.6. Now if f is surjective, then take $\mathfrak{q}_1 \neq \mathfrak{q}_2$. Then w.l.o.g. $\exists x \in \mathfrak{q}_1$ such that $x \notin \mathfrak{q}_2$. So consider $f(m) = x$ (which exists as f is surjective). Hence $m \in f^{-1}(\mathfrak{q}_1)$ but $m \notin f^{-1}(\mathfrak{q}_2)$. Hence f^{-1} is injective. To show the last claim, we have to prove that for any open subset $U \subset Kspec(N)$ there exist an open subset V of $Kspec(M)$ such that

$$f^*(U) = V \cap f^*(Kspec(N)).$$

Again, it suffices to take $U = \mathfrak{D}(b)$ for an element $b \in N$. I claim that $V = \mathfrak{D}(a)$ does the job, for any $f(a) = b$. In fact if $\mathfrak{p}$ is in LHS, then there exist $\mathfrak{q} \in \mathfrak{D}(b)$ such that $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Then $a \notin \mathfrak{p}$, otherwise $b = f(a) \in \mathfrak{q}$. So $\mathfrak{p}$ is in RHS. Conversely, assume $\mathfrak{p}$ is in RHS. Then $a \notin \mathfrak{p}$ and $\mathfrak{p} = f^{-1}(\mathfrak{q})$ for a prime ideal $\mathfrak{q}$ of N. Then $b = f(a) \notin \mathfrak{q}$, so $\mathfrak{p}$ is in LHS. □

The reader should be aware that for a monoid homomorphism $f : M_1 \to M_2$ the induced map

$$f^* : Kspec(M_2) \to Kspec(M_1)$$

is only a homomorphism of join semi-lattices, not necessarily of lattices, i.e. need not preserve the meet operation.
Since the Kato-spectrum of a finitely generated monoid is a lattice, it is important to see the relation between monoids and lattices.

4.2. **Semilattices, idempotent monoids and the monoud M^{sl}.** Note that if L is a join semi-lattice we can consider L as a monoid, with operation

$$xy := x \vee y$$

The least element of L is the unit in this monoid. It is clear that we have $x^2 = x$ for all $x \in L$. Conversely, if M is a monoid such that $m^2 = m$ holds for all $m \in M$, then one defines $x \leq y$ if $xy = y$. It is easily seen that this way we obtain a semi-lattice. Thus, there is an equivalence between semi-lattices and monoids satisfying the identity $x^2 = x$ for all x.

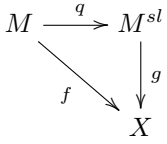
The following construction (see Theorem 1.2 of Chapter III in [14]) allows us to construct a semilattice starting from an arbitrary monoid.

Definition 4.8. *Let M be a monoid. Define the following binary relation, where $a \sim b$ if there exist natural numbers $m, n \geq 1$ and elements $u, v \in M$ such that $a^m = ub$ and $b^n = va$.*

Lemma 4.9. *(1) The relation $\sim$ is a congruence relation on M*

(2) The quotient monoid $M^{sl} = M/\sim$ is a join semi-lattice.

(3) If X is a join semi-lattice and $f : M \to X$ is a monoid homomorphism, then there exist a unique homomorphism $g : M^{sl} \to X$ such that the diagram



commutes. Here $q : M \to M^{sl}$ is the canonical projection into quotient monoid.

Proof. (1) The reflexivity and symmetry conditions for the relation $\sim$ is clear. To see the transitivity relation, assume $a \sim b$, $b \sim c$. Then there are natural numbers $n_1, m_1, n_2, m_2 \geq 1$ and elements $u_1, u_2, v_1, v_2 \in M$ such that $a^{m_1} = u_1 b$, $b^{n_1} = v_1 a$, $b^{m_2} = u_2 c$ and $c^{n_2} = v_2 a$. Then we have $a^m = uc$ and $b^n = va$, where $m = m_1 m_2$, $u = u_1^{m_2} u_2$, $n = n_1 n_2$ and $v = v_1^{n_2} v_2$. Hence $a \sim c$. This implies that $\sim$ is an equivalence relation. To see that $\sim$ is a congruence relation, assume $a \sim b$ and $c \sim d$. Then $a^m = ub$, $b^n = va$, $c^k = wd$ and $d^p = zc$, for some natural numbers $m, n, k, p \geq 1$ and $u, v, w, z \in M$. Since

$$(ac)^{mk} = u_1(bd), \quad (bd)^{np} = v_1(ac)$$

we see that $ac \sim bd$. Here $u_1 = u^k b^{k-1} w^m d^{m-1}$ and $v_1 = v^p a^{p-1} z^n c^{n-1}$. Thus $\sim$ is a congruence relation.

(2) Recall that $a \sim b$ if $a^m = ub$ and $b^n = va$. Take $b = a^2$, $u = 1$, $m = 2$, $n = 1$, $v = a$ to get $a \sim a^2$ and hence the result.

(3) Since q is surjective, uniqueness is clear. To prove existence of g, it suffice to show that if $a \sim b$ then $f(a) = f(b)$. The condition $a \sim b$ implies that $a^m = ub$ and $b^n = va$ for some $u, v, \in M$ and natural numbers n, m. By assumtion X is a semilattice considered as a monoid. Hence the identity $x^2 = x$ holds in X. It follows

that for any element x and a natural number k one has $x^k = x$. In particular we have $f(a)^m = f(a)$ and $f(b)^n = f(b)$. Based on this we can write

$$f(a) = f(a)^m = f(a^m) = f(ub) = f(u)f(b) = f(u)f(b)^n = f(u)f(v)f(a)$$

Similarly

$$f(b) = f(u)f(v)f(b)$$

We also have

$$\begin{aligned} f(u)f(v)f(a) &= f(u)f(v)f(a)^m = f(u)f(v)f(a^m) \\ &= f(u)f(v)f(u)f(b) = f(u)^2 f(v)f(b) \\ &= f(u)f(v)f(b) \end{aligned}$$

Comparing our computations we see that $f(a) = f(b)$ and the result follows.

□

4.3. **Main results.**

4.3.1. *Product of the Kato-Spectrum.* Recall that if $M_{i|i\in I}$ is s family of monoids, then the coproduct $M = \coprod_{i\in I} M_i$ is the submonoid of $\prod_{i\in I} M_i$ which consists of elements $(m_i)_{i\in I}$ such that $m_i = 1$ for all but finitely many i's. We let $\pi_i : M \to M_i$ and $\mu_i : M_i \to M$ be canonical morphisms. Thus $\pi_j\mu_i(m) = 1$ if $j \neq i$ and $\pi_i\mu_i(m) = m$ for any $m \in M_i$. For an element $x \in M$ we let $\mathbf{Supp}(\mathbf{x})$ the support of x, be the set of indexes $i \in I$ such that $\pi_i(x) \neq 1$. By definition $Supp(x)$ is always finite in the coproduct and it is empty if and only if $x = 1$. The monoid homomorphisms μ_i induce the continuous maps $Kspec(M) \to Kspec(M_i)$ for all i. Hence we have the continuous map $\mu^* : Kspec(M) \to \prod_{i\in I} Kspec(M_i)$. So we have

Proposition 4.10. *The canonical map*

$$\mu^* : Kspec(\coprod_{i\in I} M_i) \to \prod_{i\in I} Kspec(M_i)$$

is bijection.

Proof. Take any collection of prime ideals $\mathfrak{q}_i \in Kspec(M_i)$, $i \in I$. Let $\hat{\mathfrak{q}_i}$ be the subset of $M = \coprod_{i\in I} M_i$ which consists of elements $x \in M$ such that $\pi_i(x) \in \mathfrak{q}_i$. Since $\hat{\mathfrak{q}_i}$ is a prime ideal of $Kspec(M)$ for all $i \in I$, we see that $\hat{\mathfrak{q}} = \cup_{i\in I}\hat{\mathfrak{q}_i}$ is also a prime ideal of M. The inverse map

$$\nu^* : \prod_{i\in I} Kspec(M_i) \to Kspec(M)$$

is defined by $\nu^*((\mathfrak{q}_i)_{i\in I}) = \hat{\mathfrak{q}}$. To see that these maps are indeed mutually inverse maps, we proceed as follows. Take any $\hat{\mathfrak{p}} \in Kspec(M)$. Then $\mu^*(\hat{\mathfrak{p}}) = (\mathfrak{p}_i)_{i\in I}$, where

$$\mathfrak{p}_i = \{m \in M | \mu_i(m) \in \hat{\mathfrak{p}}\}$$

For all $x \in M$ we have

$$x = \prod_{i\in Supp(x)} \mu_i(\pi_i(x))$$

Hence $x \in \hat{\mathfrak{p}}$ if and only if there is an element $i \in I$ such that $\pi_i(x) \in \mathfrak{p}_i$. This shows that

$$\hat{\mathfrak{p}} = \cup_{i \in I} \hat{\mathfrak{p}_i}$$

Thus $\nu^* \circ \mu^* = id$. The equality $\mu^* \circ \nu^* = id$ is straightforward. □

4.3.2. *Reduction Theorem.* Recall that for any monoid M there is a semi-lattice M^{sl} and the quotient homomorphism

$$f : M \to M^{sl} = M/\sim \qquad \text{(see Lemma 4.9)}$$

universal among all homomorphisms into semi-lattices.

The following theorem states that M^{sl} is the greatest reduction of M we can make, such that the Kato-spectrum does not change.

Theorem 4.11. *For every monoid M we have the homeomorphism (see 4.7)*

$$f_* : Kspec(M^{sl}) \to Kspec(M)$$

induced by the canonical projection $f : M \to M^{sl}$.

Proof. Since f is surjective, the induced morphism $Kspec(M^{sl}) \to Kspec(M)$ is injective. To see that it is also surjective, we take a prime ideal $\mathfrak{p}$ of M. Since f is surjective $f(\mathfrak{p}) = \mathfrak{q}$ is an ideal of M^{sl}. Our first claim is that $1 \notin \mathfrak{q}$. For this assume $1 \in \mathfrak{q}$. Hence there exists $a \in \mathfrak{p}$ such that $1 \sim a$. But then a must be invertible, which contradicts the condition that $\mathfrak{p}$ is prime and the claim is proved. Our second claim is that $\mathfrak{q}$ is a prime ideal of M^{sl}. Suppose $f(x)f(y) \in \mathfrak{q}$. Hence there exists an element $a \in \mathfrak{p}$ such that $xy \sim a$. So $(xy)^n = au$ for some $n \geq 1$. It follows that $x^n y^n \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we can see that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. But this means that $f(x)$ or $f(y)$ belongs to $\mathfrak{q}$. It follows that $\mathfrak{q}$ is a prime ideal and the second claim is proved. Recall that f_* is taking the pre-image of a prime ideal, so it remains to show that

$$\mathfrak{p} = f^{-1}(\mathfrak{q}) = f^{-1}(f(\mathfrak{p}))$$

Take an element $x \in f^{-1}(f(\mathfrak{p}))$. So there is an element $b \in \mathfrak{p}$ such that $f(x) = f(b)$. Thus $x \sim b$. We have $x^n = bv \in \mathfrak{p}$, which implies that $x \in \mathfrak{p}$. So we have proven that $\mathfrak{p} \subset f^{-1}(f(\mathfrak{p}))$. Since $\mathfrak{p} \supset f^{-1}(f(\mathfrak{p}))$ is obvious, the result follows. □

As an immediate consequence we obtain that for any subgroup $G \subset U(M)$ the canonical homomorphism $M \to M/G$ induces a homeomorphism $Kspec(M/G) \to Kspec(M)$, the result stated as Lemma 3.1.16 in Gabber-Ramero [12].

As a consequence we can amongst other things sharpen Lemma 4.2 in [9].

Corollary 4.12. *Let B be a submonoid of A and assume for any element $a \in A$ there exists a natural number n such that $a^n \in B$. Then*

$$Kspec(A) \to Kspec(B)$$

is a bijection.

Proof. It suffices to show that $B^{sl} \to A^{sl}$ is an isomorphism. Take any element $a \in A$. Since $a \sim a^n$ for all n we see that the map in question is surjective. Take now two elements b_1, b_2 in B and assume $b_1 \sim b_2$ in A. Then there are $u, v \in A$ such that $b_1^k = ub_2$ and $b_2^m = vb_1$. Take N such that $u_1 = u^N \in B$ and $v_1 = v^N \in B$. Then $b_1^{kN} = u_1 b_2^N$ and $b_2^N = v_1 b_1^N$. Thus $b_1 \sim b_2$ in B and we are done. □

Furthermore we can give an alternative proof of Proposition 5.5 in [16].

Corollary 4.13. *Let M be a finitely generated monoid, with $\{x_i\}_{i=1}^n$ as generators. Then $|Kspec(M)| \leq 2^n$.*

Proof. Let $\{x_i\}$ be the set of generators of M. Then as $x_i \sim x_i^n$ for all n and all i in M^{sl}, result follows. □

4.3.3. *Kspec of finite semi-lattices.* Theorem 4.11 reduces the study of $Kspec(M)$ to the case when M is a join semi-lattice. Our aim now is to prove that for any finite (=finitely generated) semi-lattice L there exists a "natural" bijection $\alpha_L : L \to Kspec(L)$. Since the $Kspec(L)$ is equipped with the Zariski topology, one can ask what is a corresponding topology on L for which α_L is a homeomorphism. As we will see this a topology on L where open sets are exactly ideals of L.

Definition 4.14. *Let L be a join semi-lattice considered as a monoid. Take an element $a \in L$ and consider the set*

$$\langle a \rangle = \{x \in L | x \leq a\}.$$

Define $\alpha_L(a)$ to be the complement of $\langle a \rangle$ in L.

Then we have

Lemma 4.15. *(1) $\langle a \rangle$ is a sub semi-lattice.*
(2) $(a \leq b) \Longleftrightarrow (\langle a \rangle \subset \langle b \rangle)$
(3) $\alpha_L(a) \in Kspec(L)$
(4) $\alpha_L : L \to Kspec(L)$ is an injective map.
(5) If L is a lattice, then $\alpha_L(a \wedge b) = \alpha_L(a) \cup \alpha_L(b)$.

Proof. The first two are just definitions, the third comes from the fact that $\langle a \rangle$ is a face (see Rem.2.15). Injection is also clear from (2), as $\alpha_L(a) = \alpha_L(b) \Leftrightarrow \langle a \rangle^c = \langle b \rangle^c \Leftrightarrow \langle a \rangle = \langle b \rangle$. To see the last one, let $c = a \vee b$. Then $\alpha_L(a) \subset \alpha_L(c)$ and $\alpha_L(b) \subset \alpha_L(c)$. Also since c is the greatest among all satisfying $c \leq a$ and $c \leq b$, it follows that $\alpha_L(c)$ is the least of all satisfying $\alpha_L(a) \subset \alpha_L(c)$ and $\alpha_L(b) \subset \alpha_L(c)$. But since the union of prime ideals are prime, $\alpha_L(c) = \alpha_L(a) \cup \alpha_L(b)$. □

Theorem 4.16. *(Main theorem of affine Kato-spectrums) Let L be a finite semi-lattice.*

(1) For any prime ideal $\mathfrak{p}$, let $\beta_L(\mathfrak{p})$ be the greatest element of the sub semi-lattice $L - \mathfrak{p}$. Then for the map $\beta_L : Kspec(L) \to L$ one has $\beta_L \circ \alpha_L = id$. Thus

$$\alpha_L : L \to Kspec(L)$$

is bijective.

(2) For a subset $S \subset L$, the set $\alpha_L(S)$ is an open subset of $Kspec(L)$ if and only if S is an ideal.

(3) Let L be a finite semi-lattice and let $f : L \to L'$ be a morphism of join semi-lattices. Then one has the following commutative diagram

$$\begin{array}{ccc} L' & \xrightarrow{f^\dagger} & L \\ {\scriptstyle \alpha_{L'}}\downarrow & & \downarrow{\scriptstyle \alpha_L} \\ Kspec(L') & \xrightarrow[f^*]{} & Kspec(L) \end{array}$$

where as usual $f^\dagger$ denotes the right adjoint of f (see Definition 3.4).

Proof. Take $a \in L$. Then we have

$$\beta_L \circ \alpha_L(a) = \beta_L(L - \langle a \rangle) = Max\{x | x \in \langle a \rangle\} = a$$

and the first assertion follows. We also have

$$D(a) = \{\mathfrak{p} \in Kspec(L) | a \notin \mathfrak{p}\} = \{\alpha_L(m) | a \notin \alpha_L(m)\} =$$

$$\{\alpha_L(m) | a \in \langle m \rangle\} = \{\alpha_L(m) | m \in (a)\}.$$

Thus $\alpha_L^{-1}(D(a)) = (a)$ and the second assertion follows as well. To see that the diagram commutes it suffices to prove that

$$\beta_L \circ f^{-1} \circ \alpha_{L'} = f^\dagger$$

Take $y \in L'$. Then $\beta_L \circ f^{-1} \circ \alpha_{L'}(y) = \beta_L f^{-1}(L' - \langle y \rangle) = \beta_L(L - f^{-1}(\langle y \rangle)) = Max\{x \in L | f(x) \in \langle y \rangle\} = Max\{x \in L | f(x) \leq y\} = f^\dagger(y)$ and we are done. □

This result has the following consequences and enables us to answer the 2 questions:

Corollary 4.17. *(1) If L is a finite semi-lattice then the composite*

$$L \xrightarrow{\alpha_L} Kspec(L) \xrightarrow{\alpha_{Kspec(L}} Kspec(Kspec(L))$$

is an isomorphism of semi-lattices, which is functorial in L.

(2) Any finite semi-lattice is the Kato-spectrum of a monoid.

Proof. The second part is a direct consequence of the first one. Since the bijection α_L reverses the ordering the first part follows. □

Combine Theorem 4.16 and Theorem 4.11 to obtain the following result.

Corollary 4.18. *(1) For any monoid M there is an injective map*

$$\alpha_M : M^{sl} \to Spec(M)$$

which is bijective if M is a finitely generated monoid.

(2) *Let M be a finitely generated monoid and let $f : M \to N$ be a morphism of monoids. Then one has the following commutative diagram*

$$\begin{array}{ccc} N^{sl} & \xrightarrow{f^\dagger} & M^{sl} \\ {\scriptstyle \alpha_N}\downarrow & & \downarrow{\scriptstyle \alpha_M} \\ Kspec(N) & \xrightarrow[f^*]{} & Kspec(M) \end{array}$$

where as usual $f^\dagger$ is the right adjoint to the poset map $f_* : M^{sl} \to N^{sl}$.

(3) *For any finitely generated monoid M one has the natural bijection*

$$Kspec^3(M) \cong Kspec(M),$$

where $Kspec^3$ is the three fold composite of $Kspec$.

Example 4.19. *Consider the monoid*

$$M = \langle x_1, x_2, x_3, x_4 \rangle / (x_1^{n_1} x_3^{n_2} = x_3^{n_3}, x_2^{n_4} x_4^{n_5} = x_4^{n_6}, x_2^{n_7} x_3^{n_8} = x_1^{n_9} x_4^{n_{10}})$$

Find the Kato-spectrum of M.

<u>*Solution.*</u>

First we write out M^{sl}. For this one can assume that all powers are 1, as $x^2 \sim x$ hold for all elements. Then we write out all combinations of x_i and cross out the repeated ones modulo the relations. Note that x_3x_4 is the only one that is not straight forward. For this observe that

$$x_3x_4 = x_1x_3x_4 = x_1x_3x_2x_4 = x_1x_4x_1x_4 = x_1x_4$$

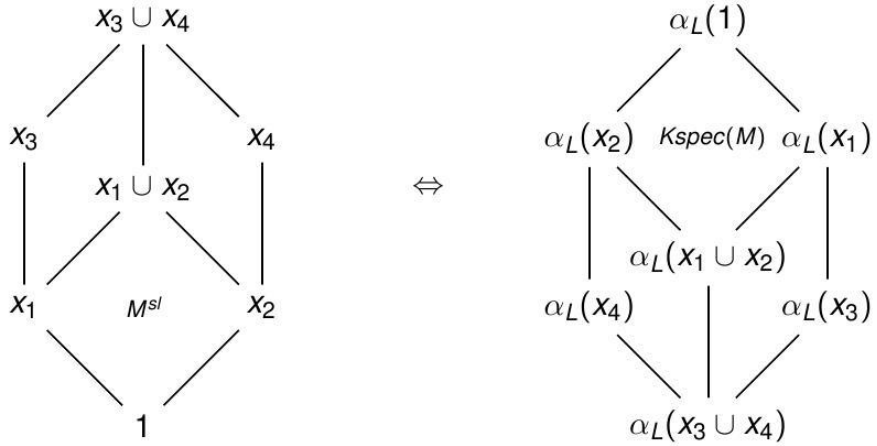
$$\Rightarrow Kspec(M) \cong M^{sl} = \{ \quad 1,$$

$$x_1, x_2, x_3, x_4,$$

$$x_1x_2, \cancel{x_1x_3}, x_1x_4, \cancel{x_2x_3}, \cancel{x_2x_4}, \cancel{x_3x_4},$$

$$\cancel{x_1x_2x_3}, \cancel{x_1x_2x_4}, \cancel{x_1x_3x_4}, \cancel{x_2x_3x_4},$$

$$\cancel{x_1x_2x_3x_4}\}.$$



5. Monoidal Schemes

5.1. **Introduction.** In this section we introduce monoidal schemes. Here we are essentially following [16] and [8]. This part is very similar to the classical theory of schemes, but due to the fact that the Kato-spectrum is significantly simpler then the classical ring-spectrum, much of the constructions will become very simple. First of all observe that since the Kato-spectrum of a finitely generated monoid is a lattice and the morphisms are semi-lattice morphisms, we can do the whole constructions with just posets, where monotone maps will be continuous maps and open sets coideals. This allows us to give a new description of the category of noetherian monoidal schemes in terms only of posets and functors defined on elements of the poset. A similar result is contained in [6]. The difference between our approach and [6] is that we not only don't need topological spaces and sheaves, but also prime ideals.

Recall that a coideal I^c is a set satisfying the condition that if $x \in I^c$ and $y \leq x$, then $y \in I^c$. A principal coideal is a coideal that can be generated with just one element, i.e satisfies that if $y \leq x$, then $y \in I^c$ for some fixed x. Note that this is in fact a join sub semi-lattice, and as such will be written as $\langle x \rangle$.

Also note that $\mathfrak{D}(f) = \{\mathfrak{p} | f \notin \mathfrak{p}\}$ is indeed a sub semi-lattice of $Kspec(M)$ and their unions are coideals, as they satisfy the property that if $f \notin \mathfrak{p}$ and $\mathfrak{q} \subset \mathfrak{p}$, then $f \notin \mathfrak{q}$

5.2. **Monoid spaces.**

Definition 5.1. *A* **monoid space** *is a pair $(X, \mathfrak{F}_X)$ where X is a topological space and $\mathfrak{F}_X$ is a sheaf of monoids on X. A* **morphism of monoid spaces** *from $(X, \mathfrak{F}_X)$ to $(Y, \mathfrak{F}_Y)$ is the following data:*

- *a continuous map: $f : X \longrightarrow Y$,*
- *a function which assigns to each open set $V \subset Y$ a morphism of monoids $\mathfrak{F}_Y(V) \to \mathfrak{F}_X(f^{-1}V)$ such that if $W \subset V$ is also an open subset, then the diagram*

$$\begin{array}{ccc} \mathfrak{F}_Y(V) & \longrightarrow & \mathfrak{F}_X(f^{-1}V) \\ {\scriptstyle \rho_{V,W}}\downarrow & & \downarrow{\scriptstyle \rho'_{V,W}} \\ \mathfrak{F}_Y(W) & \longrightarrow & \mathfrak{F}_X(f^{-1}W) \end{array}$$

  commutes

- *$\mathfrak{F}_{Y,f_x} \longrightarrow \mathfrak{F}_{X,x}$ is a local homomorphism of monoids, i.e. non invertible elements go to non invertible elements.*

5.3. **The canonical sheaf of monoids.** Let M be a monoid. Then consider $Kspec(M)$ with the Zariski topology. Then we have the following results.

Proposition 5.2. • *If M is a monoid, then there is a unique sheaf $\mathfrak{F}_M$ of monoids on $Kspec(M)$ for which $\mathfrak{F}_M(\mathfrak{D}(a))$ is the localization of M with*

respect to a. For any $\mathfrak{p} \in Kspec(M)$ one has

$$(\mathfrak{F}_M)_{\mathfrak{p}} = M_{\mathfrak{p}}$$

So $(Kspec(M), \mathfrak{F}_M)$ is a monoid space.

- *If $f : M \longrightarrow N$ is a homomorphism of monoids, then f induces a natural morphism of monoid spaces*

$$(f^*, f^{-1}) : (Kspec(N), \mathfrak{F}_N) \longrightarrow (Kspec(M), \mathfrak{F}_{Kspec(M)})$$

- *If M and N are monoids, then any morphism of monoid spaces from $(Kspec(N), \mathfrak{F}_N)$ to $(Kspec(M), \mathfrak{F}_M)$ induces a homomorphism of monoids $f : M \longrightarrow N$ as in the above (second) point.*

The proof is essentially the same as for rings [8]. In case of finitely generated monoids we will give an alternative proof. By Theorem 3.30 we have to define a functor on the oposit poset of the underlying poset of $Kspec(M)$, which is the same as the poset of prime ideals where ordering is defined by inclusion of prime ideals. Observe that if $\mathfrak{p} \subset \mathfrak{q}$ are prime ideals, then we have a canonical homomorphism $M_{\mathfrak{q}} \to M_{\mathfrak{p}}$. Hence it gives rise to a sheaf which is denoted by $\mathfrak{F}_M$ and is called the canonical sheaf of M.

5.3.1. *Definition of monoidal schemes.*

Definition 5.3. *An* **affine monoidal scheme** *is a monoid space $(X, \mathfrak{F}_X)$ which is isomorphic to $(Kspec(M), \mathfrak{F}_M)$ for some monoid M.*

Definition 5.4. *A* **monoidal scheme** *is a monoid space $(X, \mathfrak{F}_X)$ in which every point has an open neighborhood U such that the topological space U together with the inherited sheaf (under restriction) $(\mathfrak{F}_{X|U})$ is an affine scheme. In this case, X is said to be the* **underlining topological** *space and $\mathfrak{F}_X$ its* **structural sheaf**.

Definition 5.5. *A* **morphism of monoidal schemes** *is a morphism of monoid spaces. An* **isomorphism** *is a morphism with an inverse.*

Lemma 5.6. *Let $(X, \mathfrak{F}_X)$ be a monoid scheme. Then X is a T_0-space.*

Proof. Since $Spec(M)$ is a T_0-space by Lemma 4.3 it suffices to note that if a space is locally T_0, then globally it is a T_0-space. □

This fact allows us to apply the material developed in Section 3 to monoid schemes. To formulate main results we need some additional notions.

5.4. **realisation.** Any monoid M gives rise to the commutative ring $\mathbb{Z}[M]$ known as the **monoid** algebra of M. As an abelian group it if freely generated by elements of M and the multiplication is induced by the multiplication of M. This assignment can be extended to a functor known as (the realisation) from the category of monoid schemes to the category of schemes [6]. The general description of this functor is based in the inductive limit in the category of schemes, but in practice it is usually not too hard to describe it. The point is that there is a nice relationship between

localization M_m of the monoid M by an element $m \in M$ and the localization $(\mathbb{Z}[M])_m$ of the ring $\mathbb{Z}[M]$ by m considered as an element of $\mathbb{Z}[M]$. Namely, one has the following isomorphism

$$\mathbb{Z}[M_m] \cong (\mathbb{Z}[M])_m.$$

This allows us to describe the realisation in much more concrete terms. For example if we have a monoid scheme obtained by gluing two affine monoid schemes $Kspec(M)$ and $Kspec(M')$ on the main open subset $\mathfrak{D}(f) \subset Kspec(M)$ and $\mathfrak{D}(f') \subset Kspec(M')$ via isomorphism $\phi : M_f \cong M'_{f'}$, then the corresponding scheme is obtained by gluing affine schemes $Spec(\mathbb{Z}[M])$ and $\mathbb{Z}[M']$ via main subsets $D(f) \subset Spec(\mathbb{Z}[M])$ and $D(f') \subset Spec(\mathbb{Z}[M]')$ using the isomorphism of rings $\mathbb{Z}[\phi] : \mathbb{Z}[M_m] \cong \mathbb{Z}[M'_{m'}]$. This isomorphism is the extension of ϕ, which is unique since M is the basis of $\mathbb{Z}[M]$.

5.5. **Contravariant monoid sketches.**

Definition 5.7. *A* **contravariant monoid sketch** *is a pair* $(\mathfrak{P}, \mathfrak{E}_\mathfrak{P})$ *where* $\mathfrak{P}$ *is a poset and*

$$\mathfrak{E}_\mathfrak{P} : \mathfrak{P}^{op} \longrightarrow \{monoids\}$$

is a contravariant functor to the category of monoids.

Definition 5.8. *A* **morphism of contravariant monoid sketches** $f : (\mathfrak{P}, \mathfrak{E}_\mathfrak{P}) \longrightarrow (\mathfrak{R}, \mathfrak{E}_\mathfrak{R})$ *is the following data:*

- $f : \mathfrak{P} \longrightarrow \mathfrak{R}$ *is a monotone map.*
- *for all* $x \in \mathfrak{P}$ *a local homomorphism of monoids is given*

$$f_x : \mathfrak{E}_\mathfrak{R}(fx) \longrightarrow \mathfrak{E}_\mathfrak{P}(x)$$

such that if $x \leq y$*, then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{E}_\mathfrak{R}(fy) & \longrightarrow & \mathfrak{E}_\mathfrak{P}(y)) \\ \downarrow & & \downarrow \\ \mathfrak{E}_\mathfrak{R}(fx) & \longrightarrow & \mathfrak{E}_\mathfrak{P}(x) \end{array}$$

Example 5.9. *Let* M *be a monoid. Then we have a canonical contravariant monoid sketch* $(Kspec(M), \mathfrak{E}_M)$*. Here* $Kspec(M)$ *is considered as a poset and*

$$\mathfrak{E}_M(\mathfrak{p}) = M_\mathfrak{p}$$

Actually this construction can be generalised as follows:

Proposition 5.10. *Let* $(X, \mathfrak{E}_X)$ *be a monoid space. Then we obtain a contravariant monoid sketch on the underlying poset of* X*, which assigns* $\mathfrak{E}_x$ *to any point* $x \in X$*. In this way one obtains a functor from the category of monoid spaces to the category of contravariant monoid sketches.*

Proof. This follows from Proposition 3.20 and Lemma 3.28. □

One can go further and introduce the subclass of contravariant monoid sketches which corresponds to monoid schemes.

Definition 5.11. *A* **contravraint monoid system** *is a contravariant monoid sketch* $(\mathfrak{P}, \mathfrak{E}_{\mathfrak{P}})$ *such that for any point* $x \in \mathfrak{P}$ *the restriction of* $\mathfrak{E}_{\mathfrak{P}}$ *to the coideal* $\langle x \rangle$ *is isomorphic to the contravariant monoid sketch corresponding to the monoid* $\mathfrak{P}_x$ *as defined in Example 5.9.*

Then Proposition 5.10 specified to monoid schemes gives the following functor:

Proposition 5.12. *Let* $(X, \mathfrak{E}_X)$ *be a monoid scheme. Then we obtain a contravariant moniod system on the underlying poset of* X*, which assigns* $\mathfrak{E}_x$ *to any point* $x \in X$*. In this way one obtains a functor from the category of monoid schemes to the category of contravariant moniod system's.*

In the next section we show the equivalence for locally finitely generated monoids.

5.6. **Locally noetherian monoid schemes.**

Definition 5.13. *A monoidal scheme* $(X, \mathfrak{F}_X)$ *is* **locally noetherian** *provided any point* $x \in X$ *has an open neighbourhood* U *such that* $\mathfrak{F}_X(U)$ *is a finitely generated monoid. It is called* **noetherian** *if additionally* X *is quasi-compact.*

Observe that the realisation of a locally noetherian monoid scheme is a locally noetherian scheme and the realisation of a noetherian monoid scheme is a noetherian scheme.

Lemma 5.14. *A monoidal scheme* $(X, \mathfrak{F}_X)$ *is noetherian if and only if* X *has a finite open covering by affine monoidal schemes corresponding to finitely generated monoids.*

Proof. In fact, since $(X, \mathfrak{F}_X)$ is locally noetherian, it follows that it has a covering by affine monoidal schemes corresponding to finitely generating monoids. By quasi-compactness this covering is finite. The converse statement is clear. □

Lemma 5.15. *Let* $(X, \mathfrak{F}_X)$ *be a locally noetherian monoid scheme. Then* X *is a* $\mathfrak{P}$*-topology. If additionally* $(X, \mathfrak{F}_X)$ *is noetherian, then* X *is a finite* $\mathfrak{P}$*-topology.*

Proof. Take any collection of open subsets $(U_i)_{i \in I}$. Denote by V the intersection $\cap_{i \in I} U_i$. Take a point $x \in V$. By assumption there is an open set W, such that $x \in W$ and $W \cong Spec(M)$ where M is finitely generated. Hence W is a $\mathfrak{P}$-topology. Since $W \cap V = W \cap \{\cap_{i \in I} U_i\}$ is an intersection of open sets in the neighbourhood of x, which is a $\mathfrak{P}$-topology, it follows V is open and thus X is a $\mathfrak{P}$-topology. Assume now that $(X, \mathfrak{F}_X)$ is noetherian. As we have seen any point of X is contained in a finite open subspace. By quasi-compactness X is finite as well. □

These fact suggest that we can use the results from Section 3 to give an alternative description of the category of locally noetherian monoidal schemes in terms of contravariant moniod systems.

Definition 5.16. *A contravariant moniod system* $(\mathfrak{P}, \mathfrak{E}_{\mathfrak{P}})$ *is* **locally noetherian** *if for any point* x *the monoid* $\mathfrak{E}_{\mathfrak{P}}(x)$ *is finitely generated. If additionally* $\mathfrak{P}$ *is finite, then* $(\mathfrak{P}, \mathfrak{E}_{\mathfrak{P}})$ *is called a* **noetherian contravariant moniod system**.

Now we are ready to formulate the following theorem, which for noetherian monoidal schemes appears as Proposition 2.9 in [6]

Theorem 5.17. *The category of locally noetherian monoidal schemes is equivalent to the category of locally noetherian contravariant moniod system's. Under this equivalence noetherian objects goe to noetherian ones.*

Proof. This is a consequence of Theorems 3.24, 3.30 and the above discussion. □

This description of monoidal schemes has the advantage, that it does not uses sheaves, at least in case of locally noetherian objects.

5.7. **Monoid sketches.**

Definition 5.18. *A* **monoid sketch** *is a pair* $(\mathfrak{P}, \mathfrak{F}_{\mathfrak{P}})$ *where* $\mathfrak{P}$ *is a poset and*

$$\mathfrak{F}_{\mathfrak{P}} : \mathfrak{P} \longrightarrow \{monoids\}$$

is a functor to the category of monoids.

Definition 5.19. *A* **morphism of monoid sketches** $f : (\mathfrak{P}, \mathfrak{F}_{\mathfrak{P}}) \longrightarrow (\mathfrak{R}, \mathfrak{F}_{\mathfrak{R}})$ *is the following data:*

- $f : \mathfrak{P} \longrightarrow \mathfrak{R}$ *is a monotone map.*
- *for all* $x \in \mathfrak{P}$ *a local homomorphism of monoids is given*

$$f_x : \mathfrak{F}_{\mathfrak{P}}(x) \longrightarrow \mathfrak{F}_{\mathfrak{R}}(f(x))$$

such that if $x \leq y$ *then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{F}_{\mathfrak{P}}(x) & \longrightarrow & \mathfrak{F}_{\mathfrak{R}}(f(x)) \\ \downarrow & & \downarrow \\ \mathfrak{F}_{\mathfrak{P}}(y) & \longrightarrow & \mathfrak{F}_{\mathfrak{R}}(f(y)) \end{array}$$

For any monoid M we have a monoid sketch $(M^{sl}, \mathfrak{O}_M)$. To describe it recall that M^{sl} is a semi-lattice and hence a poset. If a is an element of M, we let $[a] \in M^{sl}$ be the corresponding class. The subset $[a] \subset M$ is a sub-semigroup of M. Hence we can localize M by the submonid $\{1\} \cup [a]$. The resulting monoid is denoted by $M_{[a]}$. If $a \in [a]$, then the submonoid generated by a is contained in $\{1\} \cup [a]$, so there is a canonical homomorphism $M_a \to M_{[a]}$, where as usual M_a denotes the localization of M by a. Since $M_a \to M_{[a]}$ is an isomorphism, we will write M_a instead of $M_{[a]}$. Moreover, if $[a] \leq [b]$ in M^{sl}, then $[ab] = [b]$ and we have a canonical homomorphism

$$M_a \to M_b$$

Hence if we define

$$\mathfrak{O}_M([a]) = M_a$$

then one obtains a covariant functor $\mathfrak{O}$ from the category of monoids to the category of monoid sketches.

Let $f : M \to N$ be a monoid homomorphism, it induces the homomorphism $f^{sl} : M^{sl} \to N^{sl}$, which is also a morphism of semi-lattices and in particular is a monotonic map. For any element $a \in M$ we have a homomorphism

$$f_{[a]} : M_{[a]} \to N_{[fa]}$$

such that if, $[a] \leq [b]$ then the diagram commutes:

$$\begin{array}{ccc} M_{[a]} & \longrightarrow & N_{[fa]} \\ \downarrow & & \downarrow \\ M_{[b]} & \longrightarrow & N_{[fb]} \end{array}$$

The pair $(M^{sl}, \mathfrak{O}_M)$ plays the role of affine monoidal scheme for monoid sketches.

Recall that for a poset P and an element $x \in P$, we have the following subset of P:

$$(x) = \{y \in P | x \leq y\}$$

For the next definition, we will need to introduce a notation. If $X : P \to Monoids$ is a functor defined on P by the restriction of X on the subset (x) we obtain a functor $(x) \to Monoids$ which we will denote by $Res_x X$.

Now we are in the position to define the notion of a monoid system which will corresponds to monoid schemes.

Definition 5.20. *A* **monoid system** *is a monoid sketch* $(\mathfrak{P}, \mathfrak{F}_{\mathfrak{P}})$ *such that for any element* $x \in \mathfrak{P}$ *the monoid sketch*

$$((x), Res_x \mathfrak{F}_{\mathfrak{P}})$$

is isomorphic to the monoid sketch $(M^{sl}, \mathfrak{O}_M)$*, where* $M = \mathfrak{F}_{\mathfrak{P}}(x)$*.*

A **morphism of monoid systems** is simply a morphism of monoid sketches.

Definition 5.21. *A monoid system* $(\mathfrak{P}, \mathfrak{F}_{\mathfrak{P}})$ *is* **locally noetherian** *if for any element* $x \in \mathfrak{P}$ *the monoid* $\mathfrak{F}_{\mathfrak{P}}(x)$ *is finitely generated. A locally noetherian monoid system is* **noetherian** *if* $\mathfrak{P}$ *is a finite poset.*

6. Toric Varieties

6.1. **Cones.** Let A be a free abelian group of finite rang (i.e. $A \equiv \mathbb{Z}^n$ for some $n \in \mathbb{N}$). We have the dual group $B = Hom(A, \mathbb{Z})$. For an element $b \in B$ and an element $a \in A$ the evaluation of b at a is denoted by

$$\langle a, b \rangle = b(a) \in \mathbb{Z},$$

Denote by $A_{\mathbb{R}} = A \otimes_{\mathbb{Z}} \mathbb{R}$, hence as $A = \mathbb{Z}^n \Rightarrow A_{\mathbb{R}} = \mathbb{R}^n$.

Definition 6.1. *A* **strongly rational, convex polyhedral cone** *generated by* $v_1, \cdots, v_s \in A$*, or simply referred to as a cone from now on, is a set*

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s \in A_{\mathbb{R}} : r_i \geq 0, r_i \in \mathbb{R}\}$$

such that $\{\sigma\} \cap \{-\sigma\} = (0)$.

The dual cone of σ is given by

$$\sigma^\vee = \{u \in B_\mathbb{R} : \langle u, v\rangle \geq 0 \ for\ all\ v \in \sigma\}.$$

From the theory of convex sets, it follows that $(\sigma^\vee)^\vee = \sigma$.

Definition 6.2. *By a* **face** τ *of* σ*, we mean the intersection of* σ *with any supporting hyperplane*

$$\tau = \sigma \cap u^\perp = \{v \in \tau : \langle u, v\rangle = 0\}$$

for some u *in* $\sigma^\vee$.

Proposition 6.3. *We have the following basic properties*

- *The cone is a face of the cone*
- *any face is a cone itself*
- *any intersectin of face is also a face*
- *the dual of a cone is a cone*

For the proof we refer to [11].

Definition 6.4. *Every cone* σ *defines a monoid*

$$M_\sigma = \sigma^\vee \cap B = \{u \in B : \langle u, v\rangle \geq 0 \ for\ all\ v \in \sigma\}$$

By Gordons lemma, the monoid M_σ is finitely generated [11].

Proposition 6.5. *There is a one-to-one correspondence between prime ideals of* M_σ *and faces of* σ.

Proof. Let us introduce the following equivalence relation on M_σ, where $a \sim b$ if and only if they generate the same face, i.e. $\{v_a\} = \{v_b\}$, where

$$\{v_a\} = \{v \in \sigma : \langle a, v\rangle = 0\}$$

and

$$\{v_b\} = \{v \in \sigma : \langle b, v\rangle = 0\}$$

So we have $\tau_a = v_a \cap \sigma$ and $\tau_b = v_b \cap \sigma$. It suffice to prove that

$$M/\sim = M^{sl}$$

So I have to prove that $a \sim b$ if and only if $\exists n, m \in \mathbb{N}, t, s \in M_\sigma$ such that $na = b + t$ and $mb = a + s$. Say $\exists n, m, t, s$ such that $na = b + t$ and $mb = a + s$. Then from $\langle a, v\rangle = 0 \Rightarrow \langle na, v\rangle = 0 \Rightarrow \langle b + t, v\rangle = 0 \Rightarrow \langle b, v\rangle + \langle t, v\rangle = 0$. As $t \in M_\sigma$, by definition $\langle t, v\rangle \geq 0$. Hence $\langle b, v\rangle = 0 = \langle t, v\rangle$. Similar for $mb = a + s$. It follows that $a \sim b$. The other way is a statement (11) on page 13 in [11]. Hence by Corollary 4.18, it follows that the prime ideals of M_σ and faces of σ are in a one-to-one relation. □

Since this bijection is order-reversing, it follows that the set of faces on σ is a poset, opposite to the $Kspec(M_\sigma)$.

As shown, localising with prime ideals is equivalent to localising with elements of M^{sl} i.e. $M_{[a]} \equiv M_{\alpha(a)}$ where $\alpha_M : M^{sl} \longrightarrow KSpec(M)$ is the homomorphism described in Corollary 4.18. Hence, to show that the diagram

$$\begin{array}{ccc} \text{faces of } \sigma & \xrightarrow{\sim} & \text{primes of } M_\sigma \\ \downarrow & & \downarrow \\ M_\tau & \xrightarrow{\sim} & (M_\sigma)_p \end{array}$$

commutes, it suffices to show that M_τ is indeed localisation with τ, i.e. that it makes elements of $\{u\}$, where $\tau = \sigma \cap u^\perp$ invertible, but this is exactly the statement 2 in page 13 of [11].

6.2. **Fan's.**

Definition 6.6. *By a* **fan** $\triangle$ *in* A*, we mean a set of cones* σ *in* $A_{\mathbb{R}}$ *such that*

- *Each face of a cone in* $\triangle$ *is a cone in* $\triangle$
- *The intersection of two cones in* $\triangle$ *is a face of each.*

Next we need to show that a fan induces a monoid scheme. First observe that we are given a poset where elements are faces. Next I have to define a functor from the poset to the category of monoids. This is given by

$$\tau \mapsto M_\tau = \tau^\vee \cap B.$$

This is a contravariant functor, since

$$\tau_1 \subset \tau_2 \Rightarrow \tau_2^\vee \subset \tau_1^\vee.$$

Finally I have to show that locally the fan is like a Kato-spectrum of a cone, i.e. like a dual cone. But this is clear, since for every element, the set of faces less or equal to it forms a cone, hence it is the dual cone of its dual. It follows that a fan is essentially like the spectrum of a monoid.

6.3. **Realisation of fan's.** Since a fan induces a monoid scheme we can take realisation to get a genuine scheme corresponding to a fan.

Actually realisation of a monoid scheme can be defined over any commutative ring k, simply replacing $\mathbb{Z}[M]$ by $k[M]$. Note that, for geometrical purposes, we want the k-algebra $k[M]$ to be integral domains. So for this, the monoid clearly has to be cancellative, but even if it is, it is still not enough. Consider $M = C_p$, the cyclic group, which is of course cancellative. Note that then in say $\mathbb{Z}[C_p]$,

$$x^p - 1 = 0 = (x-1)(1 + x + x^2 + \cdots + x^{p-1}).$$

So we need an extra condition, for the ring to be integrally closed, namely, since M is cancellative, we can consider the group generated by M, i.e. $M^{-1}M$ and this group has to be torsion free. In other words, if $x^m = x^n$ in M, then $m = n$. These two conditions are enough, $k[M]$ to be an integral domain.

7. Congruence co-sheaves

7.1. **Introduction.** I was interested in congruences for quite some time, and throughout the course of this project I kept thinking about it. And even though I had a few small successes every now and then, it really was never enough to mention it. It was only about one week before the deadline, that I realised that I had found a way to extend the sheaf from prime ideals to prime congruences.

While this extension is not perfect and nowhere close to a finalised version, I still think to a very important part of this whole project, giving many things a deeper meaning (such as the fact that M^{sl}, which is equivalent to the spectrum, has a covariant functor.).

In this chapter, I will explain and define thing a little different then in the above chapters, but I will try to give arguments at least, if not always a proof, as to why these will be equivalent. Also I will not go into too much detail and I will restrict myself to a very special case, namely the cancellative monoids, though I believe that after a little adjustments in definitions and proofs, this result will also hold for more general monoids as well.

This chapter can be seen as *extra* and not so much as a part of the project, but I really liked it and wanted to include it.

7.2. **Introducing the cosheaf.**

Definition 7.1. *Let M be a monoid and let K be a congruence on M. We say that K is a* **prime congruence***, if M/K is a cancellative monoid, or a cancellative monoid with a zero. Equivalently, if for elements $a, x, y \in M$ one has $ax \sim ay$ then $x \sim y$ or $az \sim a$ for any $z \in M$. s*

Definition 7.2. *The* congruence spectrum*, $Speactrum(M)$, is the set of all prime congruences and has a natural topology, where the principal open sets are*

$$\mathfrak{D}(f, g) = \{k \in Spectrum(M) | a \nsim_K b\}.$$

Unfortunately they are not closed with respect to intersection, but if we add all the intersections, then it forms a basis for the topology.

Definition 7.3. *Define a functor from $CSpec(M)$ to the category of moniods by*

$$\mathfrak{X}(\mathfrak{D}(a, b)) \equiv \mathfrak{X}(a, b) = \langle M \cup \langle x, y \rangle \rangle / (ax = b, by = a)$$

And define

$$\mathfrak{X}(\mathfrak{D}(a, b) \cup \mathfrak{D}(f, g)) \equiv \mathfrak{X}((a, b) \cup (f, g)) = \mathfrak{X}(af, ag, bf, bg),$$

where

$$\mathfrak{X}(af, ag, bf, bg) = M \cup \langle x_1, \cdots, x_{16} \rangle / (afx_1 = ag, \cdots, bfx_{16} = bg),$$

every combination of $\{af, ag, bf, bg\}$.

Definition 7.4. *Now that we associate a monoid to the union, we can use the"pullback" to associate a moniod to the intersection, i.e. the unique monoid, making the following diagram*

$$\begin{array}{ccc} \mathfrak{D}(a,b)\cup\mathfrak{D}(f,g) & \longleftarrow & \mathfrak{D}(a,b) \\ \uparrow & & \uparrow \\ \mathfrak{D}(f,g) & \longleftarrow & \mathfrak{D}(a,b)\cap\mathfrak{D}(f,g) \end{array}$$

commutative.

What I am doing in essence, is that I am adding $\frac{a}{b}$ and $\frac{b}{a}$ to M. In the later case, I am adding fractions of the type $\frac{af}{ag}$ and $\frac{af}{gb}$

In what follows we will assume that M is a cancellative monoid. We can and we will identify M as a submonoid of M^{gr}. In particular for elements $a, b \in M$ we will frequently write $\frac{a}{b}$ assuming that the resulting element is in M^{gr}. This will make everything much simpler, hence I will redefine the functor $\mathfrak{X}$.

Definition 7.5. *For $a, b \in M$ define the monoid $\mathfrak{X}(\mathfrak{D}(a,b))$ to be the submonoid of M^{gr} generated by the submonoid M and by the elements $\frac{a}{b}$ and $\frac{b}{a}$. If*

$$U = \cap_{i=1}^{n}\mathfrak{D}(a_i, b_i)$$

then we put

$$\mathfrak{X}(U) = \cap_{i=1}^{n}\mathfrak{X}(a_i, b_i) \subset M^{gr}$$

Hence we have defined $\mathfrak{X}$ on the generating open sets, if

$$U = \cup_{i\in I}U_i$$

is an open set, where U_i is a basis open subset, then we define $\mathfrak{X}(U)$ to be the smallest submonoid of M^{gr} generated by the submonoids $\mathfrak{X}(U_i)$.

For simplicity sometimes we will write $\mathfrak{X}(a,b)$ instead of $\mathfrak{X}(\mathfrak{D}(a,b))$.

Proposition 7.6. *This construction is well defined, in other words, if $\mathfrak{D}(a,b) \subset \mathfrak{D}(f,g)$. Then $\mathfrak{X}(a,b) \subset \mathfrak{X}(f,g)$. In particular $\mathfrak{X}(a,b)$ depends only on $\mathfrak{D}(a,b)$ and not a and b.*

To prove this we first need a few lemmas.

Lemma 7.7. *Let K be a prime congruence. Define*

$$\mathfrak{p}_K = \{a \in M | az \sim_K a, \text{ for all } z \in M\}$$

Then $\mathfrak{p}_K$ is a prime ideal.

Proof. If $a \in \mathfrak{p}_K$, then for any $b \in M$ and $z \in M$ we have

$$(ab)z = a(bz) \sim a \sim ab$$

where we write $\sim$ instead of $\sim_K$. This shows that $ab \in \mathfrak{p}_K$. Hence $\mathfrak{p}_K$ is an ideal. Assume $xy \in \mathfrak{p}_K$ and $y \notin \mathfrak{p}_K$. Then there exists u such that $yu \nsim y$. Since $x(yu) \sim xy$ and $\sim$ is a prime congruence it follows that $x \in \mathfrak{p}_K$. □

Lemma 7.8. *Let H be a subgroup of M^{gr}. Define the relation $\sim_H$ on M by $a \sim_H b$ if $\frac{a}{b} \in H$. Here a and b are elements of M. Then $\sim_H$ is a prime congruence.*

Proof. The fact that $\sim$ is a congruence is easy to see. Assume know $ax \sim ay$. Then $\frac{ax}{ay} \in H$. Hence $\frac{x}{y} \in H$ and $x \sim y$. □

Lemma 7.9. *Let $\mathfrak{p}$ be a prime ideal of M, then the Rees congruence $K_\mathfrak{p}$ corresponding to $\mathfrak{p}$ is a prime congruence.*

Proof. Recall that $x \sim y$ if and only if $x = y$ or $x, y \in \mathfrak{p}$. Assume now $ax \sim ay$. If $ax = ay$, then $x = y$. If $ax \neq ay$ then $ax, ay \in \mathfrak{p}$. Hence either $a \in \mathfrak{p}$ or $x, y \in \mathfrak{p}$. So we have either $x \sim y$ or $az \cong a$ for all z. This finishes the proof. □

Lemma 7.10. *For elements $a, b, f, g \in M$ one has*

$$\mathfrak{D}(a, b) \subset \mathfrak{D}(f, g)$$

if and only if $a = b$, or the the following two conditions hold:

i) There exist a natural number n such that

$$\frac{a}{b} = \left(\frac{f}{g}\right)^n$$

ii) For any prime ideal $\mathfrak{p}$ the condition $f, g \in \mathfrak{p}$ implies $a, b \in \mathfrak{p}$.

Proof. Assume $\mathfrak{D}(a, b) \subset \mathfrak{D}(f, g)$. This means that for any prime congruence K the condition $f \sim_K g$ implies $a \sim_K b$. We also assume that $a \neq b$. Take a prime ideal $\mathfrak{p}$. If $f, g \in \mathfrak{p}$, we can consider the corresponding Rees congruence $K_\mathfrak{p}$. Then $f \sim g$ and hence $a \sim_{K_\mathfrak{p}} b$. Since $a \neq b$ we have $a, b \in \mathfrak{p}$. Thus the condition ii) holds. To show i) also holds consider the subgroup H of M^{gr} generated by $\frac{f}{g}$. Then we obtain the prime congruence $\sim_H$ (Le.7.8). Since $f \sim_H g$ by the definition we obtain $a \sim_H b$. Hence $\frac{a}{b} \in H$. By our construction the subgroup H is cyclic generated by $\frac{f}{g}$ and i) follows.

Conversely, if $a = b$, then $\mathfrak{D}(a, b) = \emptyset$ and hence $\mathfrak{D}(a, b) \subset \mathfrak{D}(f, g)$. Assume now $a \neq b$ and i) and ii) hold. We have to prove that for any prime congruence K the condition $f \sim_K g$ implies $a \sim_K b$. For the prime congruence K we consider the corresponding prime ideal $\mathfrak{p}_K$. By i) we have $ag^n = bf^n$. Since $f \sim_K g$, we obtain $af^n \sim_K ag^n = bf^n$, which implies either $a \sim f$ or $f^n \in \mathfrak{p}_K$. If the first case we are done. If $f^n \in \mathfrak{p}_K$, since $\mathfrak{p}_K$ is a prime ideal, it follows that $f \in \mathfrak{p}_K$. Since $f \sim g$ we also have $g \in \mathfrak{p}_K$. By 2) we also have $a, b \in \mathfrak{p}_K$ and hence $a \sim_K b$. □

Proof. of the Proposition 7.6: □

Proof. By Proposition 7.10 we have $\frac{a}{b} = \left(\frac{f}{g}\right)^n$, which shows that $\mathfrak{X}(a, b) \subset \mathfrak{X}(f, g)$. □

7.3. **Maps between M^{sl} and $CSpec(M)$.**

First I will explain M^{sl} and the cosheaf associated to it a bit better.

Proposition 7.11. *Let $\langle [a] \rangle$ denote the set of elements of M^{sl} less then or equal to $[a]$. Recall that each element of M^{sl} is a class of elements in M. Then*

- *$\langle [a] \rangle$ seen as a subset of M is a face, i.e. the complement of a prime ideal.*
- *Every face is of the form $\langle [a] \rangle$.*

Proof. To prove the first assertion, recall that a face is a submonoid S satisfying $xy \in S \Rightarrow x \in S, y \in S$. Say $x \in \langle [a] \rangle$ and $y \in \langle [a] \rangle$, where x and y are seen as elements of M and $[a]$ is seen as a class of elements of M. Then $x \in [u]$ and $y \in [v]$ where $[u], [v] \leq [a]$. Hence $xy \in [u][v] \leq [a] \in \langle [a] \rangle$, so it is a submonoid of M. To prove that it's a face, observe that if $x \notin \langle [a] \rangle \Rightarrow x \in [b]$ such that $[b] \geq [a]$. But then it is simple to see that $xy \in [c]$ such that $[c] \geq [b], \forall y \in M$. This finishes the proof.

The second claim is trivial, as the elements of M^{sl} are in a one-to-one relation to the elements of $Kspec(M)$. □

I did not mention the notion of *open* in M^{sl}, as it was simpler to look at things as poset functors, rather then sheafs or cosheafs, but since I now want to extend M^s to the congruence spectrum, which of course can no longer be given with a lattice, it is vital to introduce a topology on M^{sl}.

So here we are given 2 different, and equal ways to introduce the topology. One is to call ideals of M^{sl} open, this would correspond to the classical contravariant sheaf on $KSpec(M)$ the other is to call coideals open, this one gives rise to the covariant functor, that I called the monoid system. Note that what I am doing here, is calling open sets in the zariski-topology of $Kspec(M)$ closed, and closed sets open. Because our set is finite, this is not a problem, and the equivalence is proven as one of the main theorems of this project.

Since we described the cosheaf in M^{sl} in the following way that if $x \leq y \Rightarrow M_x \leq M_y$, it is clear that in fact open sets here are coideals.

To give meaning to our definition of the congruence cosheaf, observe that we have many maps connecting M^{sl} and $CSpec(M)$.

Proposition 7.12. *We have the following maps from the topological space M^{sl} to the topological space $CSpec(M)$:*
(1) $\delta : M^{sl} \to Spectrum(M)$ with

$$\delta([a]) = K_{[a]}$$

where K is a congruence relations such that $u \sim_{K_{[a]}} v$ if and only if $u, v \in \langle [a] \rangle$, then $u, v \sim_{K_{[a]}} 1$.

(2) $\overline{\delta} : Open(M^{sl}) \to Open(Spectrum(M))$ *given by*

$$\delta(\langle [a] \rangle) = \bigcup_{x \in \langle [a] \rangle} \mathfrak{D}(x, 1)$$

Note that since every element of M^{sl} uniquely defines an open set, we in fact have maps from elements to elements, elements to open sets, open sets to elements and open sets to open sets, showing once again that there are many things to be clarified yet.

Proof. □

Proposition 7.13. *We have the following map from* $CSpec(M)$ *to topological space* M^{sl}*:*

(1) $\tau : Spectrum(M) \to Open(M^{sl})$*, given by*

$$\tau(K) = \{x | \overline{x} \in U(M/K)\} = \langle [a] \rangle$$

where $U(M/K)$ *is the set of invertible elements of the quotient monoid of* M *with the congruence relation* K *and* $\overline{x}$ *is* x *modulo* K*.*

(2) $\overline{\tau} : Spectrum \to M^{sl}$ *given by*

$$\tau(K) = [a]$$

where $[a]$ *is the maximal class of* $\langle [a] \rangle = \{x | x \in U(M/K)\}$

Proof. In other words, we want to show that $\tau(K) = \langle [a] \rangle = \{x | \overline{x} \in U(M/K)\}$ is a principal coideal of M^{sl}. To prove this, it suffices to prove that its complement, $Y = \{y | \overline{y} \notin U(M/K)\}$, is a prime ideal and then use Proposition. 7.11.

So say y is such that $\overline{y}$ is not invertible in M/K. Then $\overline{ya}$ is not invertible for all a. Hence Y is an ideal. 1 is invertible, hence Y is a proper ideal. Finally, if $\overline{ab}$ is not invertible, but $\overline{a}$ is invertible, then clearly $\overline{b}$ is not invertible. Since every face is a principal coideal, it has a unique maximal element, giving us the second map. □

Corollary 7.14. *If* $[a]$ *is invertible, then* $\langle [a] \rangle$ *is invertible as well.*

Proposition 7.15. $\tau \circ \delta = id$

7.4. **relation between the monoid system and the congruence cosheaf.**

Lemma 7.16. $\mathfrak{X}(f, 1) \cup \mathfrak{X}(g, 1) = \mathfrak{X}(fg, 1)$

Proof. By definition, $\mathfrak{X}(f, 1) \cup \mathfrak{X}(g, 1) = \mathfrak{X}(fg, f, g, 1)$ so I claim that $\mathfrak{X}(fg, 1) = \mathfrak{X}(fg, f, g, 1)$. Clearly as $LHS \subset RHS$. But if $N = \left\langle M \cup \frac{1}{fg} \right\rangle$, it follows that we have $\frac{f}{fg}$ in N, hence we have $\frac{1}{g}$ in N and similarly $\frac{1}{f}$. Hence $RHS \subset LHS$. □

Theorem 7.17. *The congruence cosheaf* $\mathfrak{X}$ *restricted to* M^{sl} *is the monoid system.*

Proof. Take in M^{sl} the principal open set, $\langle [a] \rangle$. The monoid associated to the monoid system is M_a. Now to $\delta(\langle [a] \rangle)$ we associate

$$\mathfrak{X}(\bigcup_{x \in \langle [a] \rangle} \mathfrak{D}(x, 1)) = \text{ by Lemma 7.16 } = \mathfrak{X}(\mathfrak{D}(\prod_{x \in \langle [a] \rangle}, 1)) = \text{ by Cor. 7.14} = \mathfrak{X}(\mathfrak{D}(\prod_{x \in [a]}, 1))$$

which means that the elements of $[a]$ become invertible, which is equivalent to localising with the element a.

The second thing to prove is that if $\langle[a]\rangle \subset \langle[b]\rangle$, then

$$\mathfrak{X}(\bigcup_{x\in\langle[a]\rangle}\mathfrak{D}(x,1)) \subset \mathfrak{X}(\bigcup_{y\in\langle[b]\rangle}\mathfrak{D}(y,1))$$

Now if $\langle[a]\rangle \subset \langle[b]\rangle$, then $[a] \leq [b]$, hence $[ab] = [b]$. Take a congruence K in the union $\bigcup_{x\in\langle[a]\rangle}\mathfrak{D}(x,1)$, hence $K \in \mathfrak{D}(x,1)$ for some $x \in [a]$ by $Cor.7.14$. In other words, K is a congruence, such that $x \nsim_K 1 \Leftrightarrow$ (since K is prime) $x^2y \nsim_K xy$. Since this is true for all $y \in M$, take $y \in [b]$. Then $x^2y \nsim_K xy$, it follows that either $x^2y \nsim_K 1$ or $xy \nsim_K 1$. And as both $xy, x^2y \in [ab] = [b], K \in \mathfrak{D}(xy,1)\cup\mathfrak{D}(x^2y,1) \subset \bigcup_{y\in\langle[b]\rangle}\mathfrak{D}(y,1)$. □

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